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Phase Analysis for Discrete-time LTI Multivariable Systems ⋆

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Abstract: In contrast to the well-developed gain analysis for multi-input-multi-output (MIMO) linear time-invariant (LTI) systems, the research on the phase analysis does not share the same status. In this paper, we introduce the phase response of a class of discrete-time (DT) LTI multivariable system by exploiting a definition of matrix phases based on the numerical range. Half-sectorial transfer matrices are defined, which can generalize the positive real and negative imaginary systems. A sectored real lemma is obtained to characterize the phase information of a half-sectorial system from a state-space realization. Motivated by finding a phasic counterpart to the small gain theorem, we develop a small phase theorem for the internal stability of closed-loop systems.

Keywords: discrete-time LTI multivariable systems, phase response, small phase theorem, sectored real lemma

1. INTRODUCTION

In classical control theory, it is common to conduct the system analysis in the frequency domain. Graphic representations such as Nyquist plot and Bode diagram offer unique intuitions to control engineers, from which the gain and phase information of a single-input-single-output (SISO) LTI system can be directly obtained. Research results related to the gain and phase of SISO systems are well-established, and often come out in parallel. For example, both the gain and phase margins provide valuable information for the robustness of control systems due to their clear physical meanings.

Their extension to MIMO systems is not trivial. The singular values are widely accepted as the gains of a matrix, thus the gain-based theory of MIMO systems has flourished in the area of robust control. The $\mathcal{H}_\infty$ norm, i.e., the peak magnitude of the frequency response, serves as a performance measure of robust control design. The well-known small gain theorem (Zhou et al., 1996, Theorem 9.1) lays the foundation for the development of $\mathcal{H}_\infty$ control theory. In contrast, the phase analysis has not shared the same extent of popularity. An important reason is that the definition of the matrix phases remains obscure over a long period of time.

That being said, the notions of positive realness (passivity) and negative imaginariness are in fact qualitative phase type characterizations. Roughly speaking, the phase of a positive real system is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and that of a negative imaginary system over positive frequencies lies in $[-\pi, 0]$. The positive real and negative imaginary properties both originate from the observation of physical system behaviour such as electrical circuits and mechanical systems, see for example, Anderson (1967), Kottenstette et al. (2014), Petersen and Lanzon (2010) and Ferrante et al. (2017). Their research provides inspiration for the phase analysis in this paper.

There are also multiple earlier attempts in defining quantitative phase related concepts. One approach is to define the principal phases of a complex matrix from its polar decomposition (Postlethwaite et al., 1981), (Bar-on and Jonckheere, 1990). Another approach is to characterize the phase information through the notion of numerical range (Owens, 1984), (Tits et al., 1999). Recently, Wang et al. (2020) proposed to adopt the canonical angles, introduced by Furtado and Johnson (2001), as the phases of a broad class of complex matrices, which can be derived from a minimax approach over the numerical range. A collection of interesting properties of such phases has been established, hinting that phases defined this way provide a suitable candidate for control system analysis and design. Chen et al. (2019) adopts this definition of matrix phases and conducts the phase analysis for a class of continuous-time LTI MIMO systems, which are sectorial at each frequency.

In this paper, we are interested in performing an analogous development of Chen et al. (2019) for DT LTI multivariable systems. This is particularly important in view of the pervasive role of digital control in modern applications. Some results in Chen et al. (2019) are generalized to the

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frequency-wise semi-sectorial systems in this paper, the concept of which will be explained in the later sections. Furthermore, the systems with poles on the unit circle are also considered. The rest of the paper is organized as follows. In Section 2, phases of a semi-sectorial matrix are introduced. In Section 3, the phase response of frequency-wise semi-sectorial systems is given. In order to analyse the closed-loop stability, a DT version of the small phase theorem is established in Section 4, complementary to the small gain theorem. The half-sectorial systems are defined in Section 5, which exhibit some good properties. A sectored real lemma is given in Section 6 to characterize the phase information of half-sectorial systems in a state space realization. The paper is finally concluded in Section 7. Due to page limit, we omit all the proofs in this paper.

