Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Design of Controller and Observer for Dynamical Network Systems Based on Weighted Degrees

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Abstract: In this paper, stability conditions based on graph theory for dynamical network systems are shown. Although many frameworks based on graph theory for analysis of dynamical systems have been proposed, there is no stability condition that can be utilized to design of controllers and observers for linear dynamical systems. In this work, to show the stability condition based on graph theory for control and estimation, the dynamical system is represented by a directed graph with weights. The proposed stability conditions are obtained as the inequality of the weighted degrees defined in this paper. As applications, equilibrium point analysis of Lotka-Volterra system and design of pinning controllers and observers for consensus systems are proposed.

Keywords: Dynamical Network System, Multi Agent System, Graph Theory

1. INTRODUCTION

A dynamical network systems is a system whose behaviors are determined by interactions over large-scale complex networks. Many phenomena, for example, a gene network, electrical power network, epidemic, hit phenomena Boccaletti et al. (2006); Mesbahi and Egerstedt (2010); Liu et al. (2011); Ishii et al. (2012) can be represented by dynamical network systems. Since it is difficult for us to apply the standard analysis for the large-scale complex networks, efficient frameworks to analyze the dynamical network systems are required. For example, to reduce a complexity of networks some model reduction methods have been proposed in Ishizaki et al. (2014); Antoulas (2005); Ishizaki and Imura (2015).

In the past decades, graph theory is utilized the analysis of the dynamical systems Liu et al. (2011); Wang et al. (2016); Gu et al. (2015); Liu et al. (2013). To represent the dynamical network system based on graph theory, a structured system representation is proposed in Dion et al. (2003). The structured system is represented by a directed graph, where nodes correspond to states, inputs, and outputs of the original state space and edges means that there exits nonzero parameter between end points of the corresponding edge in the original state space.

Properties of the graph corresponding to the structured system provide generic conditions of the original systems, for example, controllability Glover and Silverman (1976); Shields and Pearson (1976) and the number of invariant zeros van der Woude (1999). Since little information is required for the analysis, the conditions obtained from the structured system are useful to analyze the systems which are difficult to identify, for example, biological systems and social systems. However, the conditions based on the structured system are conservative. In particular, the paper Dion et al. (2003) indicates that we can not obtain stability conditions from the structured systems.

In other past works, the stability condition of the dynamical network systems based on graph theory has been proposed in Mochizuki et al. (2013); Fiedler et al. (2013); Zañudo et al. (2017); Ogura and Preciado (2017); Azuma et al. (2017). In Mochizuki et al. (2013) and Fiedler et al. (2013), the stability condition of nonlinear dynamical network systems based on a feedback vertex set is proposed. Based on the condition proposed in Mochizuki et al. (2013) and Fiedler et al. (2013), the system is stable if the states corresponding to the feedback vertex set are 0. However, the paper Zañudo et al. (2017) indicates that there is no guaranty of the existence of the controller which stabilizes the states corresponding to the feedback vertex set. Although the other works Ogura and Preciado (2017); Azuma et al. (2017) show the stability condition based on graph theory, the system is confined to a positive system and a Boolean network, respectively.

In this works, the stability condition based on graph theory for linear dynamical network systems is shown and design methods of a controller and an observer are proposed. A system in this work is expressed as a directed graph with weights. For the directed graph with weights, weighted degrees are defined to derive stability conditions of the linear dynamical systems. From the stability condition based on the weighted degrees, parameter region of the controller and observer which stabilize the system and error systems are shown.

As applications, we consider an equilibrium point analysis of Lotka-Volterra system and design problem of the controller and observer for consensus systems.

Preliminaries

We utilize $[A]_{i,j}$ as the (i,j)-entry of the matrix A. For a set $\mathcal{N}$, we utilize a cardinality of $\mathcal{N}$ as $|\mathcal{N}|$. Let $G := (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a directed graphs with weights, where $\mathcal{V} := \{1, \cdots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges and $\mathcal{W} := \{ w_{(i,j)} \ ; \ w_{(i,j)} \in \mathbb{R}\}$ is a set of the weights. The graph $\bar{G}$ is defined by $\bar{G} := (\mathcal{V}, \bar{\mathcal{E}}, \bar{\mathcal{W}})$, where $\bar{\mathcal{E}} := \{ (i,j) \ ; \ (j,i) \in \mathcal{E}\}$ and $\bar{\mathcal{W}} = \{\bar{w}_{(j,i)}; \ \bar{w}_{(j,i)} = w_{(i,j)} \in \mathcal{W}\}$. Let $G_1 := (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1, \mathcal{W}_1)$ and $G_2 := (\mathcal{V}_2 \cup \mathcal{V}_3, \mathcal{E}_2, \mathcal{W}_2)$ be directed bipartite graphs with the weights, where $\mathcal{E}_i \subseteq \mathcal{V}_i \times \mathcal{V}_{i+1}$ and $\mathcal{W}_i = \{w^i_{(j,i)}; w^i_{(i,j)} \in \mathcal{W}\}$. A product of G_1 and G_2 are defined by $G_{1\times 2} := (\mathcal{V}_1 \cup \mathcal{V}_3, \mathcal{E}_{1\times 2}, \mathcal{W}_{1\times 2})$. The set of the edges is defined by $\mathcal{E}_{1\times 2} := \{(j,i); (j,k) \in \mathcal{E}_1 \text{ and } (k,j) \in \mathcal{E}_2\}$ and $w^{1\times 2}_{(j,i)} \in \mathcal{W}_{1\times 2}$ is defined by

