Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Systems with vanishing time-variant delay: Structural properties based on the Pantograph (Scale-Delay) equation. ⋆

Erik I. Verriest *

* *Georgia Institute of Technology, Atlanta, GA 30332 USA*
erik.verriest@ece.gatech.edu

Abstract: This paper studies the scale-delay (SD) system as a paradigm for systems with vanishing time-variant delay. After a brief review of the scale-delay equation and its associated solution, we derive new identities for the zeros of the deformed exponential. A solution method for the higher order scale-delay equation is given in terms of these deformed exponentials. We then consider the inverse problem: "Given a function x, which differential equation does it solve?" but with a twist: We are interested in time-variant realizations of the solution, as described in a companion paper. This extends the well known solution for Bohl functions, where an LTI-ODE solves the problem. If the function belongs to a subclass of real-analytic functions with only single real zeros, a regular smooth second-order linear time-variant annihilator exists. Finally this is applied to obtain a second order time-variant approximation of the SD-equation.

Keywords: Systems with time delays, linear systems, time-varying systems, non-parametric methods

1. INTRODUCTION

The scale-delay (SD) equation, $\dot{x}(t) = Ax(t) + Bx(\alpha t)$ where $\alpha \in (0, 1)$, corresponds to a delay equation with delay $\tau(t) = (1 - \alpha)t$, and therefore satisfies the causality condition $\dot{\tau} < 1$ in Verriest [2011]. As a delay equation, it is infinite dimensional. This causality condition is necessary to make the problem well-posed. However, the equation with $\alpha > 1$ was used by Amburtsumian to model the absorption of light in interstellar matter. Causal models appear in certain studies of coherent states in quantum theory (Griebel [2017], Spiridonov [1996]) and models of cell growth in biology (van Brunt et al. [2018]). In both limit cases, $\alpha \in \{0, 1\}$, the system is finite dimensional. Also for all, α, the system starts at $t = 0$ with a finite dimensional initial condition at 0, and thus builds up its own memory, which is required to evolve the system for $t > 0$. We coin such a system as a *self-starting* system. The scale-delay equation, also known as the pantograph equation has been studied extensively, and interesting properties are being discovered (Liu [2018]). For this reason, we shall take it as a canonical form and starting point to study a delay system with a time-varying delay vanishing in a point-set $\mathcal{T} \subset \mathbb{R}$. We expect these system to have an (at least locally) self-starting effect. Finite-dimensional linear systems that are feedback-controlled by a digital computer through a sample and hold scheme form an important class of systems with such behavior, albeit for $\alpha = 0$.

In order to set the stage, we first present some known results about the scale-delay systems. Its solution is known by a series expansion, which in a specific case has a complexity just slightly higher than the exponential function. It is therefore referred to it as the *deformed exponential*. As we are only interested in behaviors for a short instant after the time where the delay vanishes, we shall work with finite truncations of the series expansion. In a companion paper, submitted to this conference (Verriest [2020]), we explore a realization problem that is directly useful for this purpose: Given a smooth function, represent it by its time-variant linear differential operator (a regular differential polynomial with smooth time varying coefficients) and its initial condition.

In order to make this paper self-contained, Section 3 builds on the ideas from (Verriest [2020]), after briefly introducing the scale-delay equation in Section 2. In that section we also present some new identities for the zeros of the deformed exponential, and show how a straightforward extension (using a generalized characteristic equation) from LTI system theory allows to solve the homogeneous equation in higher dimensions in terms of these deformed exponentials. We give an approximate two-dimensional linear time-variant representation for the deformed exponential in Section 4. Section 5 briefly returns to the class of self-starting systems.

2. SCALE-DELAY EQUATION

Valeev [1964] showed that the scalar functional differential equation (FDA)

$$\dot{y}(t) = \mu y(t) + \beta y(\alpha t), \quad y(0) = 1, \tag{1}$$

has a solution given by the series expansion

$$y(t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \prod_{i=0}^{k-1} (\mu + \beta\alpha^i). \tag{2}$$

⋆ The support by the NSF grant CPS-1544857 is gratefully acknowledged.

Application of the ratio test shows that this series has an infinite radius of convergence, hence it is an entire function. For $\mu \neq 0$, the solution diverges if $|\beta| > |\mu|$ and converges if $|\beta| < |\mu|$. A particularly interesting case appears for $\mu = 0$, $\beta = 1$: The unit solution ($y(0) = 1$) (a.k.a. the *deformed exponential*)

$$y(t) = \sum_{k=0}^{\infty} \frac{\alpha^{k(k-1)/2}}{k!} t^k \stackrel{\text{def}}{=} E_\alpha(t), \tag{3}$$

satisfies $0 < y(t) < \exp(t^\epsilon)$ for all $\epsilon > 0$, and $y(t) \geq t^{\frac{\ln \ln t}{2 \ln \alpha} + o(\ln t)}$. It follows that for $\mu = 0$ and arbitrary β, the unit solution is the time-scaled version $E_\alpha(\beta t)$. Perhaps surprisingly, the solution $E_\alpha(-t)$ for $\beta = -1$ is oscillatory and diverges. Its zeros are asymptotically given by $t_k = \frac{k}{\alpha^{k-1}}(1 + \phi(\alpha)k^{-2} + o(k^{-2}))$, where $\psi(\alpha)$ is the generating function of the sum-of-divisors function $\sigma(k)$ (Zhang [2016], Wang [2017]). Since the order, that is, $\inf_{r>0}\{E_\alpha(-z) \sim O(\exp|z|^r)\}$, of $E_\alpha(z)$ is zero, Hadamard's factorization theorem (see Rudin [1974]) yields the simple form in terms of the roots $\{t_k > 0\}$ of $E_\alpha(-t)$:

$$E_\alpha(-t) = \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_n}\right). \tag{4}$$

