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Omega-limit sets and robust stability for switched systems with distinct equilibria ⋆

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Abstract: This work characterizes the asymptotic behavior that results from switching among asymptotically stable systems with distinct equilibria when the switching frequency satisfies an average dwell-time constraint with a small average rate. The asymptotic characterization is in terms of the Ω-limit set of an associated ideal hybrid system containing an average dwell-time automaton with the rate parameter set equal to zero. This set is globally asymptotically stable for the ideal system. The actual switched system, including small disturbances, constitutes a small perturbation of this ideal system, resulting in semi-global, practical asymptotic stability.

Keywords: Switched systems, Stability, Hybrid systems, Practical asymptotic stability

1. INTRODUCTION

Various authors have studied the robust stability properties of switched systems with distinct equilibria. See, for example, the pioneering results of Alpcan and Basar (2010) and the recent studies (Veer and Poulakakis, 2019c), (Veer and Poulakakis, 2019a) and (Veer et al., 2017). Studying this class of switched systems is motivated by many applications, including game theory (Basar and Olsder, 1999), where the system switches between different games containing distinct Nash equilibria, and in robotics (Gregg et al., 2012),(Veer and Poulakakis, 2019b) for motion estimation of legged robots. The control of multi-cell wireless networks with mobile switching between cells (Alpcan, 2006) and modeling of non-spiking neurons in neurophysiology (Makarenkov and Phung, 2018) are other interesting applications of switched systems with distinct equilibria.

Typically, in the literature, a Lyapunov function with certain properties is assumed and boundedness of solutions is established under a sufficiently small average dwell-time switching constraint, where the dwell-time rate can be computed from the properties of the Lyapunov function. In this work, we eschew a Lyapunov-based approach, aiming to give a more precise characterization of the set to which trajectories converge while giving a more qualitative description of the required dwell-time rate. We approach the problem using a hybrid systems modeling framework (Goebel et al., 2012). We employ the notion of an Ω-limit set from a compact set of initial conditions for a hybrid system, as considered in Cai et al. (2008a). We characterize this Ω-limit set for an associated, ideal hybrid system that uses an average dwell-time switching automaton from Vu et al. (2007) or Cai et al. (2008a) with the switching rate set to zero so that only a finite number of switches is allowed. In turn, we draw conclusions for the switched system under small average dwell-time switching by using results developed on robust (semi-global, practical) asymptotic stability of a compact set for a hybrid system.

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2. PRELIMINARIES

2.1 General notation

We use $\mathbb{R}^n$ to denote n-dimensional Euclidean space, $\mathbb{R}_{\geq 0}$ for the nonnegative real numbers, and $\mathbb{Z}_{\geq 0}$ for the nonnnegative integers. We use $|x|$ for the Euclidean norm of $x \in \mathbb{R}^n$. For a closed set $K \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the symbol $|x|_K$ denotes the distance of x to K, i.e., $|x|_K := \inf_{y \in K} |x - y|$. Given $r > 0$, we use $r\mathbb{B}$ for the set $\{x \in \mathbb{R}^n : |x| \leq r\}$ and $r\mathbb{B}^\circ$ for the set $\{x \in \mathbb{R}^n : |x| < r\}$. For a set $S \subset \mathbb{R}^n$, the symbol $\overline{S}$ denotes its closure. The closure of the convex hull of S is written as $\overline{\text{co}}S$. We say that $\alpha \in \mathcal{K}^+$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and strictly increasing. Given $\alpha_1, \alpha_2 \in \mathcal{K}^+$, the symbol $\alpha_1 \circ \alpha_2$ denotes their composition, i.e., $\alpha_1 \circ \alpha_2(s) := \alpha_1(\alpha_2(s))$. We say that $\alpha \in \mathcal{K}$ if $\alpha \in \mathcal{K}^+$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(., t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $t \mapsto \beta(r, t)$ is nonincreasing and decreases to zero as $t \to \infty$ for each fixed $r \geq 0$.

2.2 Hybrid systems

We use the hybrid systems framework described in Goebel et al. (2012). A hybrid system is written formally as

$$x \in C, \qquad \dot{x} \in F(x) \tag{1a}$$

$$x \in D, \qquad x^+ \in G(x) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $C \subset \mathbb{R}^n$ is the flow set, $D \subset \mathbb{R}^n$ is the jump set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the jump map. The data (C, F, D, G) is said to satisfy the *hybrid basic conditions* if C and D are closed, the graphs of F and G are closed, F and G are locally bounded, the values of F are nonempty and convex on C and the values of G are nonempty on D. A solution of the hybrid system (1) is a hybrid arc satisfying the constraints in (1); a hybrid arc is defined through the following concepts. A *compact hybrid time domain* is a set of the form

$$\cup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\}) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$$

for some real numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$. A *hybrid time domain* is a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ having the property that, for each $(T, J) \in E$, the set $E \cap([0,T] \times \{0,\ldots,J\})$ is a compact hybrid time domain. A *hybrid arc* is a function $x : E \to \mathbb{R}^n$ where E is a hybrid time domain and $x(\cdot, j)$ is locally absolutely continuous for each nonnegative integer j. We typically use $\mathrm{dom}(x)$ to denote the domain of the hybrid arc x. A hybrid arc is a *solution of (1)* if it satisfies the constraints implicit in (1), i.e.,

