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Evolution of Self-confidence in Opinion Dynamics over Signed Network ⋆

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Abstract: This paper discusses the evolution of self-confidence of individuals in Degroot-Friedkin (D-F) model when they interact along sequences of issues over a signed network. The underlying strongly connected signed network contains both positive and negative influences on the opinions of the individuals. It has been shown in this work that the opinions of the individuals under this network polarize into two groups with the individuals attaining same asymptotic opinion value within the same group while asymptotic opinion values having same magnitude, opposite in sign for different group. The evolution of self-confidences of the individuals along sequences of issues have been shown and they are found to converge to an equilibrium contained in an n-simplex. The numerical simulations validate the theoretical results obtained in the work.

Keywords: Opinion dynamics, self-confidence, social power index, DeGroot-Friedkin (D-F) model, signed graph.

1. INTRODUCTION

Recently, the investigation of social networks and characterization of its underlying phenomena has received considerable amount of research interest from control community across the world. The popularity of online social network platforms like Facebook, Twitter, Instagram, etc. in recent times are all the more reasons why the study of social network is very pertinent over the past decade or so. The analysis of the dynamics in social network simply captures how relations and influences vary among the individuals (also called agents) and how the manifested phenomena evolve over time. The dynamics in question are mainly opinion dynamics (Parsegov et al. (2017)) and/or decision-making dynamics (Gray et al. (2018)) which are the results of the interactions between individuals. Excellent repository of literature exists in Proskurnikov and Tempo (2017) and Proskurnikov and Tempo (2018).

Opinion dynamics is related to developing, studying, and analyzing mathematical models that capture the displayed cognitive orientation of individuals towards certain issues or topics. The seminal works of French (1956) and DeGroot DeGroot (1974) gave birth to the widely celebrated model namely ‘French-DeGroot’ model where the ‘iterative opinion pooling’ mechanism brings the discussants into ‘unanimity of opinion’ or generally known as ‘consensus’. Apart from consensus models, there are also some models such as Friedkin-Johnsen model (Friedkin and Johnsen (1999)), bounded confidence models of Hegselmann-Krause (Hegselmann and Krause (2002)) and Deffuant (Deffuant et al. (2000)), Altafini model (Altafini (2013)), etc. which apprehend the ‘disagreement’ aspect of social behaviour.

Based on the reflected appraisal mechanism pertinent to sociology (Friedkin (2011)) and consensus of DeGroot model (DeGroot (1974)), recently DeGroot-Friedkin (D-F) model has been proposed in Jia et al. (2015) over a sequence of issues. This proposed model is instrumental in assessing the dynamics of self-appraisal or self-confidence and social power of an individual along the issue sequences based on reflected appraisal. Reflected appraisal simply means one individual’s self-appraisal to be influenced by the appraisals held by other interacting agents towards the individual in question. The modified self-appraisal for the individuals then becomes the self-weights in the influence matrix for the next issue. Subsequently, many modified versions of D-F model has been proposed in the literature. Jia et al. (2017) studies the model for reducible influence matrix while Ye et al. (2017) studies the model for dynamic relative interpersonal influence matrix that varies over issues. Moreover, MirTabatabaei et al. (2014) investigates the model with stubborn agents while Chen et al. (2017) studies a continuous-time version of the model. Xu et al. (2015) studies the model to update the self-appraisals of the individuals in finite-time without having to wait for the opinions to reach agreement on any issue while Jia et al. (2019) studies the model where the reflected appraisal and opinion consensus process takes place on the same timescale, and hence the name single-timescale model. Chen et al. (2019) studies the model over switching influence networks with/without environmental noise while Ye and Anderson (2019) generalizes the model to incorporate various other different individual behaviours. Friedkin et al. (2016) establishes the validity of self-appraisal mechanism empirically.

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In social network, the interactions between individuals do not always embody positive aspects of relationships as distrust, antagonism, hostility, competition, etc. are also very prevalent that symbolize negative aspects (Wasserman and Faust (1994)). In order to characterize the behaviours emerging from such social structure, signed graphs are used where the positive edges represent cooperative or friendly relations while the negative ones signify the competition or hostile relations (Xue et al. (2019)). Essentially, if the signed graph of interactions is structurally balanced, then the opinions polarize into two factions of individuals (Altafini (2013)). Along this line of research, many works have been reported, some of which are Proskurnikov et al. (2016); Shi et al. (2016); Meng et al. (2018); Bhowmick and Panja (2019).

In this paper, self-confidence or self-weight manifesting along sequences of issues from the self-appraisal mechanism of D-F model has been studied over a signed directed graph of interactions which has not been paid much focus yet. The ubiquity of negative description of interactions in social network and the emerging behaviour resulting from D-F model under both friendly and hostile ties makes the study pertinent and interesting to this field. It has been shown in this work that opinions of the individuals tend to polarize the individuals into two groups if the underlying signed interaction graph follows structural balance and strong connectedness property. The final limiting values of opinions of both the groups have been obtained and are found out to be equal in modulus, but opposite in sign. Apart from this, self-confidences of the individuals when vary along issue sequences are shown to converge to the equilibrium contained in an n-simplex while forgetting the initial self-appraisal or self-weight along issue sequences.

The remainder of this paper is organized as follows. Section 2 provides the notations used along with necessary graph theory basics. In Section 3, D-F model over signed interaction network is discussed followed by principal results of the work in Section 4. The established results are verified with numerical simulation results in Section 5. Finally, concluding remarks and future direction of study are provided in Section 6.