Zabko et al. [1997] show that the higher dimensional FDE

$$\dot{x}(t) = Ax(t) + Bx(\alpha t), \quad x \in \mathbb{R}^n, \tag{5}$$

where for $f(s) = \det(sI - A)$, and $\phi(s) = \det(A + \mathrm{e}^{s \ln \alpha} B)$ we define

$$\overline{\lambda} = \max_{f(s)=0} \mathrm{Re}\lambda$$
$$\overline{\mu} = \max_{\phi(s)=0} \mathrm{Re}\,\mu, \quad \text{setting } \overline{\mu} = -\infty \text{ if there are no roots,}$$

the system is asymptotically stable if $\overline{\lambda} < 0$ and $\overline{\mu} < 0$, and unstable if $\overline{\lambda} > 0$ or $\overline{\mu} > 0$.

Apart form its purely theoretical interest, with applications in number theory (Kato [1971]), scale-delay equations (pantograph equations) have been successful in modeling cell division (van Brunt et al. [2018]), electrodynamics, quantum calculus (Griebel [2017]), and control (Verriest [2001]).

2.1 Zero-Properties of the deformed exponential $E_\alpha(-t)$

Consider the series of inverse powers of the roots

$$S_n = \sum_{k=1}^{\infty} \frac{1}{t_k^n}, \quad n = 1, 2, \ldots. \tag{6}$$

Let also c_n denote the coefficient of t^n in $E_\alpha(t)$ in (3). Using the extensions of Newton's identities for Weierstrass products, (See Beuer [2012]) one finds

$$S_0 = 1$$
$$S_n - c_1 S_{n-1} + c_2 S_{n-2} + \cdots + (-1)^n n c_n S_0 = 0.$$

Theorem 1. The sum of inverse powers of the roots can be recursively computed from (3), and give in particular the sequential relations

$$S_1 = 1$$
$$S_2 = 1 - \alpha$$
$$S_3 = \frac{1}{2}(1-\alpha)^2(2+\alpha)$$
$$S_4 = \frac{1}{6}(1-\alpha)^3(6 + 6\alpha + 3\alpha^2 + \alpha^3)$$
$$S_5 = \frac{1}{24}(1-\alpha)^4(24 + 36\alpha + 30\alpha^2 + 20\alpha^3 + 10\alpha^4 + 4\alpha^5 + \alpha^6).$$

In addition, the zeros satisfy

$$\sum_{k=1}^{\infty} \frac{1}{\alpha t_k - t_\ell} = 0, \quad \ell = 1, 2, \ldots. \tag{7}$$

and

$$\frac{1}{t_k} \prod_{\ell \neq k} \frac{t_\ell - t_k}{t_\ell - \alpha t_k} = 1 - \alpha, \quad k = 1, 2, \ldots \tag{8}$$

Proof. The identities (7) and (8) follow by substituting Hadamard's expansion in the FDE, and evaluating respectively at $t = \frac{t_\ell}{\alpha}$ and $t = t_k$. □

To the best of our knowledge, the identities (7) and (8) are new.

3. HIGHER ORDER SCALE-DELAY REALIZATION

Given $\dot{x}(t) = Ax(\alpha t)$ with $y = cx$ of dimension 1. Let $A \in \mathbb{R}^{n \times n}$ have characteristic polynomial $a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$. Successive differentiation of the output yields

$$\begin{aligned}
\dot{y}(t) &= cAx(\alpha t) \\
\ddot{y}(t) &= cA^2 \alpha x(\alpha^2 t) \\
y^{(3)}(t) &= cA^3 \alpha^{1+2} x(\alpha^3 t) \\
&\vdots \\
y^{(n-1)}(t) &= cA^{n-1} \alpha^{1+2+\cdots+(n-1)} x(\alpha^{n-1} t) \\
y^{(n)}(t) &= cA^n \alpha^{1+2+\cdots+n} x(\alpha^n t).
\end{aligned}$$

By the Cayley-Hamilton theorem, it follows that

$$\begin{aligned}
&y^n(t) + a_1 \alpha^n y^{(n-1)}(\alpha t) + \\
&\quad + a_2 \alpha^{n+(n-1)} y^{(n-2)}(\alpha^2 t) + \cdots + a_n \alpha^{n+\cdots+1} y(\alpha^n t) = 0.
\end{aligned}$$

This insight leads to a solution method for a class of scale-delay equations:

3.1 Solution of a class of SD-equations

We use an operator form to simplify notation. Denote by $\mathbf{D}$ and $\mathbf{Q}$ are respectively the *differentiation* and *multiplication by the independent variable* operator. These operators generate an operator algebra, called the Weyl-algebra, important in quantum mechanics. The fundamental commutation relation $\mathbf{DQ} - \mathbf{QD} = 1$ (Heisenberg) is easily shown. The combination $\mathbf{DQ} - \mathbf{QD}$ is also known as the commutator of $\mathbf{D}$ and $\mathbf{Q}$ and denoted by $[\mathbf{D}, \mathbf{Q}]$. This algebra is fundamental for the theory of linear time-variant systems (Verriest [1993]). An algebraic theory of time-variant differential delay systems in the language of functional operators is discussed b Quadrat and Ushirobira in (Quadrat and Ushirobira [2016]).