Notation : Most notations used in this paper are standard. Let $\mathbb{R}$ and $\mathbb{C}$ be the set of real and complex numbers, respectively. For a matrix $A \in \mathbb{C}^{n\times n}$, A^T denotes its transpose, A^* denotes its complex conjugate transpose and $\overline{A}$ denotes its entrywise complex conjugate. The sets of eigenvalues and their angles of A are denoted by $\lambda(A)$ and $\angle\lambda(A)$. The Euclidean norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. For a vector $x \in \mathbb{C}^n$, x^* denotes its complex conjugate transpose. The identity matrix is denoted by I. Denote by $\mathcal{R}^{m\times m}$ the set of $m\times m$ real rational proper matrices and let $\mathcal{RH}_\infty^{m\times m} \subset \mathcal{R}^{m\times m}$ contain all its stable elements. In this paper, we will adopt the z-transform in discrete time. Therefore, $\mathcal{RH}_\infty^{m\times m}$ is the set of real rational transfer matrices with poles in the open unit disk. This is different from the definition in the complex function theory. We use $\ell_2(-\infty,\infty)$ to denote the Hilbert space of DT real-valued square-summable time sequences and $\mathcal{RL}_2\,(-\pi,\pi]$ to denote the Hilbert space of DT square-integrable rational frequency functions.

2. PHASES OF A MATRIX

Let a matrix $A \in \mathbb{C}^{n\times n}$. It is widely accepted that the magnitudes of A are given by its n singular values, denoted by

$$\sigma(A) = [\sigma_1(A)\ \cdots\ \sigma_n(A)]\,.$$

Without loss of generality, they are arranged in a non-increasing order such that

$$\overline{\sigma}(A) = \sigma_1(A) \geq \cdots \geq \sigma_n(A) = \underline{\sigma}(A).$$

The singular values have many useful properties, one of which is that the product of the largest singular values of A and B provides an upper bound for the largest singular value of AB (Horn and Johnson, 1991, Theorem 3.3.4), i.e.,

$$\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B). \qquad (1)$$

Note that (1) lays the foundation for the famous small gain theorem (Zhou et al., 1996, Theorem 9.1) in the control theory.

Among the various attempts in defining matrix phases, the canonical angles introduced in Furtado and Johnson (2003) based on numerical range appear to hold the key to generalize the phase results for SISO systems in the same way as the singular values do in the small gain theorem. In this paper, we adopt this definition of phases. A brief review is given below.

Given a matrix $A \in \mathbb{C}^{n\times n}$, let the numerical range of A be defined as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\}.$$

This is a convex and compact subset of the complex plane (Horn and Johnson, 1991, Section 1.2) and contains the spectrum of A. Moreover, the angular numerical range of A is defined to be

$$W'(A) = \{x^*Ax : x \in \mathbb{C}^n, x \neq 0\}.$$

The matrix A is said to be semi-sectorial if the origin is not in the interior of $W(A)$. Furthermore, it is said to be sectorial if the origin is not contained in $W(A)$. Let $\delta(A)$ denotes the field angle of A, i.e., the angle subtended by two supporting lines of $W(A)$ at the origin (Horn and Johnson, 1991, Section 1.1). Then it is clear that $\delta(A) \leq \pi$ for a semi-sectorial matrix A and $\delta(A) < \pi$ for a sectorial matrix A.

The phases can be defined for a nonzero semi-sectorial matrix. A semi-sectorial matrix A may be singular. Let $m = \operatorname{rank} A$. We denote the range of A by $\mathcal{R}(A)$ and the null space of A by $\mathcal{N}(A)$. Then A is unitarily similar to a matrix of the form

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad (2)$$

where $\tilde{A}_{11}$ is nonsingular semi-sectorial with dimension m (Furtado and Johnson, 2003). It can be implied that $\mathcal{N}(A)$ is the orthogonal complement of $\mathcal{R}(A)$. Since the phase of 0 is undefined, we only confine our attention to the nonsingular part. According to Furtado and Johnson (2003), the matrix $\tilde{A}_{11}$ can be written as

$$\tilde{A}_{11} = T^*DT, \qquad (3)$$

where T is nonsingular and D is a unique matrix up to permutation, which is of the form