$$w^{1\times 2}_{(j,i)} = \begin{cases} \sum_{(j,k)\in\mathcal{E}_1, (k,i)\in\mathcal{E}_2} w^1_{(j,k)} w^2_{(k,j)} & \text{if } (j,i) \in \mathcal{E}_{1\times 2} \\ 0 & \text{otherwise.} \end{cases}$$

2. DYNAMICAL NETWORK SYSTEMS

Let us consider the dynamical systems in which states evolve over complex networks. Nodes of the networks are categorized by the states, inputs, and outputs. Let $\mathcal{V}_x := \{x_1, \cdots, x_n\}$ be a set of the states, $\mathcal{V}_u := \{u_1, \cdots, u_p\}$ be a set of the inputs and $\mathcal{V}_y := \{y_1, \cdots, y_q\}$ be a set of the outputs. Interaction among nodes is expressed by directed graphs with weights.

The dynamics of the states are expressed by a self feedback and the interaction among the other states. Gain of self-feedback in ith state is given by a_i. The interaction among the states is defined by a directed graph with weights like $G_x := (\mathcal{V}_x, \mathcal{E}_x, \mathcal{W}_x)$, where $\mathcal{E}_x \subseteq \mathcal{V}_x \times \mathcal{V}_x$ is a set of edges and $\mathcal{W}_x := \{ w_{(i,j)} \ ; \ w_{(i,j)} \in \mathbb{R}\}$ is a set of the weights. If x_j affects dynamics of x_j, $(x_i, x_j) \in \mathcal{E}_x$. The weight $w_{(j,i)}$ corresponds to a coefficient of x_j in the dynamics of x_i. Index sets of the states which affect and are affected by x_i are denoted by $\mathcal{N}_i^{\text{in}} := \{ j \ ; \ (x_j, x_i) \in \mathcal{E}_x\}$ and $\mathcal{N}_i^{\text{out}} := \{ j \ ; \ (x_i, x_j) \in \mathcal{E}_x\}$, respectively.

The interaction between the inputs and the states are defined by a directed bipartite graph with the weights like $G_u := (\mathcal{V}_x \cup \mathcal{V}_u, \mathcal{E}_u, \mathcal{W}_u)$, where $\mathcal{E}_u \subseteq \mathcal{V}_u \times \mathcal{V}_x$ is a set of edges and $\mathcal{W}_u := \{ b_{(j,i)} \ ; \ b_{(j,i)} \in \mathbb{R}\}$ is a set of the weights. If the input u_j affects the dynamics of x_i, $(u_j, x_i) \in \mathcal{E}_u$. The weight $b_{(j,i)}$ corresponds to a coefficient of u_j in the dynamics of x_i. An index set of the inputs which affect the dynamics of x_i are denoted by $\mathcal{P}_i := \{ j \ ; \ (u_j, x_i) \in \mathcal{E}_u\}$.

The interaction between the states and the outputs are defined by a directed bipartite graph with the weights like $G_y := (\mathcal{V}_x \cup \mathcal{V}_y, \mathcal{E}_y, \mathcal{W}_y)$, where $\mathcal{E}_y \subseteq \mathcal{V}_x \times \mathcal{V}_y$ is a set of edges and $\mathcal{W}_y := \{ c_{(j,i)} \ ; \ c_{(j,i)} \in \mathbb{R}\}$ is a set of the weights. If the state x_j affects the output y_i, $(x_j, y_i) \in \mathcal{E}_y$. The weight $c_{(j,i)}$ corresponds to a coefficient of u_j in the dynamics of x_i. An index set of the states which affect y_i are denoted by $\mathcal{Q}_i := \{ j \ ; \ (u_j, x_i) \in \mathcal{E}_u\}$.