Let a be a sufficiently smooth function. Define $\mathbf{S}_\alpha$, the scaling operator: i.e., for all t and all x: $\mathbf{S}_\alpha x(t) = x(\alpha t)$. The vector scale-delay equation is in operator form

$$\mathbf{D}x = A\mathbf{S}_\alpha x.$$

The commutation rule $\mathbf{D}\mathbf{S}_\alpha = \alpha\mathbf{S}_\alpha\mathbf{D}$ and product rule $\mathbf{S}_\alpha\mathbf{S}_\beta = \mathbf{S}_{\alpha\beta}$ yield for the products of the factors, $\Omega(a) \stackrel{\text{def}}{=} (\mathbf{D} - a\mathbf{S}_\alpha)$, the commutation rule

$$(\mathbf{D} - \mu\mathbf{S}_\alpha)(\mathbf{D} - \lambda\mathbf{S}_\alpha) = (\mathbf{D} - \alpha\lambda\mathbf{S}_\alpha)(\mathbf{D} - \frac{\mu}{\alpha}\mathbf{S}_\alpha),$$

or more compactly,

$$\Omega\mu\Omega(\lambda) = \Omega(\alpha\lambda)\Omega(\frac{\mu}{\alpha}). \tag{9}$$

Define the product operator by iteration, $\overline{\Omega}(a_1, \ldots, a_n) = \Omega(a_1)\Omega(a_2)\cdots\Omega(a_n)$.

Theorem 2. If $\{\mu_1, \mu_2/\alpha, \ldots, \mu_n/\alpha^{n-1}\}$ are disjoint then

$$\overline{\Omega}(\mu_n, \ldots, \mu_1)y = 0$$

has the solution set

$$\mathrm{Sol}[\Omega(\mu_1)] \oplus \mathrm{Sol}[\Omega(\mu_2/\alpha)] \oplus \cdots \oplus \mathrm{Sol}[\Omega(\frac{\mu_n}{\alpha^{n-1}})],$$

where Sol[*operator*] is the solution set, i.e, the null-space of the [*operator*].

Proof. First note that any y, annihilated by $\Omega(a_n)$, is also a solution to $\overline{\Omega}(a_1, \ldots, a_n)y = 0$. Next, we apply the commutation (9) as follows

$$\begin{aligned}
&\overline{\Omega}(\mu_k, \mu_{k-1} \ldots, \mu_1)\\
&= \overline{\Omega}(\alpha\mu_{k-1}, \frac{\mu_k}{\alpha\mu_{k-2}}, \ldots, \mu_1)\\
&= \Omega(\alpha\mu_{k-1}, \alpha\mu_{k-2}\frac{\mu_k}{\alpha^2}, \ldots, \mu_1)\\
&= \cdots = \overline{\Omega}(\alpha\mu_{k-1}, \alpha\mu_{k-2} \ldots, \alpha\mu_2\frac{\mu_k}{\alpha^{k-1}}).
\end{aligned}$$

With this, for $k = 1, \ldots, n$ it holds that $\overline{\Omega}(\mu_n, \ldots, \mu_1) = \overline{\Omega}(\mu_n, \ldots, \mu_{k+1})\overline{\Omega}(\mu_{k-1}, \ldots, \mu_1)\Omega(\frac{\mu_k}{\alpha^{k-1}})$..

Consequently, functions annihilated by $\mathrm{Sol}[\Omega(\mu_k/\alpha^{k-1})]$ are solutions of the given FDA. For differing parameter, the solution sets are independent. □

One can deal with "repeated factors" by analogy with the LTI case, but with a 'twist'. Consider

$$(\mathbf{D} - \alpha^k\mu\mathbf{S}_\alpha)\cdots(\mathbf{D} - \alpha\mu\mathbf{S}_\alpha)(\mathbf{D} - \mu\mathbf{S}_\alpha)$$

in the following way: Let $y \in \mathrm{Sol}[\mathbf{D} - \alpha\mu\mathbf{S}_\alpha]$, and let $x = \mathbf{Q}y$. Then, using (9) again

$$\begin{aligned}
(\mathbf{D} - \mu\mathbf{S}_\alpha)\mathbf{Q} &= \mathbf{D}\mathbf{Q} - \mu\mathbf{S}_\alpha\mathbf{Q}\\
&= \mathbf{Q}\mathbf{D} + 1 - \mu\alpha\mathbf{Q}\mathbf{S}_\alpha\\
&= \mathbf{Q}(\mathbf{D} - \mu\alpha\mathbf{S}_\alpha) + 1.
\end{aligned}$$