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \qquad (4)$$

in which

$$D_1 = \begin{bmatrix} e^{j\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\theta_{k_1}} \end{bmatrix},$$

D_2 is a direct sum of $k_2 \geq 0$ copies of the block

$$e^{j\theta}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

with $\theta + \frac{\pi}{2} \geq \theta_1 \geq \cdots \geq \theta_{k_1} \geq \theta - \frac{\pi}{2}, \theta \in (-\pi,\pi]$ and $m = k_1 + 2k_2$. We call $A \in \mathbb{C}^{n\times n}$ rotationally indefinite Hermitian if there exists an α such that $e^{j\alpha}A$ is indefinite Hermitian. If 0 is on the boundary of $W(\tilde{A}_{11})$, either $\tilde{A}_{11}$ is rotationally indefinite Hermitian, in which case there are two choices for θ, or θ is uniquely determined. If $0 \notin W(\tilde{A}_{11})$, then $k_2 = 0$ and there is a continuum choices for θ. In this paper we choose $\theta = \frac{\theta_1+\theta_{k_1}}{2}$ and call it the phase center of A.

Define the phases of A, denoted by

$$\phi(A) = [\phi_1(A)\ \cdots\ \phi_m(A)]\,,$$

to be $\theta_1,\ldots,\theta_{k_1}$, k_2 copies of $\theta + \frac{\pi}{2}$ and $\theta - \frac{\pi}{2}$. Without loss of generality, assume

$$\overline{\phi}(A) = \phi_1(A) \geq \phi_2(A) \geq \cdots \geq \phi_m(A) = \underline{\phi}(A).$$

Note that the uniqueness of the phases also follows from the following minimax and maximin expressions (Horn et al., 1959):

$$\phi_i(A) = \sup_{\substack{S\subset\mathcal{R}(A),\\ \dim S=i}} \inf_{\substack{x\in S, x\neq 0,\\ x^*Ax\neq 0}} \angle x^*Ax = \inf_{\substack{S\subset\mathcal{R}(A),\\ \dim S=m-i+1}} \sup_{\substack{x\in S, x\neq 0,\\ x^*Ax\neq 0}} \angle x^*Ax.$$

Matrix phases and magnitudes are expected to have some similar properties. In view of (1), it would be desirable that there hold

$$\overline{\phi}(AB) \le \overline{\phi}(A) + \overline{\phi}(B),$$
$$\underline{\phi}(AB) \ge \underline{\phi}(A) + \underline{\phi}(B).$$

However, the inequality fails even for positive definite matrices A and B. Nevertheless, a weaker version holds by considering the angles of eigenvalues of AB instead of the phases. Let $A \in \mathbb{C}^{n\times n}$ be a semi-sectorial matrix with phases in $[\theta(A) - \frac{\pi}{2}, \theta(A) + \frac{\pi}{2}]$, where $\theta(A) \in (-\pi, \pi]$ and $B \in \mathbb{C}^{n\times n}$ be a sectorial matrix with phases in $(\theta(B) - \frac{\pi}{2}, \theta(B) + \frac{\pi}{2})$, where $\theta(B) \in (-\pi, \pi]$. Let $\lambda_i(AB), i = 1, \ldots, m$, denote the nonzero eigenvalues of AB with $m = \operatorname{rank} AB$.

Lemma 1. If $\angle\lambda_i(AB), i = 1, \ldots, m$, take values in $(\theta(A)+\theta(B) - \pi, \theta(A) + \theta(B) + \pi]$, then there hold

$$\underline{\phi}(A) + \underline{\phi}(B) \le \angle\lambda_i(AB) \le \overline{\phi}(A) + \overline{\phi}(B). \quad (5)$$

Lemma 1 underpins the development of the small phase theorem in Section 4.

3. PHASE RESPONSE OF DT LTI MIMO SYSTEMS

Let $G(z) \in \mathcal{R}^{m\times m}$ be the transfer function of an LTI system, the frequency response of $G(z)$ is defined to be $G(e^{j\omega})$ for all $\omega \in (-\pi, \pi]$. The frequency response is convenient to be used in engineering. For a SISO system, the frequency response is conventionally represented by the Bode diagram, containing both magnitude and phase plots. For a MIMO system, the magnitude response $\sigma(G(e^{j\omega}))$ is a $\mathbb{R}^m$-valued function. The magnitude Bode diagram is available in the MATLAB computing environment. Now with the definition of the matrix phases from Section 2, we can define the phase response of a system and obtain the phase Bode diagram. Before that, we first define a class of DT MIMO systems, called frequency-wise semi-sectorial systems.