Based on above definition, ith state equation is expressed by

$$\dot{x}_i = a_i x_i + \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{(j,i)} x_j + \sum_{k\in\mathcal{P}_i} b_{(k,i)} u_k \tag{1}$$

and ith output equation is expressed by

$$y_i = \sum_{j\in\mathcal{Q}_i} c_{(j,i)} x_j. \tag{2}$$

Let $x := \text{col}(x_1, \cdots, x_n) \in \mathbb{R}^{|\mathcal{V}_x|}$, $u := \text{col}(u_1, \cdots, u_p) \in \mathbb{R}^{|\mathcal{V}_u|}$ and $y := \text{col}(y_1, \cdots, y_q) \in \mathbb{R}^{|\mathcal{V}_u|}$. We set A, $D \in \mathbb{R}^{|\mathcal{V}_x|\times|\mathcal{V}_x|}$, $B \in \mathbb{R}^{|\mathcal{V}_x|\times|\mathcal{V}_u|}$ and $C \in \mathbb{R}^{|\mathcal{V}_y|\times|\mathcal{V}_x|}$ as

$$A = \text{diag}(a_1, \cdots, a_n),$$
$$[D]_{i,j} = \begin{cases} w_{(j,i)} & (x_j, x_i) \in \mathcal{E}_x \\ 0 & \text{otherwise,} \end{cases}$$
$$[B]_{i,j} = \begin{cases} b_{(j,i)} & (u_j, x_i) \in \mathcal{E}_u \\ 0 & \text{otherwise,} \end{cases}$$
$$[C]_{i,j} = \begin{cases} c_{(j,i)} & (x_j, y_i) \in \mathcal{E}_y \\ 0 & \text{otherwise.} \end{cases}$$

We can obtain the state space representation of the dynamical network system as follows:

$$\dot{x} = (A + D)x + Bu, \tag{3a}$$
$$y = Cx. \tag{3b}$$

Example 1. (Air Conditioning System). Let us consider an air conditional system for an office room which is utilized in Ido et al. (2016). To simplify dynamics, we ignore the dynamics of air conditioners and only consider thermal diffusion among the areas. The state equation of the ith area is given by

$$c_i \dot{x}_i = -\sum_{j\in\mathcal{N}_i^{\text{in}}} k_{(j,i)}(x_i - x_j) + u_i \tag{4}$$

where x_i is a temperature of ith area and u_i is a heat quantity given by air conditioners. Coefficients c_i and $k_{(j,i)} = k_{(i,j)}$ denote the thermal capacity and the thermal conductance between jth and ith area. Based on (1), the gain of self feedback a_i and weight $w_{(j,i)}$ of (1) for the air conditioning system are obtained by

$$a_i = -\frac{1}{c_i} \sum_{j\in\mathcal{N}_i^{\text{in}}} k_{(j,i)} \text{ and } w_{(j,i)} = \frac{k_{(j,i)}}{c_i}.$$

3. PROBLEM FORMULATION

Since the system (3) is classified a linear time invariant system, we can analyze stability of the system and design controllers and observers based on eigenvalues. The stability condition is that real parts of eigenvalues of $A + D$ are negative. Design problems of the controller and the observer are that find K and L such that $A + D + BK$ and $A + D + LC$ are stable, respectively. However, in large scale systems, for example electric power networks or gene networks, it is difficult to execute these calculations. In addition, eigenvalue based design of the controller and the observer sometime ignore a structure of systems, which causes high-gain controllers and an inappropriate selection of actuators and sensors.

Instead of the eigenvalue, we find a stability condition, a controller and an observer for the dynamical network system (3) based on weighted degrees in this paper.

Definition 1. (Weighted Degree of G_x). For the directed graph with the weights G_x, indegree $d_{\rm in}(i)$ and outdegree $d_{\rm out}(i)$ of the node x_i are defined by

$$d_{\rm in}(i) = \sum_{j\in\mathcal{N}_i^{\rm in}} |w_{(j,i)}|,$$

$$d_{\rm out}(i) = \sum_{j\in\mathcal{N}_i^{\rm out}} |w_{(i,j)}|.$$

In the stability analysis based on the weighted degree, we find a regions of the indegree and the outdegree which guarantee that the system (3) becomes stable.

Problem 1. For the system (3a), the indegree $d_{\rm in}(i)$ and outdegree $d_{\rm out}(i)$ are given. Then, find the regions ξ_i and η_i

$$\xi_i(d_{\rm in}(i), a_i) < 0, \tag{5a}$$

$$\eta_i(d_{\rm out}(i), a_i) < 0 \tag{5b}$$

which guarantee $\lim_{t\to\infty} x = 0$.

In the design problem of the controller, we find state feedback controller such that a closed loop system satisfies (5). An actuator network among the inputs and the states are defined by a directed bipartite graph with the weights like $G_c := (\mathcal{V}_u \cup \mathcal{V}_x, \mathcal{E}_c, \mathcal{W}_c)$, where $\mathcal{E}_c \subseteq \mathcal{V}_x \times \mathcal{V}_u$ is a set of edges and $\mathcal{W}_c := \{\ k_{(j,i)}\ ;\ k_{(j,i)} \in \mathbb{R}\}$ is a set of the weights. If the controller which calculates the input values for u_i can utilize the state x_j, $(x_j, u_i) \in \mathcal{E}_c$. The weight $k_{(j,i)}$ corresponds to feedback gain of x_j in u_i. An index set of the states which are utilized in u_i are denoted by $\mathcal{K}_i := \{\ j\ ;\ (x_j, u_i) \in \mathcal{E}_c\}$. Using the actuator network G_c, linear feedback controller for u_i are given by

$$u_i = \sum_{j\in\mathcal{K}_i} k_{(j,i)} x_j. \tag{6}$$

Then, we consider the following problem.

