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Event-based stabilization of nonlinear Lipschitz systems ⋆

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Abstract: In this article, we propose an event-triggering mechanism for stabilization of a class of nonlinear Lipschitz systems under disturbance rejection H_∞ performance. Instead of following the prevalent dwell-time approach to address the Zeno issue, we propose a novel triggering threshold which switches between a constant and a function of states norm and allows for avoiding Zeno behaviour while achieving the desired performance level. The efficiency of the proposed approach is then justified through a numerical example.

Keywords: Event-triggered control, Lipschitz systems, H_∞ control.

1. INTRODUCTION

Event-based systems have seen significant attention within the control community in recent years (see Heemels et al. (2012); Postoyan et al. (2015); Ghodrat and Marquez (2019) and the references therein). Limiting the flow of information between system components is the primary goal of the event-based controllers, making them an attractive option in applications such as systems with limited energy and/or memory supplies or network control systems with limited channel bandwidth.

From the early works, the problem of disturbance rejection was found to be more challenging compared to that of internal stability. Indeed, as shown in Borgers and Heemels (2014) the method of Tabuada (2007), which is originally stated for internal stabilizing, fails to work in presence of arbitrary disturbances as it may lead to the accumulation of triggering instants; an undesired phenomenon that is known as Zeno behaviour. The possible existence of Zeno behaviour, it turn, questions the advantage of the event-based approach. This problem is well studied both for linear systems Lemmon et al. (2007); Wang and Lemmon (2009, 2010); Wang et al. (2017) and for nonlinear systems Dolk et al. (2017); Abdelrahim et al. (2017). The majority of these results tackle the problem using a special technique called "time-regularization" method. In this approach the time-triggered and event-triggered schemes are actually cleverly combined such that checking the triggering condition is paused for a period of time, known as dwell-time, after each triggering (see Dolk et al. (2017)), thus ensuring that triggering instants are separated from each other. While the main advantage of this method is the guaranteed nonzero inter event time, it has two main limitations: First, the time-regularized event-based system may reduce to time-triggered (periodic) system in certain situations, Dolk et al. (2017). Second, the dwell-time approach sets a lower bound on the proximity of the event-based performance ($\mathcal{L}_2$-gain in this work) and its continuous-time counterpart, Ghodrat and Marquez (2020a).

Based on the above observations, the primary interest of this paper is to study the finite gain $\mathcal{L}_2$-stability of nonlinear Lipschitz systems subject to arbitrary exogeneous disturbances. To overcome the mentioned shortcomings associated with the dwell-time approach, we follow a different approach in which the triggering threshold switches between a constant and a function of the states. We exploit the following compromise in our design: While using constant threshold serves to address the Zeno issue in the presence of exogenous disturbances, the $\mathcal{L}_2$-gain stability can not be obtained, Ghodrat and Marquez (2020a). On the other hand, a relative threshold which is a function of the system's state is the key to obtain the $\mathcal{L}_2$-stability goal, but fails to efficiently address the Zeno issue, Borgers and Heemels (2014). Our proposed triggering condition in this paper, however, enjoy the benefits offered from each method; namely, excluding accumulation of triggering instants while achieving the desired $\mathcal{L}_2$-stability criterion. We can summarize the main contributions as follows.

First, our proposed solution to rule out Zeno behaviour when disturbances are in effect is different from the time regularization technique. In fact, our method is *fully* event-based and does not require the enforcement of a dwell period. This enables us to avoid the mentioned shortcomings of the time-regularization method.

Second, while solving the $\mathcal{L}_2$-stability problem of nonlinear Lipschitz systems, we reduce the conservatism exists in prior works Ghodrat and Marquez (2019, 2020a), by exploiting the Lipschitz property in designing the event-based controller. The results in these references are derived for general nonlinear systems and hence may be too conservative when applied to the Lipschitz case.

Third, contrary to the reference Ghodrat and Marquez (2020b) which addresses the $\mathcal{L}_2$-stability of event-based Lipschitz systems, the obtained results in this paper are

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tailored for analysis rather than design and the proposed conditions are stated in terms of algebraic Riccati equation which is a commonly used tool in theory of linear systems.

The paper is organized as follows. Section 2 states the problem description. Section 3 introduces the event-triggered mechanism. The main results including the design of triggering parameters and H_∞ analysis of the event-based system are given in Section 4. The theoretical findings are finally evidenced through a compelling numerical example. Proof of results are given in the Appendix Section.

Nomenclature. Throughout the paper $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}_0^+$ represent the field of real, positive real and non-negative real numbers, respectively, and $\mathbb{N}$ denotes the set of positive integers. $\mathbb{R}^n$ is the set of n-dimensional vectors with elements in $\mathbb{R}$. By $|x|$ and $|x|_\infty$ we denote the Euclidean norm and supremum norm of column vector $x \in \mathbb{R}^n$, respectively. $\langle x, y\rangle = x^\top y$ represents the inner product of vectors $x, y \in \mathbb{R}^n$ where $x^\top$ is the transpose of x. I denotes the identity matrix. We denote by $\overline{\lambda}(A)$, $\underline{\lambda}(A)$ the maximum and minimum eigenvalues of matrix A. Given $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, we denote the weighted norm of vector x by $|x|_A = \sqrt{x^\top Ax}$. $\mathcal{L}_2^n$ is the space of measurable functions $w \in \mathbb{R}^n$ with bounded 2-norm defined as $(\int_0^\infty |w(t)|^2 dt)^{\frac{1}{2}}$. A function $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ is said to be locally Lipschitz-continuous in an open set B, if for each $z \in B$ there exist $L_f > 0$ and $r > 0$ such that $|f(x) - f(\tilde{x})| \leqslant L_f|x - \tilde{x}|$ for all $x, \tilde{x} \in \{y \in B|\ |y - z| < r\}$. Given a subset $\mathcal{A} \subseteq \mathbb{R}$, we denote by $\chi_\mathcal{A}$ the characteristic function of $\mathcal{A}$, i.e.,

$$\chi_\mathcal{A}(x) = \begin{cases} 1, & x \in \mathcal{A}, \\ 0, & x \notin \mathcal{A}. \end{cases}$$