- If $(t_1, j), (t_2, j) \in \mathrm{dom}(x)$ and $t_1 < t_2$ then, for almost all $t \in [t_1, t_2]$,
$$x(t,j) \in C, \quad \dot{x}(t,j) \in F(x(t,j));$$
- If $(t, j), (t, j+1) \in \mathrm{dom}(x)$ then
$$x(t,j) \in D, \qquad x(t, j+1) \in G(x(t,j)).$$

The solution x to (1) is maximal if it cannot be extended, that is, the hybrid system has no solution x' whose domain x' contains $\mathrm{dom}\, x$ as a proper subset and x' agrees with x on $\mathrm{dom}\, x$; the solution x is complete if $\mathrm{dom}\, x$ is unbounded. Every complete solution is maximal.

Given $K \subset \mathbb{R}^n$, we use $\mathcal{S}(K)$ to denote the *set of solutions to (1) that start in* K. Given $K \subset \mathbb{R}^n$, we use $R(K)$ to denote the *reachable set from* K, i.e.,

$$R(K) := \{z \in \mathbb{R}^n : z = x(t,j),\ x \in \mathcal{S}(K), (t,j) \in \mathrm{dom}(x)\}.$$

Given $K \subset \mathbb{R}^n$, we use $\Omega(K)$ to denote the *Ω-limit set from* K, i.e.,

$$\Omega(K) := \Big\{ z \in \mathbb{R}^n : z = \lim_{i \to \infty} x_i(t_i, j_i),\ x_i \in \mathcal{S}(K), \\ (t_i, j_i) \in \mathrm{dom}(x_i), \lim_{i\to\infty} t_i + j_i = \infty \Big\}.$$

A sequence of hybrid arcs $\{x_i\}_{i=1}^{\infty}$ is said to be *locally eventually bounded* if for any $m > 0$, there exists $i_0 > 0$ and a compact set $K \subset \mathbb{R}^n$ such that for all $i > i_0$, all $(t, j) \in \mathrm{dom}\, \phi_i$ with $t + j < m$, $x_i(t, j) \in K$.

Fundamental to our analysis are results derived from the properties of the solutions to (1) for the solutions of the inflated system

$$x \in C_\delta, \qquad \dot{x} \in F_\delta(x) \tag{2a}$$
$$x \in D_\delta, \qquad x^+ \in G_\delta(x) \tag{2b}$$

where $\delta > 0$ and

$$C_\delta := \{x \in \mathbb{R}^n : (x + \delta\mathbb{B}) \cap C \neq \varnothing\} \tag{3a}$$
$$F_\delta(x) := \overline{\mathrm{co}} F((x + \delta\mathbb{B}) \cap C) + \delta\mathbb{B} \tag{3b}$$
$$D_\delta := \{x \in \mathbb{R}^n : (x + \delta\mathbb{B}) \cap D \neq \varnothing\} \tag{3c}$$
$$G_\delta(x) := G((x + \delta\mathbb{B}) \cap D) + \delta\mathbb{B}. \tag{3d}$$

We use $\mathcal{S}_\delta(K)$ to denote the solutions of (2) starting in K.

2.3 Stability concepts

We state several stability concepts for hybrid systems. They apply just as well to ordinary differential equations. The hybrid system (1) is said to be *Lagrange stable* if there exists $\alpha \in \mathcal{K}^+$ such that, for each $x_\circ \in \mathbb{R}^n$, each $x \in \mathcal{S}(x_\circ)$, and $(t, j) \in \mathrm{dom}(x)$, we have that $|x(t, j)| \leq \alpha(|x_\circ|)$.

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *(Lyapunov) stable* for the hybrid system (1) if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_\circ|_{\mathcal{A}} \leq \delta$, $x \in \mathcal{S}(x_\circ)$ and $(t, j) \in \mathrm{dom}(x)$ imply that $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$.

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *attractive* for (1) if there exists $\delta > 0$ such that each solution $x \in \mathcal{S}(\mathcal{A} + \delta\mathbb{B})$ is bounded and, if complete, satisfies $\lim_{t+j\to\infty} |x(t, j)|_{\mathcal{A}} = 0$. The *basin of attraction* for an attractive set $\mathcal{A}$ is the set of initial conditions from which each solution is bounded and, if complete, satisfies $\lim_{t+j\to\infty} |x(t, j)|_{\mathcal{A}} = 0$.