Definition 1. A system $G(z) \in \mathcal{R}^{m\times m}$ is said to be frequency-wise semi-sectorial if the following conditions are satisfied:

(1) $G(z)$ has no poles in $|z| > 1$.
(2) $\delta(G(e^{j\omega})) \le \pi$ for all $\omega \in [0, \pi]$ satisfying $e^{j\omega}$ is not a pole of $G(z)$.
(3) If $e^{j\omega_0}$ is a pole of $G(z)$, then it is at most a simple pole of each entry and the residue matrix $K_0 = \lim_{z\to e^{j\omega_0}}(z - e^{j\omega_0})G(z)$ satisfies $\phi(e^{-j\omega_0}K_0) \in (-\pi, \pi]$.

Furthermore, we can also define the frequency-wise sectorial systems.

Definition 2. A system $G(z) \in \mathcal{RH}_\infty^{m\times m}$ is said to be frequency-wise sectorial if the following conditions are satisfied:

(1) $G(z)$ has no poles in $|z| \ge 1$.
(2) $G(e^{j\omega})$ is nonsingular and $\delta(G(e^{j\omega})) < \pi$ for all $\omega \in [0, \pi]$.

Note that the Definition 1 requires the poles of frequency-wise semi-sectorial system $G(z)$ to be in the closed unit disk, while the Definition 2 requires the poles of frequency-wise sectorial system $G(z)$ to be in the open unit disk.

For a frequency-wise semi-sectorial system $G(z)$, the phase response is defined to be $\phi(G(e^{j\omega}))$ for $\omega \in (-\pi, \pi]$, which is a $\mathbb{R}^m$-valued function. Each element of $\phi(G(e^{j\omega}))$ takes values in $[\theta(G(e^{j\omega})) - \frac{\pi}{2}, \theta(G(e^{j\omega})) + \frac{\pi}{2}]$. The phase center $\theta(G(e^{j\omega}))$ is a real-valued continuous function, which takes values in $(-\pi, \pi]$. When $G(e^{j\omega_0})$ is rotationally indefinite Hermitian, the value of $\theta(G(e^{j\omega_0}))$ is chosen to be $\lim_{\omega\to\omega_0}\gamma(G(e^{j\omega}))$. The Bode diagrams now can be completed by plotting the magnitude response $\sigma(G(e^{j\omega}))$ and the phase response $\phi(G(e^{j\omega}))$.

It is clear that $G(z)$ is conjugate symmetric, i.e., $G(e^{j\omega}) = \overline{G(e^{-j\omega})}$ for all $\omega \in (-\pi, \pi]$. Therefore, $W(G(e^{j\omega}))$ and $W(G(e^{-j\omega}))$ are symmetrical about the real axis. The symmetric property suggests that the positive frequency phase response is sufficient to reflect the phase information of the system within the entire frequency range.

For a frequency-wise semi-sectorial system $G(z)$ with a set of poles $e^{j\omega_i}, \omega_i \in \mathcal{W}$, on the unit circle, denote by

$$\overline{\phi}(G) = \sup_{\omega\in[0,\pi]\setminus\mathcal{W}} \overline{\phi}(G(e^{j\omega})),$$
$$\underline{\phi}(G) = \inf_{\omega\in[0,\pi]\setminus\mathcal{W}} \underline{\phi}(G(e^{j\omega}))$$

the maximum and minimum phase of $G(z)$ over the positive frequency, respectively. Then the $\mathcal{H}_\infty$ phase sector of $G(z)$ is defined to be