Consequently,

$$(\mathbf{D} - \mu\mathbf{S}_\alpha)\mathbf{Q}y = \mathbf{Q}(\mathbf{D} - \mu\alpha\mathbf{S}_\alpha)y + y = y.$$

But then,

$$(\mathbf{D} - \alpha\mu\mathbf{S}_\alpha)(\mathbf{D} - \mu\mathbf{S}_\alpha)\mathbf{Q}y = (\mathbf{D} - \alpha\mu\mathbf{S}_\alpha)y = 0.$$

Now, denote the solution to $(\mathbf{D} - \mu\mathbf{S}_\alpha)y$ with $y(0) = 1$ as the scale exponential $E_\alpha(\mu t)$. Then the previous may be iterated to establish that the general solution to

$$(\mathbf{D} - \alpha^k\mu\mathbf{S}_\alpha)\cdots(\mathbf{D} - \alpha\mu\mathbf{S}_\alpha)(\mathbf{D} - \mu\mathbf{S}_\alpha)x = 0$$

is given by $p_k(\mathbf{Q}\mathbf{S}_\alpha)E_\alpha(\mu t)$, where $p_k(s) \in \mathbb{R}[s]$ of degree k. We formulate now the following theorem:

Theorem 3. To solve the scale-delay equation

$$\left(\mathbf{D}^n + a_1\mathbf{S}_\alpha\mathbf{D}^{n-1} + a_2\mathbf{S}_{\alpha^2}\mathbf{D}^{n-2} + \cdots + a_n\mathbf{S}_{\alpha^n}\right)x = 0,$$

define its *scale-delay characteristic polynomial* by

$$\begin{aligned}
a_\alpha(\lambda) = a_n + a_{n-1}\lambda + \alpha a_{n-2}\lambda^2 +&\\
+ \alpha^{1+2}a_{n-3}\lambda^3 \cdots + \alpha^{1+2+\cdots+(n-1)}\lambda^n&,
\end{aligned}$$

and solve the characteristic equation $a_\alpha(\lambda) = 0$ for λ. If λ_i has multiplicity k_i, then

$$\mathrm{Sol}[a_\alpha(\mathbf{D})] = \bigoplus_i p_i(\mathbf{Q}\mathbf{S}_\alpha)E_\alpha(\lambda_i t)$$

where $\deg p_i = k_i - 1$, and $\bigoplus$ denotes a direct sum.

3.2 Some useful properties

In this subsection a is an arbitrary sufficiently smooth function. We derive some additional identities in the Weyl algebra:

Lemma 4. For any sufficiently smooth function a, it holds that $(\mathbf{D} + a)\mathbf{Q}^k = \mathbf{Q}^k(\mathbf{D} + a) + k\mathbf{Q}^{k-1}$.

Proof. Simply iterate

$$\begin{aligned}
(\mathbf{D} + a)\mathbf{Q}^k &= \mathbf{Q}(\mathbf{D} + a)\mathbf{Q}^{k-1} + \mathbf{Q}^{k-1}\\
&= \mathbf{Q}^2(\mathbf{D} + a)\mathbf{Q}^{k-2} + 2\mathbf{Q}^{k-1}\\
&= \ldots\\
&= \mathbf{Q}^k(\mathbf{D} + a) + k\mathbf{Q}^{k-1} \quad \square
\end{aligned}$$

Lemma 5. If $(\mathbf{D} + a)x = 0$, then $(\mathbf{D} + a)^{k+p}\mathbf{Q}^k x = 0$, for all $k \geq 0$ and $p \geq 1$.

Proof. It suffices to prove the induction step. Suppose that for k_0 it holds that $(\mathbf{D} + a)^{k_0+p}\mathbf{Q}^{k_0}x = 0$. Then

$$\begin{aligned}
(\mathbf{D} + a)^{k_0+p+1}\mathbf{Q}^{k_0+1}x &= (\mathbf{D} + a)^{k_0+1}\left[(\mathbf{D} + a)\mathbf{Q}^{k_0+1}\right]x\\
&= (\mathbf{D} + a)^{k_0+1}\mathbf{Q}^{k_0+1}\underbrace{(\mathbf{D} + a)x}_{=0} +\\
&\quad (k_0 + 1)\underbrace{(\mathbf{D} + a)^{k_0+1}\mathbf{Q}^{k_0}x}_{=0} \quad \square
\end{aligned}$$

Lemma 6. If $(\mathbf{D} + a)x = 0$, and $\alpha \in \mathbb{R}$, then $(\mathbf{D} + a - \alpha)\,\mathrm{e}^{\alpha\mathbf{Q}}\,x = 0$.

Proof.

$$\begin{aligned}
&(\mathbf{D} + a - \alpha)\,\mathrm{e}^{\alpha\mathbf{Q}}\,x\\
&= [\mathbf{D}\,\mathrm{e}^{\alpha\mathbf{Q}} + (a - \alpha)\,\mathrm{e}^{\alpha\mathbf{Q}}]x = \mathrm{e}^{\alpha\mathbf{Q}}(\mathbf{D} + \alpha)x = 0. \quad \square
\end{aligned}$$

Lemma 5 directly leads to

Theorem 7. If $(\mathbf{D} + a)x = 0$, then $(\mathbf{D} + a)^{k+1}p_k(\mathbf{Q})x = 0$, where $p_k(s)$ is an arbitrary polynomial in s of degree k.