Problem 2. The system (3a) is given. Then find the actuator network G_c such that the closed loop system consisted by (3a) and (6) satisfies (5).

In the design problem of the observer, we consider a Luenberger type observer for the system (3). A sensor network among observers and the outputs are defined by a directed bipartite graph with the weights like $G_o := (\mathcal{V}_y \cup \mathcal{V}_x, \mathcal{E}_o, \mathcal{W}_o)$, where $\mathcal{E}_o \subseteq \mathcal{V}_y \times \mathcal{V}_x$ is a set of edges and $\mathcal{W}_o := \{\ l_{(j,i)}\ ;\ l_{(j,i)} \in \mathbb{R}\}$ is a set of the weights. Let Σ_i be the observer which estimates x_i. If Σ_i can utilize the output y_j, $(y_j, x_i) \in \mathcal{E}_o$. The weight $l_{(j,i)}$ corresponds to an observer gain of y_j in Σ_i. An index set of the output which are utilized in Σ_i are denoted by $\mathcal{L}_i := \{\ j\ ;\ (y_j, x_i) \in \mathcal{E}_c\}$. By using the sensor network G_o, we define the observer Σ_i as

$$\begin{aligned}\dot{\hat{x}}_i =& \hat{a}_i\hat{x}_i + \sum_{j\in\hat{\mathcal{N}}_i^{\rm in}} \hat{w}_{(j,i)}\hat{x}_j + \sum_{k\in\hat{\mathcal{P}}_i} \hat{b}_{(k,i)} u_k \\ &+ \sum_{m\in\mathcal{L}_i} l_{(m,i)} y_m.\end{aligned} \tag{7}$$

Then, we consider the following problem.

Problem 3. The system (3a) is given. Then find the full state observers Σ_i for all $x_i \in \mathcal{V}_x$ (i.e., find the coefficients $\hat{a}_i$, $\hat{w}_{(j,i)}$, $\hat{b}_{(k,i)}$, the index sets $\hat{\mathcal{N}}_i^{\rm in}$, $\hat{\mathcal{P}}_i$ and the sensor network G_o) such that $\lim_{t\to\infty}(x_i - \hat{x}_i) = 0$.

4. MAIN RESULT

4.1 Stability Analysis Based on Weighted Degree

In this section, we derive the stability condition for (3a) based on the weighted degree as a solution of Problem 1. To corresponds the weighted degree to the stability of the system (3a), we utilize Gershgorin circle theorem.

Theorem 1. (Gershgorin circle theorem). Let A be a $n\times n$ matrix whose entries are complex numbers. Gershgorin disc R_i and S_i are defined by

$$R_i := \left\{\ s\ ;\ |s - [A]_{i,i}| \le \sum_{i\ne j} |[A]_{i,j}|\right\},$$

$$S_i := \left\{\ s\ ;\ |s - [A]_{i,i}| \le \sum_{i\ne j} |[A]_{j,i}|\right\}.$$

Then, every eigenvalue of A lies the region

$$\left(\bigcup_{i=1}^{n} R_i\right) \cap \left(\bigcup_{i=1}^{n} S_i\right).$$

Based on Gershgorin circle theorem, the stability condition for (3a) based on the weighted degree are given by the following theorem.

Theorem 2. The system (3) are stable if either (C1) or (C2) are satisfied.

(C1) For all $x_i \in \mathcal{V}_x$, $d_{\rm in}(i) + a_i < 0$.
(C2) For all $x_i \in \mathcal{V}_x$, $d_{\rm out}(i) + a_i < 0$.

Proof 1. (C1) From the definition of the indegree, the Gershgorin disc R_i of $A + D$ in (3a) is given by

$$R_i = \{\ s\ ;\ |s - a_i| \le d_{\rm in}(i)\}.$$

From $d_{\rm in}(i)+a_i < 0$, we can obtain $R_i \subset \{s; s \in \mathbb{C}, {\rm Re}(s) < 0\}$. Therefore, we can obtain

$$\bigcup_{i=1}^{n} R_i \subset \{s; s \in \mathbb{C}, {\rm Re}(s) < 0\}.$$

From the above discussion, we can conclude that the real parts of the all eigenvalue of $A + D$ are negative if (C1) is satisfied.