$$\Phi_\infty(G) = [\underline{\phi}(G), \overline{\phi}(G)],$$

as a counterpart of the $\mathcal{H}_\infty$ norm, given by

$$\|G\|_\infty = \sup_{\omega\in[0,\pi]} \overline{\sigma}(G(e^{j\omega})).$$

Example 1. The Bode diagrams of the systems

$$G(z) = \begin{bmatrix} \frac{2z^3 + 3z^2 + 2.32z + 0.756}{z^3 + 0.8z^2 + 0.4z + 0.32} & \frac{z + 0.6}{z + 0.8} \\ \frac{z + 0.6}{z + 0.8} & \frac{z + 0.6}{z + 0.8} \end{bmatrix} \quad (6)$$

and

$$H(z) = \begin{bmatrix} \frac{z^2 - z + 0.25}{z^2 + 0.8z + 0.15} & 0 \\ 0 & \frac{z + 0.6}{z - 0.15} \end{bmatrix} \quad (7)$$

are shown in Fig. 1 and Fig. 2 respectively. And the $\mathcal{H}_\infty$ phase sectors can be calculated as $\Phi_\infty(G) = [-0.7006\pi, 0.1250\pi]$, $\Phi_\infty(H) = [-0.2415\pi, 0.5373\pi]$.

The phase response can be used to characterize the behavior of positive real and negative imaginary systems. Using the phase language, $\mathcal{H}_\infty$ phase sector in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ captures the positive real property. And the systems with $\mathcal{H}_\infty$ phase sector in $[-\pi, 0]$ are roughly the same with negative imaginary systems. In fact, the frequency-wise semi-sectorial systems contain a wider class of systems. For example, Fig. 1 and Fig. 2 show that the systems (6)

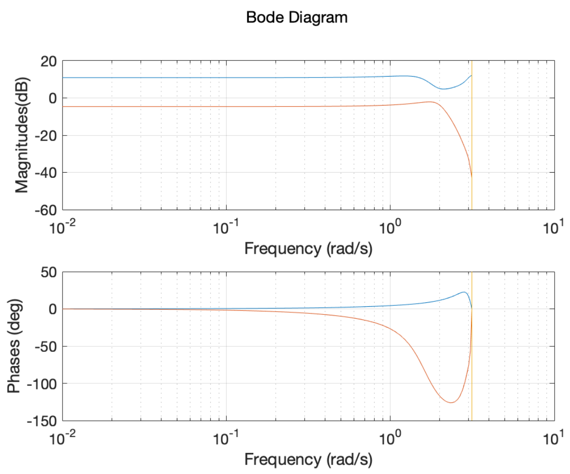

Fig. 1. Bode diagram of the DT MIMO system (6).

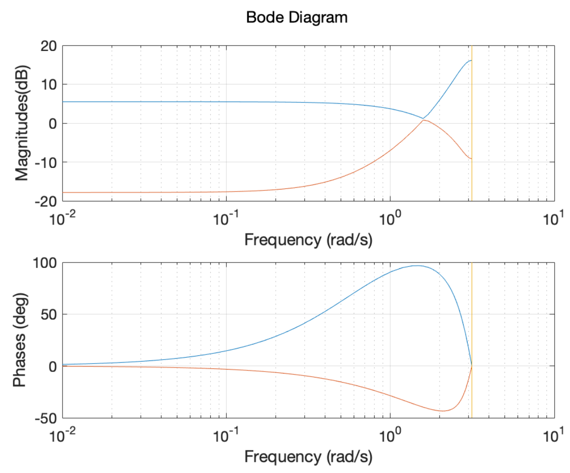

Fig. 2. Bode diagram of the DT MIMO system (7).

and (7) are neither positive real nor negative imaginary; however, they are frequency-wise semi-sectorial.

4. SMALL PHASE THEOREM

Consider a closed-loop system with negative feedback, as shown in Fig. 3.

Definition 3. The feedback interconnection of $G(z), H(z) \in \mathcal{R}^{m\times m}$ is said to be internally stable if the Gang of Four matrix

$$G\#H = \begin{bmatrix} (I+HG)^{-1} & (I+HG)^{-1}H \\ G(I+HG)^{-1} & G(I+HG)^{-1}H \end{bmatrix}$$

is stable, i.e., $G\#H \in \mathcal{RH}_\infty^{2m\times 2m}$.