Example 1: Consider the function $x(t) = (t^2 + t + 1)\,\mathrm{e}^{-t^2/2}$. Noting that $\mathrm{e}^{-t^2/2}$ is annihilated by $(\mathbf{D} + t)$, it follows from Theorem 1 that $(\mathbf{D} + t)^3$ will null $x(t)$. Its expansion is

$$(\mathbf{D} + t)^3 = \mathbf{D}^3 + 3t\mathbf{D}^2 + 3(t^2 + 1)\mathbf{D} + t^3 + 3t.$$

Theorem 8. If x solves $(\mathbf{D}+a)x=0$ and y solves $(\mathbf{D}+b)y=0$, then with $(\alpha,\beta)\in\mathbb{R}^2$, it holds that $z=\alpha x+\beta y$ solves $llcm\{\mathbf{D}+a,\mathbf{D}+b\}z=0$, where $llcm\{\mathbf{D}+a,\mathbf{D}+b\}$ denotes any left common multiple of the differential operators $\mathbf{D}+a$ and $\mathbf{D}+b$.

Proof. The *llcm* must of the form $\ell_a(\mathbf{Q},\mathbf{D})(\mathbf{D}+b)=\ell_b(\mathbf{Q},\mathbf{D})(\mathbf{D}+a)$. Hence,

$$\begin{aligned}&llcm(\alpha x+\beta y)\\ &=\alpha\ell_b(\mathbf{Q},\mathbf{D})\underbrace{(\mathbf{D}+a)x}_{=0}+\beta\ell_a(\mathbf{Q},\mathbf{D})\underbrace{(\mathbf{D}+b)y}_{=0}. \quad\square\end{aligned}$$

Theorem 9. Let χ_I denote the indicator function for the set I. If x solves $\chi_{I_1}(\mathbf{D}+a)x=0$ and $\chi_{I_2}(\mathbf{D}+b)x=0$ with $I_1\cap I_2=\emptyset$, the x solves $\chi_{I_1\cup I_2}lcm\{\mathbf{D}+a,\mathbf{D}+b\}x=0$.

Proof. *Proof*: follows directly from Theorem 8.

For instance, $x(t)=\sin t$ for $t>0$ and $x(t)=\sinh t$ for $t<0$ satisfies $(\mathbf{D}^4-1)x=0$, except at $t=0$. It is therefore a weak solution (Trentelman and Stoorvogel [2002]).

3.3 Partial state realizations for higher order equation

Let ξ be a solution to the homogeneous LTI-ODE

$$a(\mathbf{D})\xi=0, \tag{10}$$

of order (degree of $a(s)\in\mathbb{R}[s]$), n. All such functions are called Bohl-functions. We seek to express ξ as a solution to a first order LTV-ODE

$$\mathbf{D}\xi=\alpha(t)\xi. \tag{11}$$

Lemma 10. If ξ satisfies $\mathbf{D}\xi=\alpha\xi$, then the successive derivatives of ξ are given by

$$\mathbf{D}^k\xi=(\overleftarrow{\mathbf{D}}+\alpha)^k\xi \tag{12}$$

where $\overleftarrow{\mathbf{D}}$ is the derivative operator *acting to anything on the left.* (i.e, if $x(t)$ and $y(t)$ are arbitrary differentiable functions, then $x(t)\overleftarrow{\mathbf{D}}y(t)=\dot{x}(t)y(t)$).

Proof. From the definition, $\mathbf{D}\xi=(\overleftarrow{\mathbf{D}}+\alpha)\xi$. Suppose now that the above holds for k, then

$$\begin{aligned}\mathbf{D}^{k+1}\xi&=\mathbf{D}(\mathbf{D}^k\xi)\\ &=\mathbf{D}(\overleftarrow{\mathbf{D}}+\alpha)^k\xi\\ &=(\overleftarrow{\mathbf{D}}+\alpha)^k\xi\overleftarrow{\mathbf{D}}\\ &=(\overleftarrow{\mathbf{D}}+\alpha)^k\overleftarrow{\mathbf{D}}\xi+(\overleftarrow{\mathbf{D}}+\alpha)^k(\mathbf{D}\xi)\\ &=(\overleftarrow{\mathbf{D}}+\alpha)^k\overleftarrow{\mathbf{D}}\xi+(\overleftarrow{\mathbf{D}}+\alpha)^k(\alpha\xi)\\ &=(\overleftarrow{\mathbf{D}}+\alpha)^{k+1}\xi \qquad\square\end{aligned}$$

Theorem 11. The (potentially singular) first order LTV-ODE associated with a particular solution of $a(\mathbf{D})\xi=0$, where $a(s)=\sum_{i=0}^n a_is^{n-i}$ and $a_0=1$, is given by the solution to the time-variant ODE

$$\sum_{i=1}^n a_i(\overleftarrow{\mathbf{D}}+\alpha)^{n-i}=0. \tag{13}$$

Proof. follows directly from

$$0=a(\mathbf{D})\xi=\sum_{i=0}^n a_i\mathbf{D}^{n-i}\xi=\sum_{i=0}^n a_i(\overleftarrow{\mathbf{D}}+\alpha)^{n-i}\xi. \quad\square$$

Consider now the problem of the partial state realization with initialization $\xi(0)=\xi_0$, with all other derivatives dependent on it, i.e., $\xi^{(i)}(0)=\mathcal{D}_i\xi_0$ and $\mathcal{D}_i\in\mathbb{R}$ for $i=1,\ldots,n-1$.