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *asymptotically stable* for (1) if it is stable and attractive. It is said to be *globally asymptotically stable* for (1) if it is asymptotically stable with $\mathbb{R}^n$ as its basin of attraction.

The set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *semiglobally practically asymptotically stable in the parameter* $\delta > 0$ for the system (2) if there exists $\beta \in \mathcal{KL}$ and, for each $\varepsilon > 0$ and $\Delta > 0$, there exists $\delta > 0$ such that each $x \in \mathcal{S}_\delta(\mathcal{A} + \Delta\mathbb{B})$ satisfies

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j) + \varepsilon \qquad \forall (t, j) \in \mathrm{dom}(x).$$

2.4 Some useful preliminary results

The first preliminary result is contained in Exercise 4.3(b) of (Rockafellar and Wets, 1998).

Lemma 1. If the convergent sequence $\{z_i\}_{i=1}^{\infty}$ satisfies $z_i \in S_i$ for all i, where $\{S_i\}_{i=1}^{\infty}$ is a decreasing sequence of closed subsets of $\mathbb{R}^n$, i.e., $S_{i+1} \subset S_i \subset \mathbb{R}^n$ for all i, then $\lim_{i\to\infty} z_i \in \bigcap_i S_i$.

The next result is Corollary 7.7 from (Goebel et al., 2012).

Lemma 2. Suppose (C, F, D, G) satisfy the hybrid basic conditions. Let K be compact and suppose that $R(K)$ is bounded and $\Omega(K)$ is nonempty and contained in the interior of K. Then $\Omega(K)$ is asymptotically stable with basin of attraction containing K.

The next result is Lemma 7.20 from (Goebel et al., 2012).

Lemma 3. Suppose (C, F, D, G) satisfy the hybrid basic conditions. If the compact set $\mathcal{A}$ is globally asymptotically stable for (1) then that set is semiglobally practically asymptotically stable in the parameter $\delta > 0$ for the system (2).

3. PROBLEM SETTING

Let M be a positive integer and define

$$\mathcal{Q} := \{1, \ldots, M\}. \tag{4}$$

For each $q \in \mathcal{Q}$, let $f_q : \mathbb{R}^n \to \mathbb{R}^n$. Let $\delta > 0$. We analyze the asymptotic behavior of the solutions of the differential inclusion

$$\dot{z} \in \overline{\mathrm{co}} f_q(z + \delta\mathbb{B}) + \delta\mathbb{B} \tag{5}$$

where $q : \mathbb{R}_{\geq 0} \to \mathcal{Q}$ is any switching signal that satisfies an average dwell-time switching constraint parametrized by δ. In particular, letting N_0 be a positive integer, and letting $N(s, t)$ denote the number of switches of q in the interval $[s, t]$, we assume that

$$N(s, t) \leq \delta(t - s) + N_0 \qquad \forall\ 0 \leq s \leq t. \tag{6}$$

Our other assumption pertains to the family of differential equations

$$\dot{z} = f_q(z) \tag{7}$$

where $q \in \mathcal{Q}$ is constant.

Assumption 1. For each $q \in \mathcal{Q}$, f_q is continuous and the point $z_q^* \in \mathbb{R}^n$ is globally asymptotically stable for (7). ■

At times, we may also impose the following assumption, a sufficient condition for which is that the continuity in Assumption 1 is strengthened to local Lipschitz continuity.

Assumption 2. For each $q \in \mathcal{Q}$, the solution of the initial value problem

$$\dot{z} = -f_q(z) \ , \quad z(0) = z_q^* \tag{8}$$

is unique. ■

Our goal is to characterize the asymptotic behavior of (5) under the switching signal constraint (6) when $\delta > 0$ is small. To make progress toward this goal, we cast the combination of (5) and (6) as an equivalent hybrid system that employs an automaton to capture the average dwell-time switching constraint. That is, we consider the behavior of the hybrid system $\mathcal{H}_\delta$ given by

$$\mathcal{H}_\delta \begin{cases} (z,q,\tau) \in \mathbb{R}^n \times \mathcal{Q} \times [0,N_0] & \begin{cases} \dot{z} \in \overline{\mathrm{co}} f_q(z+\delta\mathbb{B})+\delta\mathbb{B} \\ \dot{q} = 0 \\ \dot{\tau} \in [0,\delta] \end{cases} \\ (z,q,\tau) \in \mathbb{R}^n \times \mathcal{Q} \times [1,N_0] & \begin{cases} z^+ = z \\ q^+ \in \mathcal{Q}\setminus\{q\} \\ \tau^+ = \tau - 1. \end{cases} \end{cases} \tag{9}$$

According to Cai et al. (2008b), the solutions to (9) are in a one-to-one correspondence with the solutions of (5) under the switching constraint (6). Also, under Assumption 1, the data of the hybrid system (9) satisfies the hybrid basic conditions spelled out in (Goebel et al., 2012, Assumption 6.5). Finally, since we pursue qualitative results, there is no loss of generality in using $\delta > 0$ to describe both the size of the perturbations to the differential equation and the maximum flow rate of the timer τ.