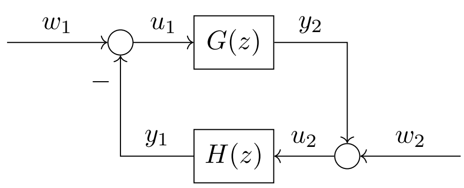

Fig. 3. Negative feedback interconnection of $G(z)$ and $H(z)$.

If both $G(z)$ and $H(z)$ are SISO systems, the internal stability of $G\#K$ can be guaranteed if the critical point $(-1, 0)$ is not encircled by the open-loop Nyquist plot according to the Nyquist criterion. This means that the Nyquist stability condition holds if the open-loop gain is smaller than 1 or the open-loop phase is never equal to $\pm\pi$. These conditions can be generalized to the MIMO systems.

The well-known small gain theorem provides a gain stabilization condition for MIMO systems, which states that for $G, H \in \mathcal{RH}_\infty^{m\times m}$, the closed-loop system $G\#H$ shown in Fig. 3 is stable if $\overline{\sigma}(G(e^{j\omega}))\overline{\sigma}(H(e^{j\omega})) < 1$. As the theorem states, the system gains should be sufficiently small while the phases can be arbitrary. The necessity of the small gain theorem holds if H is only known to be in an uncertainty set $\mathfrak{B}[r(e^{j\omega})] = \{H \in \mathcal{RH}_\infty^{m\times m} : \overline{\sigma}(H(e^{j\omega})) < |r(e^{j\omega})|\}$, where $r(e^{j\omega}) \in \mathcal{RH}_\infty$. Then the feedback system $G\#H$ is stable for all $H \in \mathfrak{B}[r(e^{j\omega})]$ if and only if $\overline{\sigma}(G(e^{j\omega})) \leq \frac{1}{|r(e^{j\omega})|}$ (Vidyasagar, 2011).

In Postlethwaite et al. (1981), the authors give a small phase theorem in an attempt to reduce the conservatism of the small gain theorem by incorporating phase information. However, the derived condition is a mixture of gain and phase information. Now based on the system phases defined in Section 3, we are able to derive the following small phase theorem for MIMO systems, as a complement to the small gain theorem.

Theorem 2. (Small phase theorem) Let G be a frequency-wise semi-sectorial system with a set of poles $e^{j\omega_i}, \omega_i \in \mathcal{W}$, on the unit circle, and H be a frequency-wise sectorial system. Then the feedback system $G\#H$ is stable if

$$\overline{\phi}(G(e^{j\omega})) + \overline{\phi}(H(e^{j\omega})) < \pi, \tag{8}$$

$$\underline{\phi}(G(e^{j\omega})) + \underline{\phi}(H(e^{j\omega})) > -\pi \tag{9}$$

for all $\omega \in [0, \pi]\backslash\mathcal{W}$.

In the small phase theorem, the gains of the systems can be arbitrarily large while the phases should be restricted to a certain range. The theorem can be treated as a generalization of passivity theorem, which states that the negative interconnected feedback system shown in Fig 3 is stable if G is positive real and H is strictly positive real.

Example 2. Consider a negative feedback interconnection as shown in Fig. 3, where $G(z)$ and $H(z)$ are given by (6) and (7) respectively. It can be easily checked from the Bode diagrams that the small gain condition is not satisfied. However, by applying the small phase condition, the closed-loop systems is guaranteed to be stable. The internal stability can also be confirmed by checking the poles of $(I + GH)^{-1}$. Since the poles of $(I + GH)^{-1}$ are all inside the unit circle, the feedback system is stable.

In analogy to the small gain theorem, the small phase theorem is necessary in the following sense. Define

$$\mathfrak{S}[f(e^{j\omega})] = \Big\{H \in \mathcal{RH}_\infty^{m\times m} : \overline{\phi}(H(e^{j\omega})) < \frac{\pi}{2} + \angle f(e^{j\omega}), \\ \underline{\phi}(H(e^{j\omega})) > -\frac{\pi}{2} + \angle f(e^{j\omega}), \omega \in [0, \pi]\Big\},$$

where $f(z) \in \mathcal{RH}_\infty$ has no zeros on or outside the unit circle.