2. PROBLEM SETUP

Consider the nonlinear plant

$$\dot{x}_p(t) = Ax_p(t)+B_1u_p(t)+\phi_1(x_p(t), u_p(t))+B_2w(t) \quad (1)$$

where $x_p \in \mathbb{R}^n$, $u_p \in \mathbb{R}^m$, $w \in \mathcal{L}_2^q$ represent the plant's state, control input and exogenous disturbance. A, B_1, B_2 are constant matrices of appropriate dimensions with (A, B_1) assumed to be a controllable pair. Also consider $z_p \in \mathbb{R}^r$ to be the plant's measured output and given by

$$z_p(t) = Cx_p(t) + Du_p(t) + \phi_2(x_p(t), u_p(t)) \quad (2)$$

where C, D are constant matrices of appropriate dimensions. The nonlinearities ϕ_1, ϕ_2 satisfy the following Lipschitz property (for $i = 1, 2$)

$$|\phi_i(x_p, u_p) - \phi_i(\tilde{x}_p, \tilde{u}_p)| \leqslant c_{\phi_i}(|x_p - \tilde{x}_p| + |u_p - \tilde{u}_p|) \quad (3)$$

for some positive constants c_{ϕ_1}, c_{ϕ_2} and all $x_p, \tilde{x}_p \in \mathbb{R}^n$, $u_p, \tilde{u}_p \in \mathbb{R}^m$. Moreover, $\phi_1(0, 0) = 0$ implying that $(x_p, u_p) = 0$ is an equilibrium point of disturbance-free system. We will also assume the state x_p to be driven from initial condition $x_p(0) = x_{p_0}$ on an open subset of $\mathbb{R}^n$ containing the origin. The static feedback controller is designed as

$$u_p(t) = Kx_p(t) \quad (4)$$

where $K \in \mathbb{R}^{m\times n}$ is the controller gain to be designed. We assume the controller receives information from the plant at discrete instants $\{t^i\}_{i\in\mathbb{N}}$ through a digital communication network, where $t^1 = 0$ by convention. Indeed, a triggering mechanism continuously monitors the plant output and if a certain condition is satisfied, a new updated signal will be sent through the communication network. Defining discrete signal $\bar{x}_p^i \doteq x_p(t^i)$, the event-based implementation of (4) gives

$$u_p(t) = K\bar{x}_p^i, \quad t \in [t^i, t^{i+1}). \quad (5)$$

The plant is controlled in an open loop between sampling instants. However, if we define sampling error $e_p \in \mathbb{R}^n$ as

$$e_p(t) = \bar{x}_p^i - x_p(t), \quad t \in [t^i, t^{i+1}) \quad (6)$$

we can rewrite (5) as $u_p = K(x_p + e_p)$, substituting which in (1) gives

$$\dot{x}_p = (A + B_1K)x_p + B_1Ke_p + \phi_1(x_p, K(x_p+e_p)) + B_2w$$

where e_p, w can be treated as the inputs of this closed-loop model. We also neglect the effects of transmission delays in the network which can be similarly addressed following the method in Tabuada (2007). To state the problem, we start with the following definition (Vidyasagar (1993)).

Definition 1. The system (1), (2) is said to be finite gain $\mathcal{L}_2$-stable from w to z_p and has $\mathcal{L}_2$-gain $\leqslant \gamma$, provided that there exist finite constants $\gamma > 0$, $\beta \geqslant 0$, called bias term, and positive semi-definite continuous function α such that for any $T \geqslant 0$, any perturbation $w \in \mathcal{L}_2^q$ and any $x_{p_0} \in \mathbb{R}^n$

$$0 \leqslant J^\gamma_{[0,T]} + \alpha(x_{p_0}) + \beta, \quad (7)$$

where $J^\gamma_{[r,s]} \doteq \int_r^s (\gamma^2|w(\tau)|^2 - |z_p(\tau)|^2)d\tau$.

The main interest of this note is to design an H_∞ event-based controller for the system (1), (2) using emulation approach. More clearly, a controller is first designed to render the resulting closed-loop network-free system finite gain $\mathcal{L}_2$-stable with $\mathcal{L}_2$-gain $\leqslant \gamma$. While in presence of communication network, the digital implementation of the designed controller does not necessarily guarantee the finite gain $\mathcal{L}_2$-stability, we will design triggering mechanism to retrieve this $\mathcal{L}_2$-gain performance for the event-based implementation.