4. ANALYSIS OF AN IDEAL SYSTEM

4.1 The model

To characterize the asymptotic behavior of the solutions of $\mathcal{H}_\delta$ in (9), we first characterize the asymptotic behavior of the the ideal system $\mathcal{H}_0$ that results from setting $\delta = 0$ in (9), i.e.,

$$\mathcal{H}_0 \begin{cases} (z,q,\tau) \in \mathbb{R}^n \times \mathcal{Q} \times [0,N_0] & \begin{cases} \dot{z} = f_q(z) \\ \dot{q} = 0 \\ \dot{\tau} = 0 \end{cases} \\ (z,q,\tau) \in \mathbb{R}^n \times \mathcal{Q} \times [1,N_0] & \begin{cases} z^+ = z \\ q^+ \in \mathcal{Q}\setminus\{q\} \\ \tau^+ = \tau - 1. \end{cases} \end{cases} \tag{10}$$

We will see that the asymptotic behavior of the solutions of this system will give an indication of the asymptotic behavior of the solutions to the system (9).

4.2 Boundedness

In this section, we establish a boundedness property for the solutions of $\mathcal{H}_0$ in (10) under Assumption 1. We start with such a boundedness result under a relaxation of Assumption 1.

Proposition 1. If, for each $q \in \mathcal{Q}$, the system (7) is Lagrange stable then the hybrid system $\mathcal{H}_0$ in (10) is Lagrange stable.

Proof. According to the assumption of the proposition, there exists a family of functions $\{\alpha_q\}_{q\in\mathcal{Q}}$ with $\alpha_q \in \mathcal{K}^+$ for each $q \in \mathcal{Q}$, such that each solution $x = (z,q,\tau)$ of the flow dynamics in (10), i.e., of

$$(z,q,\tau) \in \mathbb{R}^n \times \mathcal{Q} \times [0,N_0] \quad \begin{cases} \dot{z} = f_q(z) \\ \dot{q} = 0 \\ \dot{\tau} = 0 \end{cases} \tag{11}$$

satisfies

$$|x(t)| \leq \alpha_{q(0)}(|x(0)|) \qquad \forall t \in \mathrm{dom}(x). \tag{12}$$

Let $\mathcal{N}$ denote the family of functions obtained from k compositions of the functions α_q for any $k \in \{1,\ldots,N_0\}$ without composing the same function with itself, i.e.,

$$\begin{aligned} \mathcal{N} := \Big\{ \alpha : \alpha = \alpha_{q_k} \circ \cdots \circ \alpha_{q_1},\ & k \in \{1,\ldots,N_0\}, \\ & q_j \in \mathcal{Q} \quad \forall j \in \{1,\ldots,k\}, \\ & q_{j+1} \neq q_j \quad \forall j \in \{1,\ldots,k-1\} \Big\}. \end{aligned} \tag{13}$$

Since the composition of continuous, nondecreasing functions is continuous and nondecreasing, it follows that $\mathcal{N} \subset \mathcal{K}^+$. Note that the number of functions in the set $\mathcal{N}$ is finite. Thus we can define

$$\widetilde{\alpha}(s) := \max_{\alpha\in\mathcal{N}} \alpha(s) \qquad \forall s \geq 0, \tag{14}$$

and $\widetilde{\alpha} \in \mathcal{K}^+$ since the pointwise maximum of continuous, nondecreasing functions is continuous and nondecreasing.

Let $x = (z,q,\tau)$ be a complete solution of $\mathcal{H}_0$ in (10) and define $J := \max_{(t,j)\in\mathrm{dom}(x)} j$. Note that J is well-defined and satisfies $J \in \{0,\ldots,N_0\}$; it denotes the number of switches experienced by the solution x. Using this definition, we can write $\mathrm{dom}(x)$ as

$$\mathrm{dom}(x) = \left(\bigcup_{j=0}^{J-1} ([t_j,t_{j+1}] \times \{j\}) \right) \cup \Big([t_J,\infty) \times \{J\} \Big) \tag{15}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_J < \infty$. For notational convenience, we let $t_{J+1} > t_J$ denote an arbitrarily large positive number.