2. USED SYMBOLS AND USEFUL GRAPH THEORY NOTIONS

Notations: $\mathcal{I}$ denotes the set of nonnegative integers and $\mathcal{J}_n = \{1, 2, \ldots, n\}$. $\mathbf{R}$, $\mathbf{R}^n$ and $\mathbf{R}^{q\times n}$ are the sets of the real numbers, n-dimensional Euclidean space, and the set of $q \times n$ real matrices, respectively. By convention, all vectors considered in the paper are column vectors. $|\ |_1$ denotes the one-norm of a real vector and $|\ |$ denotes the absolute value of a scalar. $\mathbf{R}^n_\mathbf{1}$ denotes the set of all n-dimensional real vectors whose one-norm is 1. $\lambda(A)$ denotes the eigenvalues of a square matrix A. Spectral radius of a square matrix A is denoted by $\psi(A)$ while its spectrum is denoted by $sp(A)$. $\mathbf{I}_n$ denotes an $n \times n$ dimensional identity matrix while $\mathbf{1}_n$ and $\mathbf{0}_n$ are n-dimensional vectors of 1s and 0s, respectively. Superscript T denotes the transpose of a vector or matrix. A given matrix $A = [a_{ij}] \in \mathbf{R}^{n\times n}$ and a given vector $v = [v_i] \in \mathbf{R}^n$ are positive (nonnegative) and denoted by $A > 0$ ($A \geq 0$) and $v > 0$ ($v \geq 0$), respectively if $a_{ij} > 0$ ($a_{ij} \geq 0$) ($\forall i, j \in \mathcal{J}_n$) and $v_i > 0$ ($v_i \geq 0$) ($\forall i \in \mathcal{J}_n$), respectively. Further, the nonnegative matrix A is called row-stochastic if $\sum_{j=1}^{n} a_{ij} = 1$ ($\forall i \in \mathcal{J}_n$). For any given regular matrix $A = [a_{ij}] \in \mathbf{R}^{n\times n}$, its nonnegative counterpart is denoted by $abs(A) = [|a_{ij}|] \in \mathbf{R}^{n\times n}$. A diagonal matrix of $n \times n$ dimension is represented by $diag(f_i)$ where f_i ($i \in \mathcal{J}_n$) are its diagonal elements. The canonical basis in $\mathbf{R^n}$ are denoted by $e_1, e_2, \ldots, e_n$. n-simplex is denoted by $\mathcal{S}^n = \{x \in \mathbf{R}^n_\mathbf{1} : x \geq 0\}$ while $int(\mathcal{S}^n) = \{x \in \mathbf{R}^n_\mathbf{1} : x > 0\}$ and $\mathcal{S}^n_0 = \mathcal{S}^n \backslash \{e_1, e_2, \ldots, e_n\}$.

Graph Theory: A weighted signed directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{U})$ is considered to represent the opinion dynamical network where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ is the node set representing n number of individuals, $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$ is the edge set representing the interactions between two individuals, and $\mathcal{U} = [u_{ij}] \in \mathbf{R}^{n\times n}$ is the signed relative interpersonal influence matrix. In the rest of the paper, nodes, individuals, and agents will be used interchangeably. Obviously, for two distinct individuals $v_i, v_j \in \mathcal{V}$, the ordered pair $(v_j, v_i) \in \mathcal{E}$ means that $u_{ij} \neq 0$, i.e., individual v_i receives opinion information from individual v_j; $u_{ij} = 0$, otherwise. Self-loop is also considered in the graph such that $(v_i, v_i) \in \mathcal{E}$ and $u_{ii} \geq 0$ ($\forall i \in \mathcal{J}_n$). Moreover, for two distinct individuals $v_i, v_j \in \mathcal{V}$, if v_j has positive, neutral, or negative influence on v_i, then they are denoted by $u_{ij} > 0$, $u_{ij} = 0$, and $u_{ij} < 0$, respectively. The underlying interaction graph contains a directed path from node v_{p_1} to v_{p_m} if $(v_{p_k}, v_{p_{k+1}}) \in \mathcal{E}$ ($k \in \mathcal{J}_{m-1}$). A digraph is called a *star graph* if there exists a node known as 'center node' such that every directed edge is either incoming or outgoing with respect to this node. A digraph is said to contain a rooted directed spanning tree if there exists at least a node, called the root node which has a directed path to every other node in the graph. The digraph is strongly connected if there exists directed path between any pair of nonidentical nodes. The matrix $\mathcal{U}$ is irreducible if its associated digraph is strongly connected.

A graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{U})$ is said to be structurally balanced if its node set $\mathcal{V}$ can be divided into two disjoint groups $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, and $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$, such that $u_{ij} > 0$, $\forall v_i, v_j \in \mathcal{V}_q$ (or $\mathcal{V}_r$), and $u_{ij} < 0$, $\forall v_i \in \mathcal{V}_q$, $\forall v_j \in \mathcal{V}_r$ where $q \neq r$, and $q, r \in \{1, 2\}$. Signature vector $s \in \{-1, 1\}^n$ and signature matrix $S \in \{-1, 0, 1\}^{n\times n}$ corresponding to the graph are defined in such a way that $s = [s_1\ s_2 \ldots s_n]^T$ and $S = diag(s_i)$ ($\forall i \in \mathcal{J}_n$) where $s_i = 1$, $\forall v_i \in \mathcal{V}_q$ and $s_i = -1$, $\forall v_i \in \mathcal{V}_r$ with $q \neq r$. Furthermore, for $i \in \mathcal{J}_n$, the i^{th} *signature basis vector* $\overline{s}_i \in \{-1, 0, 1\}^n$ is defined in such a way that all entries of the vector is zero except the i^{th} entry which is equal to s_i. In the rest of the paper, the graph is denoted by $\mathcal{G}(\mathcal{U})$.