3. EVENT-TRIGGERING MECHANISM

Starting from $t^1 = 0$, we assume the sampling instants t^i to be decided through the following event rule

$$t^{i+1} = \inf_{t>t^i, t\in\mathbb{R}} \left\{\xi_p(t)^\top \Xi\ \xi_p(t) - \Delta(t) \geqslant 0\right\} \quad (8)$$

for $i \geqslant 2$, where t^i denotes the most recent triggering instant, $\xi_p^\top(t) = (x_p^\top(t)\ e_p^\top(t))$ and Ξ, Δ are defined as

$$\begin{cases} \Xi = \begin{pmatrix} -p_1 & p_2 \\ p_2^\top & p_3 \end{pmatrix}, \\ \Delta(t) = \begin{cases} \delta_0, & t - t^i \in \mathcal{A}, \\ \eta_0 e^{-\zeta t}, & t - t^i \notin \mathcal{A}, \end{cases} \end{cases} \quad (9)$$

for $t \in [t^i, t^{i+1})$ which can be rewritten in the closed-form $\Delta(t) = \eta_0 e^{-\zeta t}+(\delta_0-\eta_0 e^{-\zeta t})\chi_\mathcal{A}(t-t^i)$. The event-based law requires the following parameters to be designed:

- matrices p_1, p_2, p_3,
- positive constants δ_0, ζ and $\eta_0 \in \mathbb{R}_0^+$,
- (non-empty) set $\mathcal{A} \doteq [0, \hat{\tau}] \subset \mathbb{R}_0^+$.

Remark 1. To examine condition (8) from a practical perspective, we should show that triggering instants are not accumulated when implementing the event-based controller which is known as Zeno-freeness property. As will

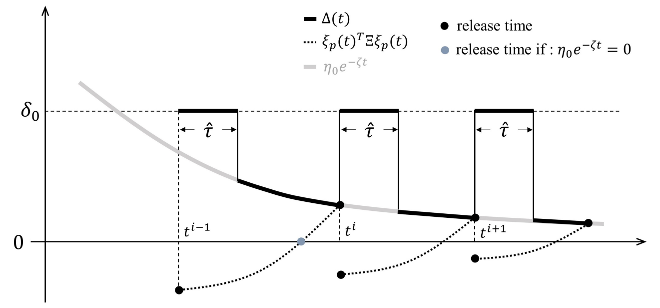

Fig. 1. Schematic plot of triggering condition

be shown in Theorem 1, this feature is indeed guaranteed due to the existence of Δ in triggering condition (8). The function Δ turns (8) into a *switching* triggering condition which regularly switches between a constant δ_0 and an exponentially decaying function $\eta_0 e^{\zeta t}$. This allows the admissible e_p to be extended by some positive constant δ_0, at least for a positive period of time $\hat{\tau}$ since the last transmission instant t^i. Without this extension, when state's norm is arbitrary small and sufficiently large disturbance is applied, e_p will suddenly grow in norm and potentially set the stage for experiencing Zeno behaviour. This scenario, however, is avoided by adding δ_0 to the triggering threshold. Therefore, with δ_0 the next sampling instant can not occur arbitrary close to the previous one. The following definition, borrowed from Borgers and Heemels (2014), is required to examine the admissibility of triggering instants.

Definition 2. Let $\tau_m = \inf\{t^{i+1} - t^i | i \in \mathbb{N}\}$ be the minimum inter-event time. The set of triggering instants has the robust semi-global event-separation property provided that some $\epsilon \in \mathbb{R}^+$ exists so that for any compact set $\mathcal{B} \subset \mathbb{R}^n$ we have $\inf\{\tau_m |\ x_{p_0} \in \mathcal{B}, |w|_\infty \leqslant \epsilon\} > 0$.

4. PRELIMINARY ANALYSIS

We start by the event-triggered closed-loop model:

$$\begin{cases} \dot{x}_p = \bar{A}x_p + B_1 K e_p + \phi_1(x_p, K(x_p+e_p)) + B_2 w, \\ z_p = Cx_p + DK(x_p+e_p) + \phi_2(x_p, K(x_p+e_p)), \\ e_p(t) = \bar{x}_p^i - x_p(t), \quad \bar{x}_p^i = x_p(t^i), \quad t \in [t^i, t^{i+1}), \\ t^{i+1} = \inf\limits_{t>t^i, t\in\mathbb{R}} \left\{ \begin{bmatrix} x_p \\ e_p \end{bmatrix}^\top \begin{bmatrix} -p_1 & p_2 \\ p_2^\top & p_3 \end{bmatrix} \begin{bmatrix} x_p \\ e_p \end{bmatrix} - \eta_0 e^{-\zeta t} \right. \\ \qquad \left. + (\delta_0 - \eta_0 e^{-\zeta t})\chi_{\mathcal{A}}(t - t^i) \geqslant 0 \right\}, (i \geqslant 2) \end{cases} \tag{10}$$

where $\bar{A} = A + B_1 K$.

The event condition (8) is designed to update the actuators whenever measurement error e_p exceeds some pre-designed threshold which depends on state x_p and Δ. Contrary to strategy in Tabuada (2007) where the threshold is merely a function of states, we introduce additional Δ term here to avoid redundant triggerings which due to existence of disturbances may lead to undesired Zeno behaviour (see Borgers and Heemels (2014) for a detailed discussion).

Assumption 1. The controller gain K is designed such that $\bar{A}$ is Hurwitz and there exist positive definite matrix P, Q and some positive γ solution to

$$\bar{A}^\top P + P\bar{A} + \gamma^{-2} P B_2 B_2^\top P + \bar{C}^\top \bar{C} + Q = 0 \tag{11}$$

where $\bar{C} = C + DK$.

Note that (11) guarantees finite gain $\mathcal{L}_2$-stability of network-free system

$$\begin{cases} \dot{x}_p = \bar{A}x_p + B_2 w \\ z_p = Cx_p + Du_p \end{cases} \tag{12}$$

obtained by (i) linearization of the model (1), (2) at $x_p = 0$, and (ii) continuous implementation of the control law (4) to this linearized model. This assumption frames the next obtained results within the emulation setting. In the following lemma we provide several conditions that can be employed as an alternative for (11).