It follows from the assertion in (12) for the solutions of the system (11) that, for each $j \in \{0,\ldots,J\}$ and each $t \in [t_j,t_{j+1}]$,

$$|x(t,j)| \leq \alpha_{q(t_j,j)}(|x(t_j,j)|). \tag{16}$$

By concatenating these bounds, it follows that, for all $k \in \{0,\ldots,J\}$ and each $t \in [t_k,t_{k+1}]$,

$$|x(t,k)| \leq \alpha_{q(t_k,k)} \circ \alpha_{q(t_{k-1},k-1)} \circ \ldots \circ \alpha_{q(0,0)}(|x(0,0)|). \tag{17}$$

By the definition of the flow map and jump map in (10), it follows that $q(t_k,k) \neq q(t_{k-1},k-1)$ for each $k \in \{1,\ldots,J\}$. Hence, for each $k \in \{0,\ldots,J\}$, we have

$$\alpha_{q(t_k,k)} \circ \alpha_{q(t_{k-1},k-1)} \circ \ldots \circ \alpha_{q(0,0)} \in \mathcal{N}. \tag{18}$$

It follows from the definition of $\widetilde{\alpha}$ in (14) that, for each $k \in \{0,\ldots,J\}$ and each $t \in [t_k,t_{k+1}]$,

$$|x(t,k)| \leq \widetilde{\alpha}(|x(0,0)|). \tag{19}$$

Since t_{J+1} is arbitrary, it follows that

$$|x(t,k)| \le \widetilde{\alpha}(|x(0,0)|) \qquad \forall (t,k) \in \mathrm{dom}(x). \tag{20}$$

Thus, the hybrid system (10) is Lagrange stable. ■

Since global asymptotic stability of a compact set implies Lagrange stability, the following corollary is a consequence of Proposition 1.

Corollary 2. Under Assumption 1, the hybrid system $\mathcal{H}_0$ defined in (10) is Lagrange stable.

4.3 The Ω-limit set for $\mathcal{H}_0$

Let $K \subset \mathbb{R}^{n+2}$. For the system $\mathcal{H}_0$ defined in (10), we use $\Omega_0(K)$ to denote the Ω-limit set from K and we use $R_0(K)$ to denote the reachable set from K. We define

$$S_q := \bigcap_{j\in\mathbb{Z}_{\ge 0}} \overline{R_0\left(\left(\{z_q^*\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q\} \times [0,N_0]\right)} \tag{21a}$$

$$S := \bigcup_{q\in\mathcal{Q}} S_q. \tag{21b}$$

The next lemma is a result of Corollary 2 and the construction of S in (21).

Lemma 3. Under Assumption 1, the set S defined in (21) is compact.

The rest of this section is devoted to establishing that $\Omega_0(K) = S$ for sufficiently large compact sets K.

Proposition 4. If Assumptions 1 and 2 hold then, for each compact set $K \subset \mathbb{R}^{n+2}$ containing the set

$$\left(\bigcup_{q\in\mathcal{Q}} \{z_q^*\} \times \{q\}\right) \times [0,N_0]$$

in its interior, $\Omega_0(K) = S$.

Proposition 4 follows from the subsequent two lemmas.

Lemma 5. If Assumption 1 holds then, for each compact set $K \subset \mathbb{R}^{n+2}$, $\Omega_0(K) \subset S$.

Proof. Let $p \in \Omega_0(K)$ and let the sequence of solutions $\phi_i \in \mathcal{S}(K)$ and times $(t_i,k_i) \in \mathrm{dom}(\phi_i)$ satisfy

$$\lim_{i\to\infty} t_i + k_i = \infty \tag{22a}$$

$$\lim_{i\to\infty} \phi_i(t_i,k_i) = p. \tag{22b}$$

Since K is compact and the system $\mathcal{H}_0$ is Lagrange stable (due to Corollary 2) the sequence $\{\phi_i\}_{i=1}^{\infty}$ is locally eventually bounded. Consequently, it contains a subsequence converging to a complete solution $\phi \in \mathcal{S}(K)$ (Goebel et al., 2012, Theorem 6.1). Henceforth, we use $\{\phi_i\}_{i=1}^{\infty}$ for the converging subsequence. Define $J := \max_{(t,j)\in\mathrm{dom}(\phi)} j$. Note that J is well-defined and satisfies $J \in \{0,\ldots,N_0\}$; it denotes the number of switches experienced by the solution ϕ. Using this definition, we can write $\mathrm{dom}(\phi)$ as

$$\mathrm{dom}(\phi) = \left(\bigcup_{j=0}^{J-1} ([t_j,t_{j+1}] \times \{j\})\right) \cup \Big([t_J,\infty) \times \{J\}\Big) \tag{23}$$

where $0 = t_0 \le t_1 \le \cdots t_J < \infty$. Moreover, with $(z,q,\tau) = \phi$, since q and τ are constant during flows, there exists $(q^*,\tau^*) \in \mathcal{Q} \times [0,N_0]$ such that $(q(t,J),\tau(t,J)) = (q^*,\tau^*)$ for all $t \in [t_J,\infty)$. Also, due to Assumption 1,