Theorem 3. Suppose that $G \in \mathcal{R}^{m\times m}$ is frequency-wise semi-sectorial with a set of poles $e^{j\omega_i}, \omega_i \in \mathcal{W}$, on the unit circle, then the feedback system $G\#H$ is stable for all $H \in \mathfrak{S}[f(e^{j\omega})]$ if and only if

$$\overline{\phi}(G(e^{j\omega})) \leq \frac{\pi}{2} - \angle f(e^{j\omega}),$$
$$\underline{\phi}(G(e^{j\omega})) \geq -\frac{\pi}{2} - \angle f(e^{j\omega})$$

for all $\omega \in [0, \pi] \backslash \mathcal{W}$.

5. TIME-DOMAIN INTERPRETATION OF HALF-SECTORIAL SYSTEMS

In this section we pay special attention to the following half-sectorial systems.

Definition 4. A frequency-wise sectorial system $G(z)$ is said to be half-sectorial if there exists an open half plane, which contains $W(e^{j\omega})$ for all $\omega \in [0, \pi]$.

A time domain interpretation can be provided for half-sectorial systems. Before giving such an interpretation, we first introduce the Hilbert transform, which is widely used in the field of signal processing (Oppenheim and Schafer, 1999, Chapter 11). It has also been used in control theory to characterize the gain-phase relationship of SISO systems (Anderson and Green, 1988). It turns out to be a powerful tool here to separate the positive frequency property from the negative frequency property and to illustrate the time-domain interpretation of the system.

Denote $\mathcal{F}$ to be the Fourier transform on $\ell_2(-\infty, \infty)$ and $\mathcal{F}^{-1}$ to be the inverse Fourier transform on $\mathcal{RL}_2(-\pi, \pi]$. Recall that the Fourier transform is an isometry between $\ell_2(-\infty, \infty)$ and $\mathcal{RL}_2(-\pi, \pi]$. Any functions in $\mathcal{RL}_2(-\pi, \pi]$ can be uniquely decomposed into the sum of functions in $\mathcal{RL}_2(-\pi, 0)$ and $\mathcal{RL}_2[0, \pi]$ as

$$\mathcal{RL}_2(-\pi, \pi] = \mathcal{RL}_2(-\pi, 0) \oplus \mathcal{RL}_2[0, \pi],$$

which is an orthogonal decomposition. It is clear that the following orthogonal decomposition holds:

$$\ell_2(-\infty, \infty) = \mathcal{F}^{-1}\mathcal{RL}_2(-\pi, 0) \oplus \mathcal{F}^{-1}\mathcal{RL}_2[0, \pi].$$

Denote $\mathcal{F}^{-1}\mathcal{RL}_2[0, \pi]$ by $\mathcal{A}$ and $\mathcal{F}^{-1}\mathcal{RL}_2(-\pi, 0)$ by $\mathcal{A}^\perp$. We will use the commutative graph Fig. 4 to illustrate the complete relationships of the signal spaces.

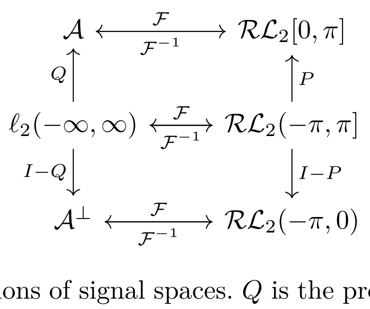

Fig. 4. Relations of signal spaces. Q is the projection onto $\mathcal{A}$ and P is the orthogonal projection onto $\mathcal{RL}_2[0, \pi]$.