Lemma 1. The network-free system (12) is finite gain $\mathcal{L}_2$-stable with zero bias and $\mathcal{L}_2$-gain $\leqslant \gamma$ if there exist a positive semi-definite matrix P, a positive definite function $V(x_p)$, some matrix K and positive $\hat{\gamma}$ so that either of the following statements holds:

a. $\langle \nabla V(x_p), \bar{A}x_p + B_2 w \rangle \leqslant \gamma^2 |w|^2 - |z|^2, \quad \forall w \in \mathbb{R}^q,$
b. $\bar{A}^\top P + P\bar{A} + \frac{1}{\gamma^2} P B_2 B_2^\top P + \bar{C}^\top \bar{C} + Q \leqslant 0,$
c. $A^\top P + PA + P\big(\frac{1}{\gamma^2} B_2 B_2^\top - \frac{1}{\hat{\gamma}^2} B_1 B_1^\top\big)P + \hat{C}^\top \hat{C} + Q \leqslant 0,$
d. $A^\top P + PA + P\big(\frac{1}{\gamma^2} B_2 B_2^\top - \frac{1}{\hat{\gamma}^2} B_1 B_1^\top\big)P + \hat{C}^\top \hat{C} + Q = 0,$

where $\hat{C} = C - \hat{\gamma}^{-2} D B_1^\top P$. Moreover, it is not difficult to verify that: a ← b ← c ← d.

Remark 2. First, for $C = I$, $D = 0$, $Q = 0$, $\hat{\gamma} = 1$, item d of Lemma 1 coincides with the one introduced in (Lemmon et al., 2007, Theorem 1). Second, (11) is weaker than item d of Lemma 1 since for $K = \frac{-1}{2\hat{\gamma}^2} B_1^\top P$ we have $\frac{-1}{\hat{\gamma}^2} P B_1^\top B_1 P = K^\top B_1^\top P + P B_1 K$ and hence if d has a solution for some P and $\hat{\gamma}$, so does (11) for $K = \frac{-1}{2\hat{\gamma}^2} B_1^\top P$. The converse, however, is not generally true.

Lemma 2. Under Assumption 1, for $t \in [t^i, t^{i+1})$ and any $i \in \mathbb{N}$ there exists positive definite $V(x_p)$ so that

$$\dot{V}(x_p) \leqslant \gamma^2 |w|^2 - |z_p|^2 + |\theta|^2_{\Theta_3} - (1-\rho)|x_p|^2_Q \tag{13}$$

for some $\rho \in (0, 1)$ where

$$\begin{cases} \theta^\top = (x_p^\top \ e_p^\top \ \phi_1^\top \ \phi_2^\top), \\ \Theta_3 = \begin{pmatrix} \Pi_1^1 + \bar{c}\Pi_1^2 & \Pi_2 \\ \Pi_2^\top & -\Pi_3 \end{pmatrix}, \\ \Pi_1^1 = \begin{pmatrix} -\rho Q & P B_1 K + \bar{C}^\top D K \\ K^\top B_1^\top P + \bar{C} D^\top K^\top & K^\top D^\top D K \end{pmatrix}, \\ \Pi_1^2 = \begin{pmatrix} I + K^\top K & K^\top K \\ K^\top K & K^\top K \end{pmatrix}, \\ \Pi_2 = \begin{pmatrix} P & \bar{C}^\top \\ 0 & K^\top D^\top \end{pmatrix}, \ \Pi_3 = \begin{pmatrix} \epsilon_1 I & 0 \\ 0 & (\epsilon_2 - 1)I \end{pmatrix}, \\ \bar{c} = \sum_{i=1}^2 2\epsilon_i c_{\phi_i}^2, \ \epsilon_1 > 0, \ \epsilon_2 > 1. \end{cases}$$

5. DESIGN OF TRIGGERING PARAMETERS

The design procedure entails two steps. First, we design set $\mathcal{A}$ (or equivalently $\hat{\tau}$) to ensure the $\mathcal{L}_2$ stability of (10) for $t - t^i \in \mathcal{A}$. Second, we design matrix Ξ (or equivalently matrices p_1, p_2, p_3) to guarantee the $\mathcal{L}_2$ stability of (10) for $t - t^i \notin \mathcal{A}$.

5.1 Design of Set $\mathcal{A}$

When $t - t^i \in \mathcal{A}$, the upper bound on measurement error e_p offered by event condition (8) is *useless* due to existence of constant term δ_0, which has unbounded integral over $[0,\infty)$. Therefore, to calculate the upper bound on e_p in this interval, we exploit the fact that in presence of exogenous disturbance, the measurement error is not only driven by state but also by disturbance. In fact, from (10) and definition of measurement error we may write

$$\frac{d}{dt}|e_p(t)| \leqslant |\dot{e}_p(t)| \leqslant \lambda_1|x_p(t)| + \lambda_2|e_p(t)| + \lambda_3|w(t)| \quad (14)$$

where $\lambda_1 = |\bar{A}| + c_{\phi_1}(1+|K|)$, $\lambda_2 = |B_1K| + c_{\phi_1}|K|$, $\lambda_3 = |B_2|$. We solve this inequality for $|e_p(t)|$ to obtain the following proposition which introduces an upper bound on measurement error in terms of state x_p and disturbance w.