$$\lim_{t\to\infty} |z(t,J) - z_{q^*}^*| = 0. \tag{24}$$

Thus, there exists an increasing, unbounded sequence of times $\{s_j\}_{j\in\mathbb{Z}_{\ge 0}}$ such that, for each $j \in \mathbb{Z}_{\ge 0}$,

$$t_J \le s_j \ , \quad |z(s_j,J) - z_{q^*}^*| \le \frac{1}{2(j+1)}. \tag{25}$$

For each $j \in \mathbb{Z}_{\ge 0}$, let $i^*(j) \in \mathbb{Z}_{\ge 0}$ be such that, for all $i \ge i^*(j)$, we have

$$t_i + k_i \ge s_j + J + 0.5 \tag{26}$$

and there exists $\hat{t}_i$ such that $(\hat{t}_i, J) \in \mathrm{dom}(\phi_i)$ satisfying

$$|\hat{t}_i - s_j| \le \frac{1}{2(j+1)} \tag{27a}$$

$$|\phi_i(\hat{t}_i,J) - \phi(s_j,J)| \le \frac{1}{2(j+1)}. \tag{27b}$$

By combining (25)-(27), it follows that

$$t_i + k_i \ge \hat{t}_i + J \tag{28a}$$

$$|z_i(\hat{t}_i,J) - z_{q^*}^*| \le \frac{1}{j+1}. \tag{28b}$$

It follows that

$$\phi_{i^*(j)}(t_{i^*(j)},k_{i^*(j)}) \in \overline{R_0\left(\left(\{z_{q^*}^*\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q^*\} \times \{\tau^*\}\right)}. \tag{29}$$

Without loss of generality, we may assume that

$$i^*(j+1) \ge i^*(j) + 1 \tag{30}$$

so that $i^*(j)$ grows unbounded in j, and hence, using (22),

$$\lim_{j\to\infty} t_{i^*(j)} + k_{i^*(j)} = \infty \tag{31a}$$

$$\lim_{j\to\infty} \phi_{i^*(j)}(t_{i^*(j)},k_{i^*(j)}) = p. \tag{31b}$$

It follows from Lemma 1 that

$$p \in \bigcap_{j\in\mathbb{Z}_{\ge 0}} \overline{R_0\left(\left(\{z_{q^*}^*\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q^*\} \times \{\tau^*\}\right)} \subset S_{q^*} \subset \bigcup_{q\in\mathcal{Q}} S_q = S. \tag{32}$$

This containment establishes the result. ■

Lemma 6. If Assumptions 1 and 2 hold then, for each compact set $K \subset \mathbb{R}^{n+2}$ containing the set

$$K_0 := \left(\bigcup_{q\in\mathcal{Q}} \{z_q^*\} \times \{q\}\right) \times [0,N_0] \tag{33}$$

in its interior, $S \subset \Omega_0(K)$.

Proof. Since K_0 belongs to the interior of K, there exists $\varepsilon > 0$ such that $K_0 + \varepsilon\mathbb{B} \subset K$. Let $p \in S$. According to (21b), we have

$$p \in \bigcup_{q\in\mathcal{Q}} S_q \ . \tag{34}$$

Thus, there exists a $q^* \in Q$ such that

$$p \in S_{q^*} = \bigcap_{j\in\mathbb{Z}_{\ge 0}} \overline{R_0\left(\left(\{z_{q^*}^*\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q^*\} \times [0,N_0]\right)}. \tag{35}$$

As a result we have that, for all $j \in \mathbb{Z}_{\geq 0}$,

$$p \in \overline{R_0\left(\left(\left\{z^*_{q^*}\right\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q^*\} \times [0, N_0]\right)}. \quad (36)$$

It follows that there exist a solution ϕ^*_j and $(t^*_j, l^*_j) \in \operatorname{dom}\phi^*_j$ such that

$$\phi^*_j(0,0) \in \left(\left(\left\{z^*_{q^*}\right\} + \frac{1}{j+1}\mathbb{B}\right) \times \{q^*\} \times [0, N_0]\right) \quad (37)$$

and

$$|\phi^*_j(t^*_j, l^*_j) - p| \leq \frac{1}{j+1}. \quad (38)$$

Let z^*_j and τ^* be such that $\phi^*_j(0,0) = (z^*_j, q^*, \tau^*_j)$. Let z_j be a solution to the system $\dot{z} = -f_{q^*}(z)$ with the initial condition $z_j(0) = z^*_j$. Define

$$h(j) := \min\{j, \inf\{t \in \operatorname{dom} z_j : |z_j(t) - z^*_q| = \varepsilon\}\}. \quad (39)$$