(C2) From the definition of the outdegree, the Gershgorin disc S_i of $A + D$ in (3a) is given by

$$S_i = \{\ s\ ;\ |s - a_i| \le d_{\rm out}(i)\}.$$

From $d_{\rm out}(i) + a_i < 0$, we can obtain $S_i \subset \{s; s \in \mathbb{C}, {\rm Re}(s) < 0\}$. Therefore, we can obtain

$$\bigcup_{i=1}^{n} S_i \subset \{s; s \in \mathbb{C}, {\rm Re}(s) < 0\}.$$

From the above discussion, we can conclude that the real parts of the all eigenvalue of $A + D$ are negative if (C2) is satisfied.

4.2 Design of Controller and Observer Based on Weighted Degree

In this subsection, the controller which stabilizes the system (3a) and the observer which estimates the state of

(3) are discussed as a solution of Problem 2 and 3. Previous subsection provides the stability condition of (3a) based on the weighted degree. Therefore, we find G_c such that the closed loop system satisfies Theorem 2 and G_o such that dynamics of e_i satisfies Theorem 2.

An interaction among the states generated by actuator networks is given by $G_u \times G_c := (\mathcal{V}_x, \mathcal{E}_{u\times c}, \mathcal{W}_{u\times c})$. An index set of the states which affect and are affected by x_i over $G_u \times G_c$ are denoted by $\mathcal{C}_i^{\text{in}} := \{ j \ ; \ (x_j, x_i) \in \mathcal{E}_{u\times c}\}$ and $\mathcal{C}_i^{\text{out}} := \{ j \ ; \ (x_i, x_j) \in \mathcal{E}_{u\times c}\}$, respectively. Let $d_{(j,i)} \in \mathcal{W}_{u\times c}$ be the weight which corresponds $(x_j, x_i) \in \mathcal{E}_{u\times c}$. Based on $d_{(j,i)}$ and $\mathcal{C}_i^{\text{in}}$, a state feedback of u_i is given by

$$\sum_{j\in\mathcal{C}_i^{\text{in}}} d_{(j,i)} x_j.$$

Then, the stability conditions of the closed loop system (1) with (6) are given by the following theorem.

Theorem 3. The closed loop system (1) with (6) is stable if either (8) or (9) are satisfied.

$$\begin{aligned}&\sum_{j\in\mathcal{N}_i^{\text{in}}} |w_{(j,i)} + d_{(j,i)}| + \sum_{j\in\mathcal{C}_i^{\text{in}}\backslash\mathcal{N}_i^{\text{in}}} |d_{(j,i)}| \\ &+ a_i + d_{(i,i)} < 0 \text{ for all } x_i \in \mathcal{V}_x.\end{aligned} \tag{8}$$

$$\begin{aligned}&\sum_{j\in\mathcal{N}_i^{\text{out}}} |w_{(i,j)} + d_{(i,j)}| + \sum_{j\in\mathcal{C}_i^{\text{out}}\backslash\mathcal{N}_i^{\text{out}}} |d_{(i,j)}| \\ &+ a_i + d_{(i,i)} < 0 \text{ for all } x_i \in \mathcal{V}_x.\end{aligned} \tag{9}$$

Proof 2. The state equation of closed loop system of (1) with (6) are expressed by

$$\begin{aligned}\dot{x}_i =& (a_i + d_{(i,i)})x_i + \sum_{j\in\mathcal{N}_i^{\text{in}}} (w_{(j,i)} + d_{(j,i)})x_j \\ &+ \sum_{j\in\mathcal{C}_i^{\text{in}}\backslash\mathcal{N}_i^{\text{in}}} d_{(j,i)} x_j.\end{aligned}$$

Based on Definition 1, the indegree and the outdegree of x_i in the closed loop system are expressed by

$$\begin{aligned}d_{\text{in}}(i) &= \sum_{j\in\mathcal{N}_i^{\text{in}}} |w_{(j,i)} + d_{(j,i)}| + \sum_{j\in\mathcal{C}_i^{\text{in}}\backslash\mathcal{N}_i^{\text{in}}} |d_{(j,i)}|, \\ d_{\text{out}}(i) &= \sum_{j\in\mathcal{N}_i^{\text{out}}} |w_{(i,j)} + d_{(i,j)}| + \sum_{j\in\mathcal{C}_i^{\text{out}}\backslash\mathcal{N}_i^{\text{out}}} |d_{(i,j)}|.\end{aligned}$$

Therefore, from Theorem 2, we can obtain (8) and (9) as the stability condition.