Example 2: Consider $\dddot{\xi}=0$ with $\xi(0)=\xi_0$ and $\dot{\xi}(0)=\mathcal{D}\xi_0$. It turns out that $\dot{\alpha}(t)=-\alpha^2(t)$, with $\alpha(0)=\mathcal{D}$. This yields the first order representation

$$\dot{x}(t)=\frac{\mathcal{D}}{1+\mathcal{D}t}x(t),\quad x(0)=\xi_0,$$

which is singular at $t=-\mathcal{D}^{-1}$.

More generally, consider the homogeneous system $\dot{x}=Ax$, with partial state $y(t)=Cx(t)$ of dimension n and $r<n$ respectively. Assume then that $y(0)=y_0$, and $x(0)=By_0$, so that for all y_0, $CBy_0=y_0$, i.e., $CB=I_r$. It follows that

$$\dot{y}(t)=CA\,\mathrm{e}^{At}\,By_0.$$

On the other hand, expressing the r-th order system as solving a time-variant system $\dot{y}(t)=F(t)y(t)$, we can identify

$$CA\,\mathrm{e}^{At}\,By_0=F(t)C\,\mathrm{e}^{At}\,By_0,$$

so that

$$F(t)=CA\,\mathrm{e}^{At}\,B(C\,\mathrm{e}^{At}\,B)^{-1}. \tag{14}$$

Successive differentiation yields the representation

$$y^{(k)}=(\overleftarrow{\mathbf{D}}+F)^ky.$$

thus generalizing the formula obtained in Theorem 12. The reduced linear time-variant representation for the Bohl function is only well-defined if the $r\times r$ matrix function $C\,\mathrm{e}^{At}\,B$ has a nonvanishing determinant. It is shown in Verriest [2020] that a scalar Bohl function whose zeros have multiplicity one can be represented as the solution to a *second order* time-variant ODE with analytic coefficients, hence it is a regular ODE. More generally, if the highest multiplicity of *any* of the zeros is m, then the order of the linear time-variant ODE increases to $m+1$. This implies the vector extension:

Theorem 12. Let y be the r-dimensional output of a homogeneous linear time-invariant system of order $n\geq r$, i.e.,

$$\dot{x}=Ax,\quad y=Cx,\dim x=n,\ \dim y=r. \tag{15}$$

then y is the solution of a second-order regular linear homogeneous vector ODE if the matrix $[y(t),\dot{y}(t)]$ has rank 2 for all t.

Proof. By definition, y is an r-dimensional vector of Bohl functions. Let $h_i,\ldots,h_r$ be an arbitrary basis in $\mathbb{R}^r$. Then the scalar functions $\xi_i(t)=h_i^\top y(t)$ are Bohl functions. Moreover, because of the rank condition, they all have zeros of multiplicity one. By the above cited result, it follows that all ξ_i satisfy a second order regular linear time-variant homogeneous ODE

$$\ddot{\xi}_i(t)+a_{1i}(t)\dot{\xi}_i(t)+a_{2i}(t)\xi_i(t)=0.$$

Let $H=[h_1,h_2,\cdots,h_r]^\top$, and combine the r equations:

$$H\ddot{y}(t)+\mathrm{diag}[a_1(t)]H\dot{y}(t)+\mathrm{diag}[a_2(t)]Hy(t)=0.$$

and thus, setting $A_i(t)=H^{-1}\mathrm{diag}[a_i(t)]H$ we obtain

$$\ddot{y}+A_1(t)\dot{y}(t)+A_2(t)y(t)=0. \qquad\square$$

In principle, this gives a vector system of dimension $2r$ by concatenating y and $\dot{y}$ into a single state vector. However a

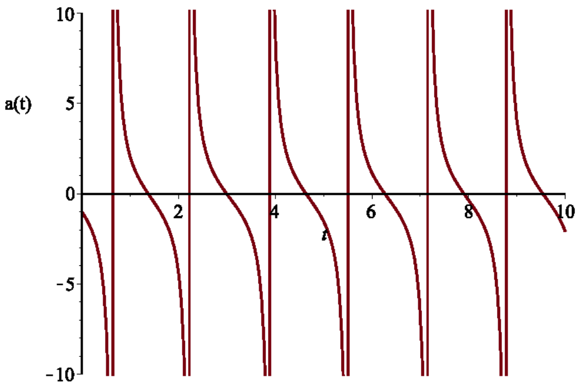

Fig. 1. Coefficient for the first order differential operator associated with the third order LTI system

reduction in the state may then be possible, so that $2r$ may not be the minimal dimension. We leave the problem of the *minimal time-variant realization* for further investigation.