3. D-F MODEL OVER SIGNED NETWORK

In this section, the opinion dynamical model namely D-F model is discussed which underpins the evolution mechanism of self-confidence and social power. It is considered that a group of $n \geq 3$ individuals discuss a number of issues sequentially. Consequently, for an issue $m \in \mathcal{I}$, it is considered that $p_i(m, k) \in \mathbf{R}$ be the opinion of i^{th} individual ($i \in \mathcal{J}_n$) at k^{th} instant ($k \in \mathcal{I}$) whose dynamical equation is

$$p_i(m,k+1) = a_{ii}(m)p_i(m,k) + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}p_j(m,k) \qquad (1)$$

where the diagonal elements $a_{ii}(m) \in [0,1]$ of signed influence matrix $\mathcal{A}(m) \in \mathbf{R}^{n\times n}$ represents the self-confidence of i^{th} individual on issue m while the off-diagonal elements $a_{ij}(m) = (1 - a_{ii}(m))u_{ij}$ represents individual i's weight assignment to j's opinion. Moreover, for $\mathcal{U} = [u_{ij}]$, u_{ij} is termed as relative interpersonal influence weight that individual i grants to the opinion of j. 'Relative' is used here to state that $u_{ii} = 0$ $(\forall i \in \mathcal{J}_n)$ and thus relative influence between two nonidentical individuals is considered. It is to be noted that since the interaction graph considered in the paper is a signed one, therefore u_{ij} can be positive, negative, or zero depending upon the corresponding type of influences, i.e positive, negative, or neutral influence, respectively. Evidently, the sign of a_{ij} is the same as that of the corresponding u_{ij} $(\forall i,j \in \mathcal{J}_n)$. However, it is assumed that $\sum_{j=1}^{n} |u_{ij}| = 1$ with $u_{ii} = 0$ $(\forall i \in \mathcal{J}_n)$ that further ensures that $\sum_{j=1}^{n} |a_{ij}| = 1$ $(\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I})$. Therefore, quite clearly, 1 is an eigenvalue of $\mathcal{U}$ and the signature vector s of graph $\mathcal{G}(\mathcal{U})$ is its corresponding right eigenvector. Moreover, with respect to eigenvalue 1, κ^T is considered to be the normalized left eigenvector of $\mathcal{U}$. For notational brevity, $a_{ii}(m) = x_i(m)$ is used. Incidentally, (1) can be written in the following compact form

$$p(m,k+1) = \mathcal{A}(m)p(m,k) \qquad (2)$$

where $p(m,k) = [p_1(m,k)\ p_2(m,k) \ldots p_n(m,k)]^T$ is the stacked vector of opinions of the individuals. Furthermore, it is easy to observe that

$$\mathcal{A}(m) = X(m) + (\mathbf{I}_n - X(m))\mathcal{U} \qquad (3)$$

where $X(m) = diag(x_i(m)) \in \mathbf{R}^{n\times n}$. It is to be noted that $\mathcal{A}(m)$ will be denoted by $\mathcal{A}$ sometimes for notational brevity.

Assumption 1: The underlying interaction graph $\mathcal{G}(\mathcal{U})$ is (non-star) strongly connected and structurally balanced.

Assumption 2: Initial self-confidence $0 \le x_i(0) < 1$ $(\forall i \in \mathcal{J}_n)$ and $\exists j \in \mathcal{J}_n$ such that $x_j(0) > 0$.

Remark 1. A star-graph represents the emergence of an autocratic power configuration with the center node being the autocrat, and the asymptotic opinion consensus value is equal to the initial opinion value of the autocrat. A non-star graph, on the other hand, ensures that there is no autocrat leader influencing the final opinion formation.

4. PRINCIPAL TECHNICAL RESULTS

Lemma 1. The underlying signed interaction digraph $\mathcal{G}(\mathcal{U})$ is considered that satisfies Assumption 1. Then, the following statements hold good for signed influence matrix $\mathcal{A}$:

(i) The spectral radius of $\mathcal{A}$ is $\psi(\mathcal{A}) = 1$ which is simple.
(ii) The (normalized) left and right eigenvectors of $\mathcal{A}$ corresponding to its $\psi(\mathcal{A}) = 1$ are $\mu^T(m)$ $(\mu(m) \in \mathbf{R}^n_{\mathbf{1}})$ and signature vector (of $\mathcal{G}(\mathcal{U})$) $s \in \{-1,1\}^n$ such that $\mu^T(m)\mathcal{A} = \mu^T(m)$ and $\mathcal{A}s = s$. Moreover, $\mu(m) = S\sigma(m)$ where $\sigma^T(m)$ is the (normalized) left eigenvector of $abs(\mathcal{A})$ and S is the signature matrix of $\mathcal{G}(\mathcal{U})$.
(iii) The steady state asymptotical value of influence matrix $\mathcal{A}$ is given as $\lim_{k\to\infty} \mathcal{A}^k = s\mu^T(m)$.

Proof. For structurally balanced, strongly connected signed graph $\mathcal{G}(\mathcal{U})$ associated with signed relative interpersonal influence matrix $\mathcal{U}$, its node set $\mathcal{V}$ can be divided into two disjoint subsets of nodes. Then, it can be easily observed that $a_{ij} = |a_{ij}|s_is_j$, $\forall i,j \in \mathcal{V}$. Therefore, $S\mathcal{A}S = abs(\mathcal{A})$. Obviously, $abs(\mathcal{A})$ is a nonnegative, row-stochastic, irreducible matrix. Moreover, from Gershgorin's circle theorem (Horn and Johnson (2013)), $|\lambda(\mathcal{A}) - a_{ii}| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = 1 - a_{ii}$, $\forall \lambda(\mathcal{A}) \in sp(\mathcal{A})$ which clearly shows that $|\lambda(\mathcal{A})| \le 1$. However, according to Perron-Frobenius theorem, the nonnegative, row-stochastic, irreducible matrix $abs(\mathcal{A})$ has a simple eigenvalue of 1 which is its spectral radius. Due to similarity transformation, $abs(\mathcal{A})$ and $\mathcal{A}$ have the same spectrum (Altafini (2013)). Thus, it can be concluded that influence matrix $\mathcal{A}$ has spectral radius $\psi(\mathcal{A}) = 1$ which is simple. This proves point (i).