Lemma 3. Let $\psi_{(\cdot,\cdot)}$ to be defined as:

$$\psi_{(s,r)} \doteq \frac{1}{2\lambda_2^2}(e^{2\lambda_2(r-s)}-1) - \frac{1}{\lambda_2}(r-s). \quad (15)$$

The following holds for $t \in [t^i, t^{i+1})$ and any $i \in \mathbb{N}$

$$\int_{t^i}^{t}|e_p(s)|^2 ds \leqslant \psi_{(t^i,t)}(\lambda_1^2\int_{t^i}^{t}|x_p(s)|^2 ds + \lambda_3^2\int_{t^i}^{t}|w(s)|^2 ds).$$

Remark 3. The coefficients of $\int_{t^i}^{t}|x_p(s)|^2 ds$ and $\int_{t^i}^{t}|w(s)|^2 ds$ terms in Lemma 3 are time dependent due to the presence of $\psi_{(t^i,t)}$. This plays a key role in designing $\mathcal{A}$ in (9). In fact, these coefficients are permitted to grow until the stability of the resulting event-based system with desired $\mathcal{L}_2$ gain is not violated. This is discussed in detail below.

Lemma 4. Under Assumption 1, for $t \in [t^i, t^{i+1})$ and any $i \in \mathbb{N}$ there exists positive definite $V(x_p)$ so that

$$V(x_p(t)) - V(\bar{x}_p^i) \leqslant J_{[t^i,t]}^{\gamma_\circ} + \int_{t^i}^{t}|\theta(\tau)|_{\Theta_4}^2 d\tau - (1-\rho)\int_{t^i}^{t}|x_p(\tau)|_Q^2 d\tau, \quad (16)$$

where

$$\begin{cases} \Theta_4 = \begin{pmatrix} \Pi_1^1 + \bar{c}\Pi_1^2 + \epsilon_\circ\Pi_1^3 & \Pi_2 \\ \Pi_2^\top & -\Pi_3 \end{pmatrix}, \\ \Pi_1^3 = \begin{pmatrix} \psi_{(t^i,t)}\lambda_1^2 I & 0 \\ 0 & -I \end{pmatrix}, \\ {\gamma_\circ}^2 = \gamma^2 + \epsilon_\circ\psi_{(t^i,t)}\lambda_3^2, \end{cases}$$

for some positive constant $\epsilon_\circ$.

Lemma 4 suggests that to guarantee finite gain $\mathcal{L}_2$-stability of event-based system (10) we have to ensure

- $\Theta_4 \leqslant 0$,
- $\gamma_\circ \leqslant \gamma_d$.

In the light of definitions of Θ_4 and $\gamma_\circ$, the above conditions impose the following restrictions on $\psi_{(t^i,t)}$ term and hence on set $\mathcal{A}$. First, to design $\mathcal{A}$ based on the first condition, we define τ_1 as

$$\tau_1 \doteq \max\{t-t^i |\ \Pi_1^1 + \bar{c}\Pi_1^2 + \epsilon_\circ\Pi_1^3 + \Pi_6 \leqslant 0\} \quad (17)$$

where

$$\Pi_6 = \begin{pmatrix} \frac{1}{\epsilon_1}P^2 + \frac{1}{\epsilon_2-1}\bar{C}^\top\bar{C} & \frac{1}{\epsilon_2-1}\bar{C}^\top DK \\ \frac{1}{\epsilon_2-1}K^\top D^\top\bar{C} & \frac{1}{\epsilon_2-1}K^\top D^\top DK \end{pmatrix}.$$

Indeed, since $\Pi_6 = \Pi_2\Pi_3^{-1}\Pi_2^\top$, the inequality in (17) can be equivalently written as

$$\Pi_1^1 + \bar{c}\Pi_1^2 + \epsilon_\circ\Pi_1^3 + \Pi_2\Pi_3^{-1}\Pi_2^\top \leqslant 0 \quad (18)$$

which by using Schur Lemma reads as

$$\Theta_4 = \begin{pmatrix} \Pi_1^1 + \bar{c}\Pi_1^2 + \epsilon_\circ\Pi_1^3 & \Pi_2 \\ \Pi_2^\top & -\Pi_3 \end{pmatrix} \leqslant 0.$$

Second, to achieve the desired disturbance rejection bound, $\mathcal{A}$ has to be designed so that for $t - t^i \in \mathcal{A}$, we have $\gamma_\circ \leqslant \gamma_d$. To this end, we define τ_2 as the solution to

$$\psi_{(t^i,t^i+\tau_2)} = \frac{1}{\epsilon_\circ\lambda_3^2}(\gamma_d^2 - \gamma^2). \quad (19)$$

The above observations lead us to design $\mathcal{A}$ so that (17), (19) simultaneously hold, *i.e.*,

$$\mathcal{A} = [0, \hat{\tau}],\ \hat{\tau} = \min\{\tau_1, \tau_2\}. \quad (20)$$

Note that from (20) we conclude $\psi_{(t^i,t^i+\hat{\tau})} \leqslant \psi_{(t^i,t^i+\tau_1)}$ and $\psi_{(t^i,t^i+\hat{\tau})} \leqslant \psi_{(t^i,t^i+\tau_2)}$. Therefore, for $t \in [t^i, t^i+\hat{\tau})$ and any $i \in \mathbb{N}$, we have $J_{[t^i,t]}^{\gamma_\circ} \leqslant J_{[t^i,t]}^{\gamma_d}$ and $\Theta_4 \leqslant 0$. Hence for $t \in \mathcal{A}$, (16) reduces to

$$V(x_p(t)) - V(\bar{x}_p^i) \leqslant J_{[t^i,t]}^{\gamma_d} - (1-\rho)\int_{t^i}^{t}|x_p(\tau)|_Q^2 d\tau. \quad (21)$$

Remark 4. While τ_1, τ_2 are functions of $\psi_{(t_i,t)}$, they are independent of the triggering instant value t^i. This is the case since according to Lemma 3, $\psi_{(s,r)}$ is a function of $r-s$ and not the particular values of r, s. Therefore, $\hat{\tau}$ is a positive number independent of triggering index i and in this sense, is defined globally.