For a real signal $u(k)$, its Hilbert transform is defined to be

$$\mathcal{H}u(k) = \sum_{l=-\infty}^{\infty} h(k-l)u(l),$$

where

$$h(k) = \begin{cases} \dfrac{2}{k\pi}, & k \text{ is odd}, \\ 0, & k \text{ is even}. \end{cases}$$

The projection of u onto $\mathcal{A}$ is $u_+ = \frac{1}{2}(u(k) + j\mathcal{H}u(k))$ and the projection of u onto $\mathcal{A}^\perp$ is $u_- = \frac{1}{2}(u(k) - j\mathcal{H}u(k))$. Here u_+ is the complex-valued positive frequency component of u. Note that $\|u_+\|_2 = \|u_-\|_2 = \frac{1}{\sqrt{2}}\|u\|_2$. Denote by $\boldsymbol{G}$: $\ell_2(-\infty, \infty) \to \ell_2(-\infty, \infty)$ the bounded time-domain operator corresponding to $G \in \mathcal{RH}_\infty^{m \times m}$. Define the positive frequency angular numerical range $\boldsymbol{G}$

$$W'_+(\boldsymbol{G}) = \{\langle u_+, \boldsymbol{G}u \rangle : u \in \ell_2(-\infty, \infty), u \neq 0\}.$$

Theorem 4. Given $G \in \mathcal{RH}_\infty^{m \times m}$ and its corresponding time-domain operator $\boldsymbol{G} : \ell_2(-\infty, \infty) \to \ell_2(-\infty, \infty)$, we have

$$\text{cl. } W'_+(\boldsymbol{G}) = \text{cl. Co}\{W'(G(e^{j\omega})) : \omega \in [0, \pi]\},$$

where cl. denotes closure and Co denotes convex hull.

From Theroem 4, it can be inferred that a frequency-wise sectorial system $\boldsymbol{G}$ is half-sectorial if cl. $W'_+(\boldsymbol{G})$ is contained in an open half plane. Furthermore, we have $\overline{\phi}(G) = \sup_{y \in W'_+(\boldsymbol{G})} \angle y$, $\underline{\phi}(G) = \inf_{y \in W'_+(\boldsymbol{G})} \angle y$.

6. SECTORED REAL LEMMA

In gain analysis, how to compute the $\mathcal{H}_\infty$ norm is an important issue. The well-known bounded real lemma gives an efficient way to compute it in state space by establishing an equivalence between frequency domain inequalities and an LMI condition (Zhou et al., 1996, Section 21.3). Likewise, to solve the computation problem of $\mathcal{H}_\infty$ phase sector, we wish to derive an analogus LMI condition. The recently developed generalized Kalman-Yakubovich-Popov (KYP) lemma appears to be the key in finding such conditions (Iwasaki and Hara, 2005), as it can deal with finite frequency intervals. The following sectored real lemma provides a state-space characterization of $\mathcal{H}_\infty$ phase sector for half-sectorial systems.

Theorem 5. (Sectored real lemma) Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of $G(z) \in \mathcal{RH}_\infty^{m \times m}$ and $\alpha, \beta \in (-\frac{3\pi}{2}, \frac{3\pi}{2}]$ with $0 < \beta - \alpha \leq \pi$. Then G is half-sectorial and $\Phi_\infty(G) \subset (\alpha, \beta)$ if and only if there exist Hermitian matrices P_i and $Q_i, i = 1, 2$, satisfying

$$Q_i > 0, \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P_i & jQ_i \\ -jQ_i & -P_i \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + M_i < 0, \quad (10)$$

where

$$M_1 = \begin{bmatrix} 0 & -e^{-j\alpha_1}C^T \\ -e^{j\alpha_1}C & -\left(e^{-j\alpha_1}D^T + e^{j\alpha_1}D\right) \end{bmatrix},$$

and

$$M_2 = \begin{bmatrix} 0 & -e^{-j\beta_1}C^T \\ -e^{j\beta_1}C & -\left(e^{-j\beta_1}D^T + e^{j\beta_1}D\right) \end{bmatrix},$$

with $\alpha_1 = -\frac{\pi}{2} - \alpha$ and $\beta_1 = \frac{\pi}{2} - \beta$.

7. CONCLUSION

In this paper, we introduce the notion of phase response for frequency-wise semi-sectorial system by using a definition of matrix phases based on the numerical range. A sectored real lemma is derived to provide a state-space characterization of $\mathcal{H}_\infty$ phase sector for half-sectorial systems. A small phase theorem is also developed for the analysis of feedback stability, as a counterpart to the small gain theorem.

Further research will be conducted on the $\mathcal{H}_\infty$ phase synthesis of DT LTI MIMO systems. The potential extension of phase analysis to sampled-data systems will also be an interesting direction.

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