Post-multiplying $S\mathcal{A}S = abs(\mathcal{A})$ by $\mathbf{1}_n$ on both sides and using $S\mathbf{1}_n = s$, and $abs(\mathcal{A})\mathbf{1}_n = \mathbf{1}_n$, one obtains $S\mathcal{A}s = \mathbf{1}_n$ which further yields $\mathcal{A}s = s$ after noting that $S^2 = \mathbf{I}_n$. Therefore, s is a right eigenvector of $\mathcal{A}$ corresponding to $\psi(\mathcal{A}) = 1$. Moreover, $\sigma^T(m)$ is the normalized left eigenvector of $abs(\mathcal{A})$ corresponding to its spectral radius 1 such that $\sigma^T(m)abs(\mathcal{A}) = \sigma^T(m)$ which yields $\sigma^T(m)S\mathcal{A} = \sigma^T(m)S$, i.e. $\mu^T(m) = \sigma^T(m)S$ is the left eigenvector of $\mathcal{A}$ corresponding to $\psi(\mathcal{A}) = 1$. Furthermore, since according to Perron-Frobenius theorem, $\sigma(m) > 0$ and normalized such that $|\sigma(m)|_1 = 1$, then $\mu(m)$ is also normalized such that $|\mu(m)|_1 = 1$. This proves point (ii).

Moreover, let $\lim_{k\to\infty} \mathcal{A}^k = \Xi$. Then, pre- and post-multiplying both sides by S and noting that $S^2 = \mathbf{I}_n$, one obtains $\lim_{k\to\infty} abs(\mathcal{A}^k) = S\Xi S$ which further yields $S\Xi S = \mathbf{1}_n\sigma^T(m)$ for row-stochastic, nonnegative, irreducible matrix $abs(\mathcal{A})$. Further, pre- and post-multiplication on both sides by S results in $\Xi = s\mu^T(m)$. This proves point (iii).

From the above discussion, the steady-state value of the opinion vector $\forall m \in \mathcal{I}$ over a signed graph can be given as

$$\lim_{k\to\infty} p(m,k) = (\lim_{k\to\infty} \mathcal{A}^k(m))p(m,0) = (\mu^T(m)p(m,0))s. \qquad (4)$$

Remark 2. Obviously, the steady state opinion values of the individuals can be seen to be polarized into two values having same magnitude but opposite sign. This is in stark contrast to opinion consensus reaching in D-F model over unsigned graph where the limiting value of opinion consensus in an issue is the convex combination of the individual's initial opinion.

Remark 3. From (4), it can be observed that the vector $\mu(m)$ embodies the contribution of the individuals to the final limiting value of the opinion vector on issue m. More specifically, for $\mu(m) = [\mu_i(m)]$ $(i \in \mathcal{J}_n)$, an individual i holds $|\mu_i(m)|$ as his/her social power on issue m that contributes to the final decision-making process, and can be considered to be social power index. However, since the quantification of self-confidence cannot be negative from an individual's own perception of himself/herself, therefore evolution of self-confidence along sequences of issues can be captured by

$$x(m+1) = S\mu(m) \tag{5}$$

where $x(m) = [x_1(m)\ x_2(m) \dots x_n(m)]^T$ is the vector of self-confidence on m^{th} issue. Since, $|\mu|_1 = 1$ ($\forall m \in \mathcal{I}$), thus it is obvious that nonnegative self-confidence vector follows $|x|_1 = 1$, i.e., $x \in \mathcal{S}^n$ ($\forall m \in \mathcal{I}$). However, social power index that can be positive, negative, or zero signifies that the net appraisals held by other agents towards a particular agent over a signed graph of interactions can be positive, negative, or even zero depending on that agent's positive, negative, or neutral influence on other agents' decision-making.

It is easily discernible that the normalized left eigenvector $\mu^T(m)$ of influence matrix $\mathcal{A}$ which is the social power index and the left eigenvector κ^T of relative interpersonal influence matrix $\mathcal{U}$ which is known as centrality measure (Jia et al. (2015)) are very informative to characterize the aspects of opinion evolution, self-confidence, and social power. To that end, the following proposition is stated that shows the relationship between these two indices.

Proposition 2. If Assumptions 1 and 2 hold for the interaction graph and initial self-confidence, respectively, then the evolution of social power along issue sequences is dictated by the following discrete-time system

$$\mu(m+1) = f(x(m+1)),\ \forall m \in \mathcal{I} \tag{6}$$

where the function $f : \mathcal{S}^n \to \mathbf{R}^n_{\mathbf{1}}$ is given by

$$f(x) = \begin{cases} \bar{s}_i & \text{if } x_i = 1\ (\forall i \in \mathcal{J}_n) \\ \dfrac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i}} \begin{bmatrix} \dfrac{\kappa_1}{1-x_1} \\ \dfrac{\kappa_2}{1-x_2} \\ \vdots \\ \dfrac{\kappa_n}{1-x_n} \end{bmatrix} & \text{otherwise } (\forall i \in \mathcal{J}_n) \end{cases} \tag{7}$$

and $\bar{s}_i$ is i^{th} the signature basis vector, $s_i \in \{\pm 1\}$, and κ_i is the i^{th} element of κ^T.