We find the sensor network G_o such that dynamics of e_i becomes stable based on the weighted degree. Let us give the observer Σ_i as

$$\begin{aligned}\dot{\hat{x}}_i =& a_i \hat{x}_i + \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{(j,i)} \hat{x}_j + \sum_{k\in\mathcal{P}_i} b_{(k,i)} u_k \\ &+ \sum_{m\in\mathcal{L}_i} l_{(m,i)} (\hat{y}_m - y_m), \\ \hat{y}_i =& \sum_{j\in\mathcal{Q}_i} c_{(j,i)} \hat{x}_j.\end{aligned} \tag{10}$$

Based on (10), the dynamics of estimation error e_i is expressed by

$$\begin{aligned}\dot{e}_i &= a_i e_i + \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{(j,i)} e_i + \sum_{m\in\mathcal{L}_i} l_{(m,i)} \alpha_m, \\ \alpha_m &= \sum_{j\in\mathcal{Q}_m} c_{(j,m)} e_j.\end{aligned} \tag{11}$$

An interaction among the estimation error generated by sensor network is given by $G_o \times G_y := (\mathcal{V}_x, \mathcal{E}_{o\times y}, \mathcal{W}_{o\times y})$. An index set of the estimation error which affect and are affected by error of x_i over $G_o \times G_y$ are denoted by $\mathcal{O}_i^{\text{in}} := \{ j \ ; \ (x_j, x_i) \in \mathcal{E}_{o\times y}\}$ and $\mathcal{O}_i^{\text{out}} := \{ j \ ; \ (x_i, x_j) \in \mathcal{E}_{o\times y}\}$, respectively. Let $h_{(j,i)} \in \mathcal{W}_{o\times y}$ be the weight which corresponds $(x_j, x_i) \in \mathcal{E}_{o\times y}$. By using $G_o \times G_y$, (11) can be expressed by

$$\begin{aligned}\dot{e}_i =& (a_i + h_{(i,i)})e_i + \sum_{j\in\mathcal{N}_i^{\text{in}}} (w_{(j,i)} + h_{(j,i)})e_j \\ &+ \sum_{j\in\mathcal{O}_i^{\text{in}}/\mathcal{N}_i^{\text{in}}} h_{(j,i)} e_j.\end{aligned} \tag{12}$$

Then, the stability conditions of (12) are given by the following theorem.

Theorem 4. The error system (12) is stable if either (13) or (14) are satisfied.

$$\begin{aligned}&\sum_{j\in\mathcal{N}_i^{\text{in}}} |w_{(j,i)} + h_{(j,i)}| + \sum_{j\in\mathcal{C}_i^{\text{in}}\backslash\mathcal{N}_i^{\text{in}}} |h_{(j,i)}| \\ &+ a_i + h_{(i,i)} < 0 \text{ for all } x_i \in \mathcal{V}_x.\end{aligned} \tag{13}$$

$$\begin{aligned}&\sum_{j\in\mathcal{N}_i^{\text{out}}} |w_{(i,j)} + h_{(i,j)}| + \sum_{j\in\mathcal{C}_i^{\text{out}}\backslash\mathcal{N}_i^{\text{out}}} |h_{(i,j)}| \\ &+ a_i + h_{(i,i)} < 0 \text{ for all } x_i \in \mathcal{V}_x.\end{aligned} \tag{14}$$

Proof 3. Based on Definition 1, the indegree and the outdegree of e_i in (12) are expressed by

$$\begin{aligned}d_{\text{in}}(i) &= \sum_{j\in\mathcal{N}_i^{\text{in}}} |w_{(j,i)} + h_{(j,i)}| + \sum_{j\in\mathcal{C}_i^{\text{in}}\backslash\mathcal{N}_i^{\text{in}}} |h_{(j,i)}|, \\ d_{\text{out}}(i) &= \sum_{j\in\mathcal{N}_i^{\text{out}}} |w_{(i,j)} + h_{(i,j)}| + \sum_{j\in\mathcal{C}_i^{\text{out}}\backslash\mathcal{N}_i^{\text{out}}} |h_{(i,j)}|.\end{aligned}$$

Therefore, from Theorem 2, we can obtain (13) and (14) as the stability condition of (12).

Let $\Xi := (A, G_u, G_x, G_y)$ denote the dynamical network system which is expressed by (3). We define $\bar{\Xi}$ as a dual system of Ξ and $\bar{\Xi}$ is expressed by $(A, \bar{G}_y, \bar{G}_x, \bar{G}_u)$. Therefore, the controller based on the outdegree for Ξ becomes the observer based on the indegree for $\bar{\Xi}$.

5. APPLICATIONS

5.1 Equilibrium Point Analysis of Lotka-Volterra System

Theorem 2 provides a relation between the stability condition and the interaction among the states over the graph. In this subsection, we apply Theorem 2 for an equilibrium point analysis of Lotka-Volterra system to discuss the characterization of competition among species. Lotka-Volterra system represents a competition among some species and is expressed by

$$\dot{x}_i = \left(\epsilon_i - \mu_{(i,i)} x_i - \sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)} x_j \right) x_i, \tag{15}$$

where $x_i > 0$ is a population of ith species, $\epsilon_i > 0$ is an intrinsic rate of natural increase of ith species, $\mu_{(i,i)} > 0$ is a self-interacting coefficient of ith species and $\mu_{(j,i)} > 0$ is an interacting coefficient from jth to ith species. The equilibrium point of Lotka-Volterra system (15) can be characterized as

- Shakeout: $x_i = 0$
- Subsistence: $x_i > 0, \epsilon_i - \mu_{(i,i)}x_i - \sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)}x_j = 0.$

In this paper, we find the condition for all species to be subsistence under (15) as the following problem.