5.2 Design of Matrices p_1, p_2, p_3

We define the matrix Ξ in (8) as

$$\Xi = \Pi_1^1 + \bar{c}\Pi_1^2 + \Pi_2\Pi_3^{-1}\Pi_2^\top. \quad (22)$$

Comparing (22) with Ξ in (9) we conclude that

$$\begin{cases} p_1 = \rho Q - \bar{c}(I + K^\top K) - \epsilon_1^{-1}P^2 - (\epsilon_2-1)^{-1}\bar{C}^\top\bar{C}, \\ p_2 = PB_1K + \bar{C}^\top DK + \bar{c}K^\top K + (\epsilon_2-1)^{-1}\bar{C}^\top DK, \\ p_3 = K^\top D^\top DK + \bar{c}K^\top K + (\epsilon_2-1)^{-1}K^\top D^\top DK. \end{cases}$$

To unravel the idea behind this choice we introduce the following Lemma.

Lemma 5. Under Assumption 1, for $t \in [t^i, t^{i+1})$ and any $i \in \mathbb{N}$ there exists positive definite $V(x_p)$ so that

$$\dot{V}(x_p) \leqslant \gamma^2|w|^2 - |z_p|^2 + \left|\begin{bmatrix} x_p \\ e_p \end{bmatrix}\right|_\Xi^2 - (1-\rho)|x_p|_Q^2. \quad (23)$$

Therefore, applying triggering condition (8) to (23) with Ξ defined in (22) and integrate the results gives

$$V(x_p(t)) - V(x_p(t^i+\hat{\tau})) \leqslant J_{[t^i+\hat{\tau},t]}^{\gamma} + \int_{t^i+\hat{\tau}}^{t}\Delta(\tau)d\tau - (1-\rho)\int_{t^i+\hat{\tau}}^{t}|x_p(\tau)|_Q^2 d\tau. \quad (24)$$

Note that contrary to the interval $t - t^i \in \mathcal{A}$, when $t - t^i \notin \mathcal{A}$ we have that $\Delta(t) = \eta_0 e^{-\zeta t}$ which has finite integral over $[0,\infty)$. This ensures the $\mathcal{L}_2$ stability of (10) for $t - t^i \notin \mathcal{A}$.

6. H_∞ ANALYSIS

We first prove the isolation of triggering instants. To do so, we need the following assumption.

Assumption 2. The set of parameters $\rho, \epsilon_1, \epsilon_2, \bar{c}$ are chosen so that the following holds:

$$\rho Q > \bar{c}(I + K^\top K) + \epsilon_1^{-1}P^2 + (\epsilon_2-1)^{-1}\bar{C}^\top\bar{C}. \quad (25)$$

Now we use the fact that $\underline{\lambda}(p_1) \in \mathbb{R}_0^+$ and $\overline{\lambda}(p_3) \in \mathbb{R}_0^+$, where the former is concluded from Assumption 2, to prove the separation of triggering instants as stated below.

Theorem 1. Under execution rule (8) with Ξ, $\mathcal{A}$ defined in (22) and (20), respectively, and for any $\delta_0, \zeta \in \mathbb{R}^+$, $\eta_0 \in \mathbb{R}_0^+$, the set of transmission instants has robust semi-global event-separation property (Definition 2).

Remark 5. Proof of Theorem 1 suggests that the event-separation property of the triggering instants is independent of the choice of matrix Ξ in (8).

Theorem 2. Under Assumptions 1, 2 and triggering condition (8) with $\mathcal{A}$, Ξ as defined in (20), (22) and any positive choice of δ_0, ζ and any non-negative η_0, the event-triggered system (10) is finite gain $\mathcal{L}_2$-stable and has $\mathcal{L}_2$-gain $\leqslant \gamma_d$ for some $\gamma_d > \gamma$.

7. NUMERICAL SIMULATIONS

We consider the following mass-spring-damper model borrowed from Seuret et al. (2016). In comparison with (1), (2) we have

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, C = [1 \ 0], D = 0,$$

where $k = 1$, $m = 0.1$, $c = 0.01$. The nonlinearities ϕ_1, ϕ_2 are assumed to satisfy (3) with $c_{\phi_1} = 0.1$, $c_{\phi_2} = 0$. We consider the state feedback gain $K = [-0.5 \ -2]$ which ensures Assumptions 1, 2 hold for $\epsilon_1 = 20$, $\epsilon_2 = 10$, $\rho = 0.85$ and

$$Q = \begin{bmatrix} 1.062 & 0.388 \\ 0.388 & 3.428 \end{bmatrix}, \quad P = \begin{bmatrix} 2.375 & 0.137 \\ 0.137 & 0.183 \end{bmatrix}.$$

The triggering parameters in (9) are then designed as:

$$p_1 = \begin{bmatrix} 0.0085 & -0.088 \\ -0.088 & 0.911 \end{bmatrix}, \; p_2 = \begin{bmatrix} -0.586 & -2.343 \\ -2.343 & -2.062 \end{bmatrix},$$
$$p_3 = \begin{bmatrix} 0.1 & 0.4 \\ 0.4 & 1.6 \end{bmatrix},$$

and $\delta_0 = 1$, $\eta_0 = 0$, $\zeta = 1$, $\hat{\tau} = 0.001$. Taking the initial condition $x_{p0} = (5, -2)$ the simulations are conducted for 20 seconds with the disturbance $w = 0.5$ in this interval. Here is a summary of the obtained results: In the interval $[0, 20]$ only 32 triggerings are generated. The intersampling intervals are bounded below by 0.3080. Obviously, choosing positive η_0 will enlarge the intersamplings and reduce the total number of triggerings (see Fig. 1).