Proof. Without any loss of generalization, it is first assumed that $x_1(m) = 1$ for some $m \in \mathcal{I}$. Consequently, the corresponding influence matrix can be given as

$$\mathcal{A}(m) = \begin{bmatrix} 1 & \mathbf{0}^T_{n-1} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \tag{8}$$

where $\mathcal{A}_{21} \in \mathbf{R}^{n-1}$ and $\mathcal{A}_{22} \in \mathbf{R}^{(n-1)\times(n-1)}$. Incidentally, it is easy to observe that $\mathcal{G}(\mathcal{U})$ has a rooted directed spanning tree, rooted at node v_1 with no incoming edges. Moreover, since the normalized left eigenvector $\mu^T(m)$ of signed influence matrix $\mathcal{A}(m)$ is not necessarily nonnegative, therefore $\mu(m) = [s_1\ 0 \dots 0]^T = \bar{s}_1$. In the same vein of discussion, it can be concluded that if $x_i(m) = 1$ ($\forall i \in \mathcal{J}_n$), then the graph $\mathcal{G}(\mathcal{U})$ has a rooted directed spanning tree rooted at node v_i with no incoming edges. Hence the evolution of social power index can be dictated by the evolution of self-confidence along issue sequences only. Thus, the model given by (6) is correctly claimed for $f(x) = \bar{s}_i$ if $x_i = 1$ ($\forall i \in \mathcal{J}_n$).

Next, it is assumed that $\nexists i$ such that $x_i(m+1) = 1$ ($\forall m \in \mathcal{I}$). Then, from (3) and after considering $\kappa^T\mathcal{U} = \kappa^T$ where κ^T is the left eigenvector of $\mathcal{U}$, the following can be obtained

$$\begin{aligned} &\kappa^T(\mathbf{I}_n - X(m+1))^{-1}\mathcal{A}(m+1) \\ &= \kappa^T(\mathbf{I}_n - X(m+1))^{-1}X(m+1) + \kappa^T\mathcal{U} \\ &= \kappa^T(\mathbf{I}_n - X(m+1))^{-1} \end{aligned} \tag{9}$$

which means that $\kappa^T(\mathbf{I}_n - X(m+1))^{-1}$ is the left eigenvector of $\mathcal{A}(m+1)$. Obviously, $(\mathbf{I}_n - X(m+1))^{-1} > 0$ and the elements in κ are positive or negative depending on the corresponding sign patterns of the signature matrix S. Consequently, after normalization of $\kappa^T(\mathbf{I}_n - X(m+1))^{-1}$ with the scaling factor $\frac{1}{\sum_{i=1}^{n}\frac{s_i\kappa_i}{1-x_i}}$, the model (6) is obtained with the function (7).

Remark 4. The function (7) clearly shows the relationship between two important parameters namely social power index and centrality measure, and how social power index varies with self-confidence. Incidentally, self-confidence has its own evolution along issue sequences that is captured by another nonlinear function. To that end, the following theorem is proposed.

Theorem 3. Assumptions 1 and 2 are considered to hold for the interaction graph and initial self-confidence, respectively. Then the evolution of self confidence along issue sequences is captured by the following dynamical model

$$x(m+1) = g(x(m)),\ \forall m \in \mathcal{I} \tag{10}$$

where the continuous function $g : \mathcal{S}^n \to \mathcal{S}^n$ is given by

$$g(x) = \begin{cases} e_i & \text{if } x_i = 1\ (\forall i \in \mathcal{J}_n) \\ \dfrac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i}} \begin{bmatrix} \dfrac{s_1\kappa_1}{1-x_1} \\ \dfrac{s_2\kappa_2}{1-x_2} \\ \vdots \\ \dfrac{s_n\kappa_n}{1-x_n} \end{bmatrix} & \text{otherwise } (\forall i \in \mathcal{J}_n). \end{cases} \tag{11}$$

Furthermore, $\forall j \in \mathcal{J}_n$, $\forall m \in \mathcal{I}$, $\exists \omega_j > 0$ such that

$$\omega_j \le \min_{\forall j \in \mathcal{J}_n} (1 - x_j, \frac{1 - 2s_j\kappa_j}{1 - s_j\kappa_j}). \tag{12}$$

Then, $x^* \in int(\mathcal{S}^n)$ is a unique fixed point of the dynamical system expressed by (10) and (11). Moreover, $\forall x(0) \in \mathcal{S}^n_0$ and $m \in \mathcal{I}$, $\lim_{m\to\infty} x(m) = x^*$.

Proof. It can be seen that the function g is continuous in $\mathcal{S}^n$ as it is analytical in $\mathcal{S}^n_0$, and it is locally Lipschitz continuous at e_i ($\forall i \in \mathcal{J}_n$). Then, since $x(m) \in \mathcal{S}^n$ ($\forall m \in \mathcal{I}$), therefore $\forall i \in \mathcal{J}_n$, $x_i = 1$ means $x(m) = e_i$. This implies that the associated graph has a rooted directed spanning tree rooted at node v_i. From Proposition 3, this further gives $\mu(m) = \bar{s}_i$. Using (5), it can be readily observed that $x(m+1) = e_i$ ($\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I}$). In other words, if an individual holds absolute/autocratic power for the decision-making on a particular issue, he/she will continue to enjoy the same power over subsequent issues.