Problem 4. For the system (15), the equilibrium point $\bar{x} := (\bar{x}_1, \cdots, \bar{x}_n)^\top > 0$ is given. Then, find the region ζ

$$\zeta(\{\mu_{(i,j)}\ ;\ (x_j, x_i) \in \mathcal{E}_x\}, \{\mu_{(i,i)}\ ;\ x_i \in \mathcal{V}_x\}) < 0$$

such that $\bar{x}$ becomes a stable equilibrium point of (15).

The region ζ is given by the following theorem.

Theorem 5. If $\sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)} < \mu_{(i,i)}$ for all $x_i \in \mathcal{V}_x$ are satisfied, $\bar{x}$ is the stable equilibrium point of (15). In other word, if an affect of self-interacting outraces interacting among neighbor species for all species, $\bar{x}$ is the stable equilibrium point of (15).

Proof 4. A linear approximated system of (15) around $\bar{x}$ is expressed by

$$\dot{\tilde{x}}_i = -\mu_{(i,i)}\bar{x}_i\tilde{x}_i - \sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)}\bar{x}_i\tilde{x}_j, \tag{16}$$

where $\tilde{x}_i = x_i - \bar{x}_i$. From the definition of indegree, $d_{\text{in}}(i)$ of (16) is expressed by

$$d_{\text{in}}(i) = \sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)}\bar{x}_i.$$

From Theorem 2, we can get a stability condition of (16) as

$$\sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)}\bar{x}_i - \mu_{(i,i)}\bar{x}_i < 0.$$

Due to $\bar{x} > 0$, we can get $\sum_{j\in\mathcal{N}_i^{\text{in}}} \mu_{(j,i)} < \mu_{(i,i)}$.

5.2 Design of Controller and Observer for Consensus System

In Section 4.2, the condition of the controller and the observer for a general dynamical network system is shown. In this subsection, we show the actual design of actuator and sensor network for the control and estimation of a consensus system as an example of a dynamical network system. Let $G := \{\mathcal{V}, \mathcal{E}\}$ be undirected graph, where $\mathcal{V} := \{1, \cdots, n\}$ is a set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. The neighbor nodes of ith nodes over G are given by $\mathcal{N}_i := \{\ j\ ;\ (j,i) \in \mathcal{E}\}$. Based on G, the consensus system is defined by

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i} (x_i - x_j). \tag{17}$$

If G is connected, x_i for all $i \in \mathcal{V}$ converges to

$$x_i(\infty) = \frac{1}{n}\sum_{i=1}^{n} x_i(0). \tag{18}$$

In this subsection, we consider the pinning controller which changes consensus values from (18) to another value. Let $\mathcal{P} \subset \mathcal{V}$ be set of the pinning nodes which are applied external inputs. The dynamics of the pinning node $p \in \mathcal{P}$ is expressed by

$$\dot{x}_p = -\sum_{j\in\mathcal{N}_p} (x_p - x_j) + u_p. \tag{19}$$

We consider following problem.

Problem 5. For the system constructed by (17) and (19), a new consensus values x_{con} is given. Then, find pinning controller

$$u_p(t) = g(\{x_j; j \in \mathcal{K}_p\}, x_{\text{con}})$$

such that

$$\lim_{t\to\infty} (x(t) - x_{\text{con}}) = 0.$$

By using $\tilde{x}_i = x_i - x_{\text{con}}$, the system constructed by (17) and (19) can be expressed by

$$\begin{cases} \dot{\tilde{x}}_i = -\sum_{j\in\mathcal{N}_i} (\tilde{x}_i - \tilde{x}_j) + u_i & i \in \mathcal{P} \\ \dot{\tilde{x}}_i = -\sum_{j\in\mathcal{N}_i} (\tilde{x}_i - \tilde{x}_j) & \text{otherwise.} \end{cases} \tag{20}$$

Let $s_i^{\text{out}} := d_{\text{out}}(i) + a_i$ and s_i^{out} for all $i \in \mathcal{V}$ of (20) is expressed by

$$s_i^{\text{out}} = |\mathcal{N}_i| - |\mathcal{N}_i| = 0.$$

To stabilize (20) based on Theorem 2, we design an actuator network such that $s_i^{\text{out}} < 0$. Let $G_c := (\mathcal{V}_c, \mathcal{E}_c, \mathcal{W}_c)$ be the actuator network for the pinning control of (20), where $\mathcal{V}_c := \{\tilde{x}_1, \cdots, \tilde{x}_n\} \cup \bigcup_{i\in\mathcal{P}}\{u_i\}$, $\mathcal{E}_c := \bigcup_{i\in\mathcal{P}}\{\bigcup_{j\in\mathcal{K}_i}\{(x_j, u_i)\}\}$ and $\mathcal{W}_c := \{\ k(j,i)\ ; k(j,i) \in \mathbb{R}\}$. Assume that $\mathcal{K}_i = \mathcal{N}_i \cup \{i\}$ and the pinning controller generated by the actuator network G_c for (20) is expressed by