Example 3: Consider the third order system with A-matrix in the reachable canonical form

$$A = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Choosing the first state as the partial state of interest, a singular first order representation follows, the scalar $a(t)$ is shown in figure 1. One should then proceed to a higher order time-variant ODE model, as explained in (Verriest [2020]). Choosing instead the first and second state as the partial states yields a second order time varying representation with the time-varying 2×2 dynamic matrix given in the (time-varying) reachable canonical form ,

$$A_2(t) = \begin{bmatrix} -a_1(t) & -a_2(t) \\ 1 & 0 \end{bmatrix}.$$

with $a_1(t)$ and $a_2(t)$ shown in Figure 2.

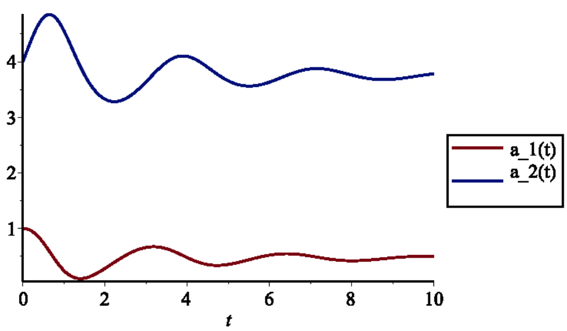

Fig. 2. Coefficients for the second order differential operator associated with the third order LTI system

In Verriest [2020] we show that if $x(t)$ is *analytic* in an interval (α, β), and possesses only real roots of multiplicity one, then $x(t)$ satisfies a regular (highest derivative has coefficient one) second order time-variant ODE, with coefficients that are *analytic* in (α, β).

A similar extension can be made here, thus generalizing Theorem 12 to non-Bohl vector functions.

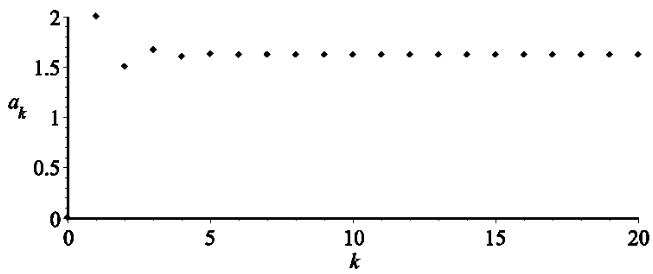

Fig. 3. a_k for equivalent first order model of the Fibonacci sequence

3.4 Discrete Case

One can also formulate a discrete version of this time-variant realization problem. We give a simple example: Consider the second order discrete LTI system specified by

$$x_{k+1} = x_k + y_k$$
$$y_{k+1} = x_k$$

where x_0 is given and y_0 is set to zero. The general solution is the Fibonacci sequence

$$x_k = A\left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \frac{3-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^k\right].$$

This satisfies the first order time-variant recursion

$$x_{k+1} = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \frac{3-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \frac{3-\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k}}\right] x_k.$$

Denoting the term in brackets by $a(k)$, the first 20 samples are shown in Figure 3.

4. SELF-STARTING SYSTEM

The time-variant realization theory we sketched in the previous Section 3 may be applied to the deformed exponential. Since the unit solution to $\dot{x}(t) = -x(\alpha t)$, for $0 < \alpha < 1$ has only positive real zeros all with multiplicity one, x must obey a second order linear time-variant ODE:

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + a_2(t)x(t) = 0.$$

From the numerical solution of the FDE: a_1 and a_2 can be approximated. See Figure 4 for $\alpha = 0.5$. In fact it suffices to consider only finitely many of the zeros if the approximation is only to hold over a finite interval. Note also that, although the original scale-delay equation is scalar, and propagates form a one-dimensional state at $t = 0$, the linear time-variant realization is consistently initialized with $\dot{x}(t) = -x(\alpha t) = -x_0$ at $t = 0$.

5. SYSTEMS WITH VANISHING DELAY

Let us now consider a homogeneous system of the form

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)), \tag{16}$$

and suppose that that at t_0, the delay $\tau(t_0) = 0$. At this point, the right hand side of (16) is $(A + B)x(t_0)$, and thus completely characterized by the finite dimensional state, $x(t_0)$. With the consistency condition $\dot{\tau}(t) < 1$, it

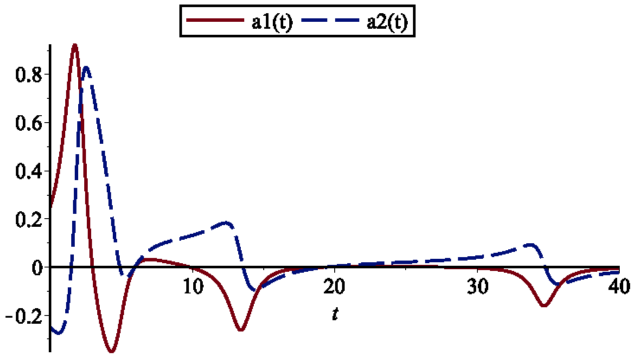

Fig. 4. Coefficients a_1 and a_2 of the LTV-ODE equivalent to the scale delay equation ($\alpha = 0.5$)

follows that for small $\epsilon > 0$, it holds in $[t_0, t_0 + \epsilon]$, that $t - \tau(t) = t_0 + \epsilon - \dot{\tau}(t_0)(\epsilon) \approx t_0 + (1 - \dot{\tau}(t_0))\epsilon$. Hence the system may be approximated after time t_0 (locally) by a scale-delay system with $\alpha = 1 - \dot{\tau}(t_0)$.