Moreover, it is noted that since $\mu^T(m)$ is the normalized left eigenvector of signed influence matrix $\mathcal{A}(m)$ for issue m ($m \in \mathcal{I}$) and from (5), $\mu(m) = Sx(m+1)$ can be obtained, thus one gets $\mathcal{A}^T(m)Sx(m+1) = Sx(m+1)$ which after using (3) further yields $(X(m) + \mathcal{U}^T(\mathbf{I}_n - X(m)) - \mathbf{I}_n)Sx(m+1) = 0$. This in turn reduces to $(\mathbf{I}_n - X(m))Sx(m+1) = \mathcal{U}^T(\mathbf{I}_n - X(m))Sx(m+1)$, i.e., $x^T(m+1)S(\mathbf{I}_n - X(m))\mathcal{U} = x^T(m+1)S(\mathbf{I}_n - X(m))$. Therefore, it can be concluded that $x^T(m+1)S(\mathbf{I}_n - X(m))$ is the left eigenvector of signed relative interpersonal influence matrix $\mathcal{U}$. Since, κ^T is the normalized left eigenvector of $\mathcal{U}$ and $x(m+1) \in \mathcal{S}^n$ ($\forall m \in \mathcal{I}$), therefore one obtains the following

$$x_i(m+1)s_i(1-x_i(m)) = \frac{\kappa_i}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i}}, \ \forall i \in \mathcal{J}_n \tag{13}$$

which further gets simplified to

$$x_i(m+1) = \frac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i(m)}} \frac{s_i\kappa_i}{(1-x_i(m))}, \ \forall i \in \mathcal{J}_n \tag{14}$$

with $\frac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i(m)}}$ being the scaling factor to ensure that $|x(m+1)|_1 = 1$.

It can be clearly observed that $x(m) = e_i$ ($\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I}$) which are the vertices of $\mathcal{S}^n$ are the fixed points of the function given by (11). However, in this paper the focus is on finding a fixed point x^* such that $x^* \in \mathcal{S}_0^n$, i.e., non-vertex fixed points. To that end, a convex, compact set $\Omega = \{x \in \mathcal{S}^n : 1-\omega \geq x_i, \forall i \in \mathcal{J}_n\}$ where $0 < \omega \leq \omega_j$ ($\forall j \in \mathcal{J}_n$) with ω_j given in (12). Moreover, since $1-x_i < 1$, then scaling factor $\frac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i}} > 0$ which ensures that $x_i > 0$ ($\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I}$). Therefore, there does not exist any fixed point on the boundary of $\mathcal{S}^n$. Consequently, the j^{th} entry of $g(x)$ is considered

$$\begin{aligned} g_j(x) &= \frac{1}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-x_i}} \frac{s_j\kappa_j}{1-x_j}, \ \forall j \in \mathcal{J}_n, \ \forall m \in \mathcal{I} \\ &= \frac{1}{\frac{s_j\kappa_j}{1-x_j}\left(1 + \frac{\sum_{i\neq j}^{n} \frac{s_i\kappa_i}{1-x_i}}{\frac{s_j\kappa_j}{1-x_j}}\right)} \frac{s_j\kappa_j}{1-x_j} \\ &\leq \frac{1}{1 + \frac{\omega}{s_j\kappa_j} \frac{\sum_{i\neq j}^{n} s_i\kappa_i}{1-x_i}} \end{aligned} \tag{15}$$

where $0 < \omega \leq \omega_j$ ($\forall j \in \mathcal{J}_n$) is used from (12). Since $1-x_i < 1$ ($\forall i \in \mathcal{J}_n$) and $\sum_{i\neq j}^{n} s_i\kappa_i = 1 - s_j\kappa_j$, therefore, (15) further yields

$$\begin{aligned} g_j(x) &< \frac{1}{1 + \frac{\omega(1-s_j\kappa_j)}{s_j\kappa_j}} \\ &= \frac{s_j\kappa_j}{s_j\kappa_j + \omega(1-s_j\kappa_j)} \\ &= \frac{s_j\kappa_j - (1-\omega)(s_j\kappa_j + \omega(1-s_j\kappa_j))}{s_j\kappa_j + \omega(1-s_j\kappa_j)} + (1-\omega) \\ &= \frac{\omega(1-s_j\kappa_j)(\omega - \frac{1-2s_j\kappa_j}{1-s_j\kappa_j})}{\omega + (1-\omega)s_j\kappa_j} + (1-\omega). \end{aligned} \tag{16}$$

Since $|\kappa_j| < \frac{1}{2}$ ($\forall j \in \mathcal{J}_n$) for non-star graph topology (Jia et al. (2015)) and $0 < \omega \leq \omega_j$, therefore $g_j(x) < (1-\omega)$. This implies that $g(\Omega) \subset \Omega$. Consequently, Brouwer's fixed point theorem states that for g which is a continuous function on compact set Ω, there exists at least a fixed point $x^* \in \Omega$. However, since $x_i > 0$ ($\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I}$), therefore it is further established that $x^* \in int(\mathcal{S}^n)$, i.e., there does not exist any fixed point on the boundary.