8. CONCLUSION

In this paper, we established a new triggering scheme for a class of nonlinear Lipschitz systems which employs a novel switching triggering threshold. The switching takes place between a constant and a function of states norm and is controlled using a parameter which needs to be carefully designed so that (i) the desired H_∞ performance level is ensured, and (ii) the Zeno behaviour in presence of arbitrary disturbances is prevented. Our method follows an emulation approach in which a controller is first designed to stabilize the network-free closed-loop system, and then the triggering condition is proposed to retrieve similar performance under event-based implementation. A possible future research topic is to study the co-design problem where the controller gain and triggering parameters are simultaneously designed. Also, designing event rule based on the output signal rather than the states is another interesting topic.

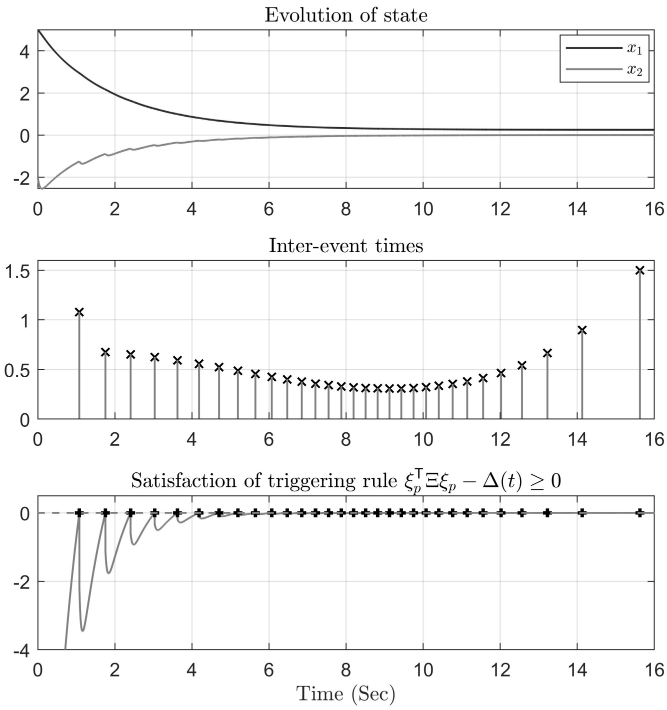

Fig. 2. Simulation graphs.

9. APPENDIX

Proof of Lemma 2. Let us define positive definite function $V(x_p) = x_p^\top P x_p$. It is then straight forward to calculate the derivative of V alongside the trajectories of (10) and employ (11) to obtain

$$\dot{V}(x_p) \leqslant \gamma^2|w|^2 - |\bar{C}x_p|^2 - |x_p|_Q^2 + \langle \nabla V(x_p), B_1 K e_p + \phi_1 \rangle.$$

In view of the following observations

$$|\bar{C}x_p|^2 = |\bar{C}x_p + DKe_p|^2 - |DKe_p|^2 - 2\langle \bar{C}x_p, DKe_p \rangle$$
$$|\bar{C}x_p + DKe_p|^2 = |z_p|^2 - |\phi_2|^2 - 2\langle \bar{C}x_p + DKe_p, \phi_2 \rangle$$

we can write

$$\dot{V}(x_p) \leqslant \gamma^2|w|^2 - |z_p|^2 + |\theta|_{\Theta_1}^2 - (1-\rho)|x_p|_Q^2 \quad (26)$$

where

$$\Theta_1 = \begin{pmatrix} \Pi_1^1 & \Pi_2 \\ \Pi_2^\top & \Pi_4 \end{pmatrix}, \; \Pi_4 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Moreover, one can write (3) in the following LMI form

$$\theta^\top \Theta_{2,i} \theta \leqslant 0, \; i \in \{1, 2\} \quad (27)$$

where

$$\Theta_{2,i} = \begin{pmatrix} -2c_{\phi_i}^2 \Pi_1^2 & 0 \\ 0 & \Pi_5 \end{pmatrix}, \; \Pi_5 = \begin{pmatrix} r_{i,1} I & 0 \\ 0 & r_{i,2} I \end{pmatrix}$$

and $r_{1,1} = r_{2,2} = 1$, $r_{1,2} = r_{2,1} = 0$. We now multiply (27) by ϵ_i, $i = 1, 2$ and add the results to (26) to obtain (13) where we use the fact that $\Theta_3 = \Theta_1 - \epsilon_1 \Theta_{2,1} - \epsilon_2 \Theta_{2,2}$. $\square$

Proof of Lemma 3. Solving (14) and apply comparison lemma gives

$$|e_p(t)| \leqslant \lambda_1 \int_{t^i}^{t} e^{\lambda_2(t-\tau)} |x_p(\tau)| d\tau + \lambda_3 \int_{t^i}^{t} e^{\lambda_2(t-\tau)} |w(\tau)| d\tau$$

for $t \in [t^i, t^{i+1})$. Defining $f_{(a,b,c,x)} \doteq \int_a^b e^{c(b-s)}|x_p(s)|ds$, we have $|e_p(t)|^2 \leqslant 2\lambda_1^2 f^2_{(t^i,t,\lambda_2,x_p)} + 2\lambda_3^2 f^2_{(t^i,t,\lambda_2,w)}$. Thus, after applying Cauchy-Schwartz inequality to $f^2_{(t^i,t,\lambda_2,w)}$ term we obtain

$$|e_p(t)|^2 \leqslant 2f_{(t^i,t,2\lambda_2,1)}\big(\lambda_1^2 \int_{t^i}^t |x_p(\tau)|^2 d\tau + \lambda_3^2 \int_{t^i}^t |w(\tau)|^2 d\tau\big).$$