6. CONCLUSION

This paper summarized some known results regarding the scale-delay equation and its solution for a special case: the deformed exponential. New identities for the zeros of this function were derived. Next we illustrated, using algebraic properties, how these deformed exponentials generate the solutions to a class of higher order SD equations. We provided also some alternative methods to solve the time-varying realization problem for Bohl-functions, which are solutions to homogeneous LTI ODE's. The inverse problem for obtaining differential annihilators for an analytic function is then applied to the deformed exponential. A two-dimensional representation as the solution to a time-variant ODE resulted. This approximate modeling by an ODE, is applicable to model the behavior of a system with time-variant delay in the neighborhood of points where the delay vanishes. We leave the study of the data requirements and accuracy of the approximation in comparison to standard methods, such as direct Taylor expansion, for future work (See Saray et al. [2018]).

REFERENCES

Florian Breuer. *Identities for Weierstrass product. The American Mathemtical Monthly*, Vol. 119, No 9, (November 2012), pp. 796-799.

Ralph Philip Boas. *Entire Functions*, Academic Press, 1954.

Gregory Derfel, Peter J. Grabner, Robert F. Tichy. On the asymptotic behaviour of the zeros of the solutions of a functional-differential equation with rescaling. In *Indefinite Inner Product Spaces, Schur Analysis, and Differential Equations - A Volume Dedicated to Heinz Langer*, Daniel Alpay and Bernd Kirstein (Eds.) Springer-Verlag, 2018, pp. 281–295.

Thomas Griebel. The pantograph equation in quantum calculus. Master's Thesis, Dept. of Mathemeatics and Statistics, Missouri S&T. 2017.

Tosio Kato and J.B. McLeod. The functional equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. AMS*, Vol. 17, No. 6, Nov. 1971, pp. 891–937.

Cheng-shi Liu. Basic theory of a kind of linear pantograph equations. arXiv:1605.06734v4 26 Jan 2018.

Yunkang Liu. On some conjectures by Morris *et al.* about zeros of an entire function. *J. Math. Anal. Appl.* Vol. 226, 1998, pp. 1-5.

Alban Quadrat and Rosane Ushirobira Algebraic analysis for the Ore extension ring of differential time-varying delay operators. MTNS 2016.

Walter Rudin. *Real and Complex Analysis,* 2-nd edition. McGraw-Hill 1974.

Behzad N. Saray and J. Manafian. Sparse representation of delay differential equation of Pantograph type using multi-wavelets Galerkin methods. *Engineering Computations,* Vol. 35 No. 2, pp. 887-903, 2018.

Vyacheslav Spiridonov. Universal Superpositions of Coherent States and Self-Similar Potentials. *Phys. Rev. A* 52, 1909 (1995); Erratum Phys. Rev. A 53, 2903 (1996).

Harry Trentelman and Anton Stoorvogel. *Control theory for linear systems* Springer-Verlag, 2002.

K.G. Valeev. Linear differential equations with linear time-delay. *Siberian Math. Journal* 5 (2), pp. 290–309 (in Russian).

Bruce van Brunt, A.A. Zaidi and T. Lynch. Cell division and the pantograph equation *ESAIM: Proceedings and Surveys.* Vol. 62, September 2018, pp. 158–167.

Erik I. Verriest. Algebraic Theory for time-variant linear systems: Modes, minimality and reachability and observability of interconnected systems. Proc. 32nd IEEE Conf. Decision and Control, San Antonio, TX, pp. 1349–1354, December 1993.

Erik I. Verriest. Stability of systems with varying scale delays. Proceedings of the IFAC Symposium on System Structure and Control, Prague, Czech Republic, August 29-31, 2001.

Erik I. Verriest. Inconsistencies in systems with time varying delays and their resolution. *IMA Journal of Mathematical Control and Information,* Special issue: "Time-Delay Systems and Their Applications." Vol. 28, pp. 147-–162, 2011.

Erik I. Verriest. Time-Varying Realizations for Arbitrary Functions. Proceedings of the 21-st IFAC-World Congress 2020, Berlin, Germany, July 12-17 2020.

Liuquan Wang and Cheng Zhang. Zeros of the deformed exponential function. arXiv:1709.04357v1, 13 Sep 2017.

Alexey P. Zhabko, A.A. Laktionov and V.I. Zubov. Robust stability of differntial-difference systems with linear time-delay. IFAC Robust Control Design, Budapest, Hungary 1997, pp. 97–101.

Cheng Zhang. An asymptotic formula for the zeros of the deformed exponential function. *J. Math. Anal. Appl.* 441, No. 2 2016, pp. 565–573.