The fixed point x^* is unique. To show this, let two distinct vectors $\hat{x} = [\hat{x}_1 \ \hat{x}_2 \dots \hat{x}_n]^T \in \mathcal{S}_0^n$ and $\hat{y} = [\hat{y}_1 \ \hat{y}_2 \dots \hat{y}_n]^T \in \mathcal{S}_0^n$ satisfy $\hat{x} = g(\hat{x})$ and $\hat{y} = g(\hat{y})$, respectively. It can be easily observed that $\forall i \in \mathcal{J}_n$, $\hat{x}_i(1-\hat{x}_i) = \alpha\hat{y}_i(1-\hat{y}_i)$ where $\alpha = \frac{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-\hat{y}_i}}{\sum_{i=1}^{n} \frac{s_i\kappa_i}{1-\hat{x}_i}}$ with $\alpha = 1$ or $\alpha > 1$ or $\alpha < 1$. Firstly, $\alpha = 1$ is considered. Apparently, either $\hat{x}_i = \hat{y}_i$ or $\hat{x}_i = 1 - \hat{y}_i$. It is assumed that there exists at least a $j \in \mathcal{J}_n$ ($j \neq i$) such that $\hat{x}_j = 1 - \hat{y}_j \neq \hat{y}_j$ and for all other $i \in \mathcal{J}_n$ ($i \neq j$), $\hat{x}_i = \hat{y}_i$. It is to be noted that $\sum_{i=1}^{n} \hat{x}_i = 1$ and $\sum_{i=1}^{n} \hat{y}_i = 1$. Then, from $\sum_{i=1}^{n} \hat{x}_i = 1$, one obtains $\sum_{i=1, i\neq j}^{n} \hat{x}_i + \hat{x}_j = 1$ which further yields $\sum_{i=1,i\neq j}^{n} \hat{y}_i + 1 - \hat{y}_j = 1$ and this implies $1 - \hat{y}_j = \hat{y}_j$ which is a contradiction. Further, it is assumed that there exists another $k \in \mathcal{J}_n$ ($k \neq j \neq i$) such that $\hat{x}_k = 1 - \hat{y}_k \neq y_k$. Then one obtains, $\sum_{i=1,i\neq j,k}^{n} \hat{x}_i + \hat{x}_j + \hat{x}_k = 1$ which yields $\sum_{i=1,i\neq j,k}^{n} \hat{y}_i + 1 - \hat{y}_j + 1 - \hat{y}_k = 1$ and this implies that $1 - \hat{y}_j + 1 - \hat{y}_k = \hat{y}_j + \hat{y}_k$ which is again a contradiction. Hence, $\nexists \hat{j} \in \mathcal{J}_n$ such that $\hat{x}_{\hat{j}} = 1 - \hat{y}_{\hat{j}}$. Therefore, $\forall i \in \mathcal{J}_n$, $\hat{x}_i = \hat{y}_i$, i.e., $\hat{x} = \hat{y}$ which implies that the fixed point $\hat{x} \in \mathcal{S}_n^0$ is unique for $\hat{x} = g(\hat{x})$. The uniqueness of the fixed point can be proved similarly for $\alpha > 1$ and $\alpha < 1$ and are omitted here.

The convergence of the self-confidences to x^* can be shown in the following way. Let, $\tilde{x}_i(m) = \frac{x_i(m)}{x_i^*}$ and $\tilde{g}_i(x) = \frac{g_i(x)}{g_i(x^*)}$ ($\forall i \in \mathcal{J}_n, \forall m \in \mathcal{I}$). Then, let, $\overline{x}(m) = \max_{j\in\mathcal{J}_n}\{\tilde{x}_j(m)\}$ and $\underline{x}(m) = \min_{j\in\mathcal{J}_n}\{\tilde{x}_j(m)\}$. Similarly, $\overline{g}(x) = \max_{j\in\mathcal{J}_n}\{\tilde{g}_j(x)\}$ and $\underline{g}(x) = \min_{j\in\mathcal{J}_n}\{\tilde{g}_j(x)\}$ are considered. A Lyapunov function $V(x(m)) = \frac{\overline{x}(m)}{\underline{x}(m)}$ is considered for analysis. Then, essentially, $V(g(x)) = \frac{\overline{g}(x)}{\underline{g}(x)}$ can be written. Without any loss of generalization, let, $\overline{x}(m) = \tilde{x}_k(m)$ and $\underline{x}(m) = \tilde{x}_l(m)$ ($k, l \in \mathcal{J}_n$). Then, one obtains $V(x(m)) = \frac{x_k(m)/x_k^*}{x_l(m)/x_l^*}$ and $V(g(x)) = \frac{(1-x_l(m))/(1-x_l^*)}{(1-x_k(m))/(1-x_k^*)}$ ($\forall m \in \mathcal{I}$). Since, $\frac{x_k(m)}{x_k^*} \geq \frac{x_l(m)}{x_l^*}$ implies $\sum_{\mathcal{J}_n\ni i\neq l} \frac{x_k(m)x_i^*}{x_k^*} \geq \sum_{\mathcal{J}_n\ni i\neq l} x_i(m)$ which eventually yields $\frac{x_k(m)}{x_k^*} \geq \frac{1-x_l(m)}{1-x_l^*}$. In a similar fashion, it can be shown that $\frac{x_l(m)}{x_l^*} \leq \frac{1-x_k(m)}{1-x_k^*}$. Consequently, Lyapunov function $V(x)$ is strictly decreasing in $int(\mathcal{S}^n)\backslash\{x^*\}$ ($\forall m \in \mathcal{I}$). Then, any sublevel set of $V(x)$ namely $\Gamma_\theta = \{x \in int(\mathcal{S}^n) | V(x) \leq \theta\}$ for some $\theta \geq 1$ is closed, bounded, and invariant. Moreover, both $V(x)$ and $g(x)$ are continuous functions. Then, applying LaSalle's invariance principle (Khalil (2002)), any trajectory starting from Γ_θ converges to equilibrium x^* asymptotically. Moreover, $\forall x(0) \in \mathcal{S}_0^n$ satisfies $V(g(x(0))) \leq \nu$ for some $\nu \geq 1$. Then, considering $\theta = \nu$, it can be concluded that any trajectory starting from $\mathcal{S}_0^n$ converges to equilibrium point x^*. In other words, $\forall x(0) \in \mathcal{S}_0^n$ and $m \in \mathcal{I}$, $\lim_{m\to\infty} x(m) = x^* \in int(\mathcal{S}^n)$.