It follows from Assumptions 1 and 2 that

$$\lim_{j\to\infty} h(j) = \infty. \quad (40)$$

Next, we define a hybrid arc $\overline{\phi}_j$ with the domain

$$\operatorname{dom}\overline{\phi}_j := \big([0, h(j)] \times \{0\}\big) \bigcup \big(\operatorname{dom}\phi^*_j + (\{h(j)\} \times \{0\})\big), \quad (41)$$

given by

$$\overline{\phi}_j(t,k) := \quad (42)$$

$$\begin{cases} (z_j(h(j)-t), q^*, \tau^*_j) & \forall (t,k) \in [0, h(j)] \times \{0\} \\ \phi^*_j(t-h(j), k) & \forall (t,k) \in \operatorname{dom}\phi^*_j + (\{h(j)\} \times \{0\}). \end{cases}$$

It can be verified that $\overline{\phi}_j$ is a solution of system (10) starting at $(z_j(h(j)), q^*, \tau^*_j)$. This point belongs to K, due to (39) and the definition of ε.

Next, we define

$$t_j := t^*_j + h(j), \qquad l_j := l^*_j \quad (43)$$

so that, due to (40),

$$\lim_{j\to\infty} t_j + l_j = \infty. \quad (44)$$

It follows from (43), (42) and (38) that

$$\begin{aligned} |\overline{\phi}_j(t_j, l_j) - p| &= |\overline{\phi}_j(t_{j^*} + h(j), l_{j^*}) - p| \\ &= |\phi^*_j(t^*_j, l^*_j) - p| \leq \frac{1}{(j+1)}. \end{aligned} \quad (45)$$

As a result, we have that

$$\lim_{j\to\infty} \overline{\phi}_j(t_j, l_j) = p. \quad (46)$$

Now follows from (44) and (46) that $p \in \Omega_0(K)$. ■

5. MAIN RESULT

We are now ready to state our main results.

Theorem 7. Under Assumptions 1 and 2, the set S defined in (21) is semiglobally, practically asymptotically stable in the parameter $\delta > 0$ for the system $\mathcal{H}_\delta$ defined in (9).

Proof. Let the compact set $K \subset \mathbb{R}^{n+2}$ be such that the set S defined in (21), which is compact according to Lemma 3, is contained in the interior of K. According to Proposition 4, $\Omega(K)$ is contained in the interior of K. According to Lemma 2, the compact set $\Omega(K)$ is asymptotically stable with basin of attraction containing K for the system $\mathcal{H}_0$ defined in (10). Since K can be taken to be arbitrarily large, it follows that S is globally asymptotically stable for the system $\mathcal{H}_0$ defined in (10). It then follows from Lemma 3 that the set S is semiglobally practically asymptotically stable in $\delta > 0$ for the system $\mathcal{H}_\delta$ defined in (9). ■

In the case where Assumption 2 does not hold, we still have the following result:

Theorem 8. Let Assumption 1 hold and let $r > 0$ be such that $S \subset r\mathbb{B}^0$. Then the set $\Omega_0(r\mathbb{B})$ is compact, contained in S, and semiglobally, practically asymptotically stable in the parameter $\delta > 0$ for the system $\mathcal{H}_\delta$ defined in (9).

Proof. According to Lemma 5, $\Omega_0(r\mathbb{B}) \subset S$. Then, since $S \subset r\mathbb{B}^\circ$, we get $\Omega_0(r\mathbb{B}) \subset r\mathbb{B}^\circ$. It follows from Lemma 2 that the set $\Omega_0(r\mathbb{B})$ is asymptotically stable with basin of attraction containing $r\mathbb{B}$. We claim that the basin of attraction is $\mathbb{R}^{n+2}$. Indeed, for any value $r' > r$, we again have that $\Omega_0(r'\mathbb{B}) \subset S$ is asymptotically stable with basin of attraction containing $r'\mathbb{B}$. It follows from the containment $S \subset r\mathbb{B}^\circ$ that each complete solution from $r'\mathbb{B}$ reaches $r\mathbb{B}$ in finite time, and thus each point in $r'\mathbb{B}$ belongs to the basin of attraction of the asymptotically stable set $\Omega_0(r\mathbb{B})$. []In fact, $\Omega(r'\mathbb{B}) = \Omega(r\mathbb{B})$ for each $r' > r$, though this is not needed for the proof.] Since $r' > r$ was arbitrary, this observation establishes that the set $\Omega_0(r\mathbb{B})$ is globally asymptotically stable for the system $\mathcal{H}_0$ defined in (10). It then follows from Lemma 3 that the set S is semiglobally practically asymptotically stable in $\delta > 0$ for the system $\mathcal{H}_\delta$ defined in (9). ■

6. EXAMPLES

In this section, we consider an example with $N_0 = 1$, to ease the visualization of the ideal Ω-limit set. Consider the following linear time-invariant systems

$$\dot{x} = A_1 x + b_1 \quad (47a)$$
$$\dot{x} = A_2 x + b_2, \quad (47b)$$

where $x \in \mathbb{R}^2$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 10 \\ -1 & -1 \end{bmatrix}$$
$$b_1 = \begin{bmatrix} -20 \\ 4 \end{bmatrix}; \qquad b_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

The matrices A_1 and A_2 are invertible, yielding the unique equilibrium points for (47a) and (47b), respectively, at $x^*_1 = [-1.6, 20]^T$ and $x^*_2 = [3.9, 0.1]^T$. Each equilibrium is exponentially stable since A_1 and A_2 are Hurwitz.