Thus $\int_{t^i}^t |e_p(s)|^2 ds \leqslant 2\int_{t^i}^t f_{(t^i,s,2\lambda_2,1)}(\lambda_1^2 \int_{t^i}^s |x_p(\tau)|^2 d\tau + \lambda_3^2 \int_{t^i}^s |w(\tau)|^2 d\tau)ds$, which finally by defining $\psi_{(a,b)} \doteq 2\int_a^b f_{(a,s,2\lambda_2,1)} ds =$ (15) completes the proof. □

Proof of Lemma 4. From Lemma 2 we can integrate (13) between t^i and t to conclude

$$V(x_p(t)) - V(\bar{x}_p^i) \leqslant J^\gamma_{[t^i,t]} + \int_{t^i}^t |\theta(\tau)|_{\Theta_3} d\tau - (1-\rho)\int_{t^i}^t |x_p(\tau)|_Q d\tau$$

which by applying Lemma 3, gives the desired result. □

Proof of Lemma 5. We start from the fact that

$$\left|\Pi_3^{-\frac{1}{2}}\Pi_2^\top \begin{bmatrix} x_p \\ e_p \end{bmatrix} - \Pi_3^{\frac{1}{2}} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\right|^2 \geqslant 0,$$

from which we conclude

$$\theta^\top \begin{pmatrix} \Pi_2\Pi_3^{-1}\Pi_2^\top & -\Pi_2 \\ -\Pi_2^\top & -\Pi_3 \end{pmatrix} \theta \geqslant 0. \quad (28)$$

Adding (28) to (13) gives (23). □

Proof of Theorem 1. We will assume $t^{i+1} \leqslant t^i + \hat{\tau}$ since otherwise $\tau_m = \hat{\tau}$ and hence the proof is immediate. Therefore, (8) suggests that the lower bound on intersampling intervals is obtained whenever $-|x_p|^2_{p_1} + x_p^\top p_2 e_p + e_p^\top p_2^\top x_p + |e_p|^2_{p_3} \geqslant \delta_0$. Defining the triggering condition $|e_p| \geqslant a|x_p| + b$, where

$$a = \alpha_3^{-\frac{1}{2}}\Big(\Big(\frac{\alpha_1 + \frac{\alpha_2^2}{4\alpha_3}}{1+\alpha^{-1}}\Big)^{\frac{1}{2}} - \frac{\alpha_2}{2\alpha_3^{\frac{1}{2}}}\Big),\ b = \Big(\frac{\delta_0}{\alpha_3(1+\alpha)}\Big)^{\frac{1}{2}},$$

$\alpha_1 = \underline{\lambda}(p_1)$, $\alpha_2 = 2|\overline{\lambda}(p_2)|$, $\alpha_3 = \overline{\lambda}(p_3)$ and $\alpha > \alpha_2^2/(4\alpha_1\alpha_3)$ is an arbitrary parameter, it is then not difficult (Ghodrat and Marquez (2020b)) to verify that the triggering threshold in this new condition would be reached sooner than that of condition (8). Hence, smaller intersampling intervals are expected. In the rest we will show that the minimum intersampling times obtained from $|e_p| \geqslant a|x_p| + b$ (which is denoted by τ^*) is bounded away from zero, and hence the same property holds under condition (8). To this end, we define $\eta = |e_p|/(a|x_p| + b)$. Thus, we have

$$\begin{aligned}\dot{\eta} &\leqslant \frac{|\dot{e}_p|}{a|x_p|+b} + \frac{a|e_p||\dot{x}_p|}{(a|x_p|+b)^2} \\ &\leqslant \Big(1 + \frac{|e_p|}{a|x_p|+b}\Big)\frac{|\dot{x}_p|}{a|x_p|+b} \\ &\leqslant (1+a\eta)\Big(\frac{\lambda_1|x_p| + \lambda_2|e_p| + \lambda_3|w|}{a|x_p|+b}\Big) \\ &\leqslant (1+a\eta)(\kappa + \lambda_2\eta)\end{aligned}$$

where $\kappa = \max\{\frac{\lambda_1}{a}, \frac{\lambda_3\bar{w}}{b}\}$. Since at the triggering instants $\eta = 1$, we solve the above inequality for τ^* where $\eta(t^i) = 0$ and $\eta(t^i + \tau^*) = 1$ and obtain

$$\tau^*(\eta) = \begin{cases} \frac{1}{\lambda_2 - \kappa a}\ln(\frac{\kappa+\lambda_2\eta}{\kappa+\kappa a\eta}), & \kappa \neq \frac{\lambda_2}{a}, \\ \frac{a\eta}{\lambda_2(1+a\eta)}, & \kappa = \frac{\lambda_2}{a}. \end{cases}$$

Therefore, we conclude $\tau_m \geqslant \tau^*(1)$ and hence for any $|w| \leqslant \bar{w}$ and any $x_{p_0} \in \mathbb{R}^n$, the intersampling intervals are uniformly bounded away from zero. □

Sketch of the proof of Theorem 2. The proof of Theorem 2 can be easily obtained by splitting the interval $[t^i, t^{i+1})$ into two subintervals $\mathcal{A} = [t^i, t^i + \hat{\tau})$ and $[t^i + \hat{\tau}, t^{i+1})$, and apply (21) when $t \in \mathcal{A}$ and (24) when $t \notin \mathcal{A}$. However, we skip the proof due to space restrictions. □

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