Remark 5. The preceding analysis shows that individuals' self-confidences are evolved as part of their repetitive deliberations on sequences of issues, thereby converging to an equilibrium structure within the interior of the n-simplex while non-autocratic initial self-weights are forgotten along the discussion of issues.

5. SIMULATION RESULTS

A social network of six individuals is considered for simulation results discussing over ten issues. The corresponding interaction network is shown in Fig. 1. Solid lines indicate positive relationships between agents, dashed lines indicate negative relations, and self-loops indicate self-weights. Clearly, the graph is structurally balanced and strongly connected. Specifically, for simulations, initial signed influence matrix for $m = 0$ issue is considered to be
$$\mathcal{A}(0) = \begin{bmatrix} 2/5 & 0 & -3/5 & 0 & 0 & 0 \\ -1/10 & 9/10 & 0 & 0 & 0 & 0 \\ 0 & 3/10 & 2/5 & 1/5 & 0 & -1/10 \\ -1/5 & 3/10 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -2/5 & 3/5 & 0 \\ 2/5 & 0 & 0 & 0 & 2/5 & 1/5 \end{bmatrix}$$ and signed relative interpersonal influence matrix is
$$\mathcal{U} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/3 & 0 & -1/6 \\ 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}.$$ The left eigenvectors $\mu^T(0) = [0.1379\ -0.6208\ -0.1379\ -0.0690\ 0.0172\ 0.0172]$ and $\kappa^T = [0.2927\ -0.2195\ -0.2927\ -0.1219\ 0.0244\ 0.0488]$ of $\mathcal{A}(0)$ and $\mathcal{U}$, respectively are obtained. Initial opinion vector of the individuals is considered to be $\overline{p}_0 = [0.2\ -0.4\ -0.3\ -0.7\ 0.1\ 0.6]$. Fig. 2 shows the evolution of opinions of six individuals in continuous-time for $m = 0^{th}$ issue exhibiting opinion polarization which conveys the fact that opinions settle into two groups with the asymptotic opinion values being equal in magnitude, but opposite in sign for the groups. From (4), the polarized opinions should settle at values 0.38 and -0.38 which can be verified from Fig 2. Moreover, Fig. 3 depicts the evolution of self-confidence of six agents along sequences of ten issues. It can be seen that every agent's self-confidence is confined in the range $(0, 1)$, and all the self-confidence values converge to the equilibrium configuration that lies in the interior of the n-simplex $\mathcal{S}^n$, i.e., $int(\mathcal{S}^n)$.

6. CONCLUSIONS

In this work, study of evolution of self-confidence along sequences of issues has been carried out on D-F model under signed digraph that contains both both and negative relationships between the individuals. It has been shown that under the structural balance, strongly connected properties of the underlying interaction topology, the opinions of the individuals polarize into two groups such that the limiting values of the final opinions being same in modulus, opposite in sign for the two groups. On the other hand, it has been shown that the evolution of self-confidence of individuals stay in the range $(0, 1)$ as they vary along sequences of issues and converge to the equilibrium point contained in the interior of the $n-$simplex. As future scope of work, the study will be further extended for dynamic interaction topology with focus on the effects of noise on the opinion and self-confidence evolution. Moreover, it will be interesting to investigate opinion and self-confidence evolution over a graph that is not necessarily a star graph but contains a star subgraph.

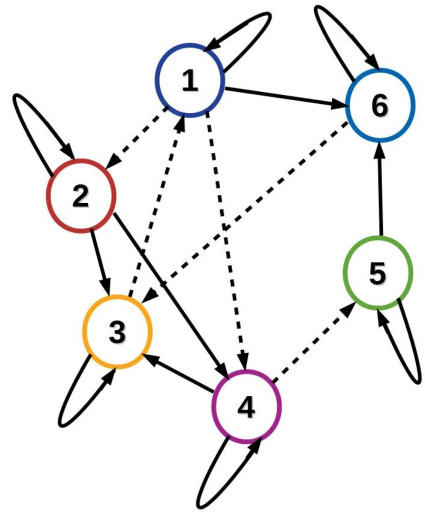

Fig. 1. Signed interaction network topology

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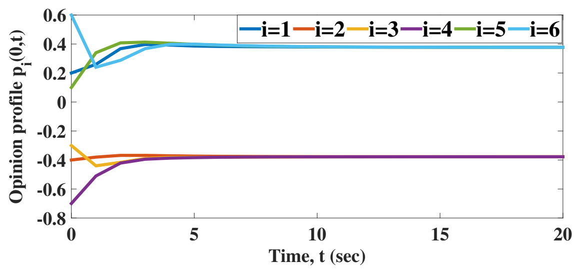

Fig. 2. Polarization of opinions with initial opinion vector $\overline{p}_0 = [0.2\ -0.4\ -0.3\ -0.7\ 0.1\ 0.6]$

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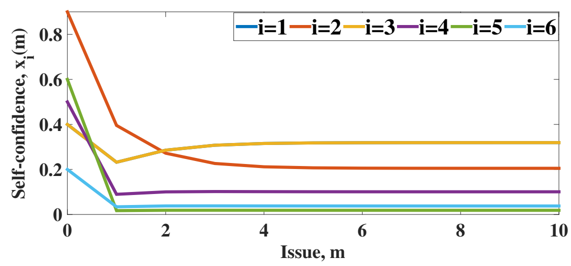

Fig. 3. Evolution of self-confidence of individuals along issue sequences

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