Figure 1 shows the Ω-limit set for the ideal hybrid system. In Figure 2, we allow for a small dwell-time parameter and switches at random times that are compatible with the average dwell-time constraint. A small disturbance has been added to both subsystems. The simulations are consistent with the fact, established in Theorem 7, that the solutions converge to a small neighborhood of the ideal Ω-limit set illustrated in Figure 1 under persistent switching.

7. CONCLUSION

This paper provides a characterization of the asymptotic behavior of a perturbed, switched system with distinct equilibria under average dwell-time switching with a small

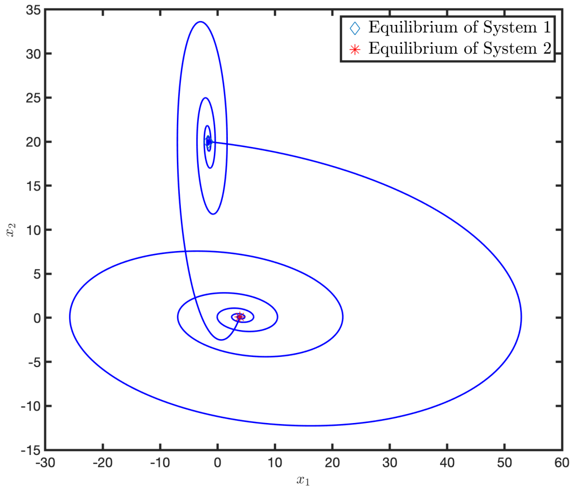

Fig. 1. The Ω-limit set corresponding to the ideal system (10) for the example (47)

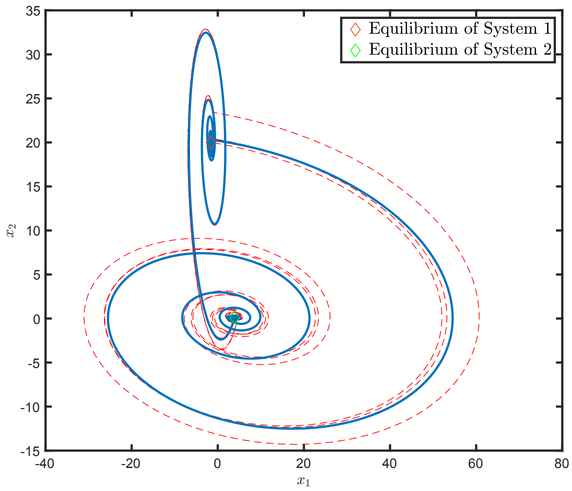

Fig. 2. "Steady-state" behavior near the ideal Ω-limit set for dwell-time switching with $\delta = \frac{1}{10}$ and disturbances

rate parameter. The asymptotic behavior of an ideal hybrid system without disturbances and without persistent switching was analyzed first. It was shown that the solutions of such a system are bounded if each subsystem is Lagrange stable. Subsequently, the Ω-limit for the ideal hybrid system was characterized and was shown to be semiglobally practically asymptotically stable in the average dwell-time parameter for the switched system. Finally, an example for a system with two equilibria was provided.

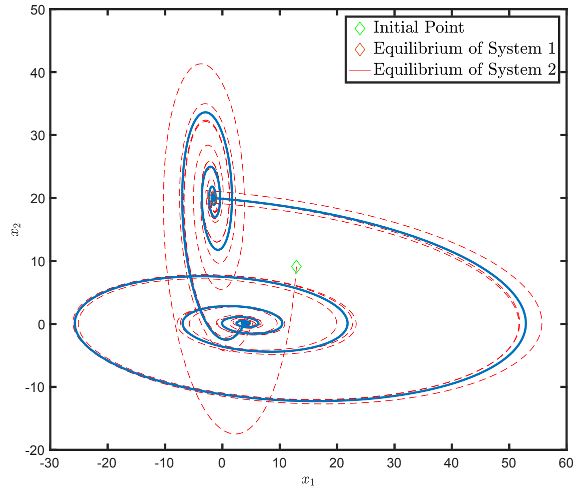

Fig. 3. Transient response from a non-equilibrium initial condition

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