$$\begin{aligned} u_i =& k_{(i,i)}\tilde{x}_i + \sum_{j\in\mathcal{N}_i} k_{(j,i)}\tilde{x}_j \\ =& k_{(i,i)}x_i + \sum_{j\in\mathcal{N}_i} k_{(j,i)}x_j \\ &- \left(k_{(i,i)} + \sum_{j\in\mathcal{N}_i} k_{(j,i)}\right) x_{\text{con}}. \end{aligned} \tag{21}$$

Then, the following theorem holds.

Theorem 6. If (C1) and (C2) are satisfied, the closed loop system (20) with (21) becomes stable.

(C1) $\bigcup_{i\in\mathcal{P}} \mathcal{N}_i \cup \mathcal{P} = \mathcal{V}$
(C2) $k_{(i,i)} < 0$ and $-2 < k_{(j,i)} < 0$

Proof 5. The state equation of the pinning node i with (21) is expressed by

$$\dot{\tilde{x}}_i = (k_{(i,i)} - |\mathcal{N}_i|)\tilde{x}_i + \sum_{j\in\mathcal{N}_i} (1 + k_{(j,i)})\tilde{x}_j.$$

From $(k_{(i,i)} - |\mathcal{N}_i|) < -|\mathcal{N}_i|$, $s_i^{\text{out}} < 0$ for $i \in \mathcal{P}$. From $|1 + k_{(j,i)}| < 1$, $d_{\text{out}}(j)$ for $j \in \mathcal{N}_i$ and $i \in \mathcal{P}$ of the closed loop system (20) with (21) is smaller than $|\mathcal{N}_i|$, which means $s_j^{\text{out}} < 0$ for $j \in \mathcal{N}_i$ and $s_i^{\text{out}} < 0$ for $i \in \mathcal{P}$. In addition, from $\bigcup_{i\in\mathcal{P}} \mathcal{N}_i \cup \mathcal{P} = \mathcal{V}$, we can conclude that $s_i < 0$ for all $i \in \mathcal{V}$ holds.

Based on the duality of the controller and the observers, we can design the observer from Theorem 6. For the system (17), we consider the following outputs

$$y_m = x_m, m \in \mathcal{Q} \subset \mathcal{V}, \tag{22}$$

where $\mathcal{Q}$ is a set of sensor nodes. Let $G_o := (\mathcal{V}_o, \mathcal{E}_o, \mathcal{E}_o)$ be the sensor network for the estimation of (17), where $\mathcal{V}_o := \{\hat{x}_1, \cdots, \hat{x}_n\} \cup \bigcup_{i\in\mathcal{Q}}\{y_i\}$, $\mathcal{E}_o := \bigcup_{i\in\mathcal{Q}}\{\bigcup_{j\in\mathcal{L}_i}\{(y_j, \hat{x}_i)\}\}$ and $\mathcal{W}_o := \{\ l_{(j,i)}\ ; l_{(j,i)} \in \mathbb{R}\}$. Based on G_o, an observer for the system (18) with (22) can be expressed by

$$\dot{\hat{x}}_i = -\sum_{j\in\mathcal{N}_i}(\hat{x}_i - \hat{x}_j) + \sum_{m\in\mathcal{L}_i} l_{(m,i)}(\hat{x}_m - y_m). \tag{23}$$

Then, we consider the following problems.

Problem 6. The system constructed by (17) and (22) is given. Then, find G_o such that $\lim_{t\to\infty}(x_i - \hat{x}_i) = 0$.

Let G_u be an actuator network for pinning controller of (17), where pinning node is selected $\mathcal{Q}$ and $\mathcal{K}_i = \mathcal{N}_i \cup \{i\}$. Then, we obtain the observer for (17) from the following theorem.

Theorem 7. If G_u satisfies Theorem 6, the estimation errors of the observer (23) based on $\bar{G}_u$ converge to 0.

Proof 6. Since G is the undirected graph, $G = \bar{G}$. From the duality, the sensor network $\bar{G}_u$ becomes the observer for (17).

6. CONCLUSION

In this works, the graph-theoretic stability condition based on the weighted degrees for linear dynamical network systems is shown. Based on the stability condition proposed in this paper, we can design a controller and an observer. As applications, an equilibrium point analysis of Lotka-Volterra system and design problem of the controller and observer for consensus systems are discussed. Finally numerical simulation show that the graph-theoretic stability based on the weighted degrees can be utilized in the pinning control and the observer for the consensus systems.

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