**Preprints of the 21st IFAC World Congress (Virtual)**
**Berlin, Germany, July 12-17, 2020**

# Joint Identification and Control in Hybrid Linear Systems

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**Abstract:** We propose a theoretical framework for joint system identification and control on a class of stochastic linear systems. We investigate optimization algorithms for inferring endogenous and environmental parameters from data, part of which are used for control purposes. A number of non-trivial interplays among stability and performance, as well as computational challenges and fundamental limits in identification rate emerge. Our results are validated via simulation example on a quadcopter control problem.

## 1. INTRODUCTION

Modern robotic systems are expected to perform in presence of uncertain and often challenging environments. Nowadays technology enables autonomous agents with sensing and processing information modules that can operate in real-time. It is thus important to investigate data-driven algorithms for general identification and diagnosis purposes. The problem becomes more challenging when parameters inferred in real-time and also use for control purposes that affect the identification process. Unlike mainstream offline algorithms, the objective in real-world scenarios is for the engineers to devise solutions that perform in parallel with the data acquisition and processing. A problem of interest regards a system-environment interaction scenario where only some parameters of either end are known. This would be either because some of the parameters are not directly observable or because e.g. those parameters are at fault state and need to be diagnosed.

*Related Literature* The present work bears resemblance to contributions in the field of system identification. A classic textbook on the general theory is this of (Ljung, 1998) to which we shall frequently refer for standard results. Notable works in the field contribute on problems that involve linear regressions (Goodwin and Mayne, 1987), multivariate linear systems with numerical robustness or stability certificates (Gibson and Ninness, 2005), (Umenberger et al., 2018) as well nonlinear systems (Schön et al., 2011). Another very related research thrust is this that looks at problems of identification for control purposes. We refer to (Gevers, 2005) and (Hof and Schrama, 1995) for further analysis and references. Due to the abundance of works in the field we saw proper to cite (Ninness, 2009) for a particularly thorough review of classic results and recent advances. Finally a third, and more contemporary, line of contributions traverses system identification and machine learning algorithms, (Pillonetto et al., 2014).

*Contribution* We consider a class of linear systems subject to multiple sources of noise (environmental as well as measurement). Maximum likelihood estimation methods are used to infer on unknown parameters, some of which are essential for internal control. We discuss the various theoretical and practical challenges the emerge. We address problems that span the non-stationary cost function, controllability of the surrogate model, as well as the stabilization of closed loop dynamics. To this end, we propose projected gradient descent optimization algorithms with vanishing regularization terms of heavy-ball type. In the practical side, computational challenges put a barrier on a brute force application of the proposed algorithms. We resort to a number of heuristics and approximations in order to formulate an algorithmic implementation of the proposed theory. Our case study regards joint identification and control on a simplified quad-copter model developed in (Bouabdallah and Siegwart, 2007). The objective is to design a data-driven method of simultaneous identification of parameters of interest (exogenous noise and controller feedback gains), in a safe and fast manner. Due to space limitations, proofs of technical results will be presented in the extended version of the work.

## 2. NOTATIONS & DEFINITIONS

By $\|\cdot\|$ we understand the Euclidean norm. The spectral radius of a matrix $F$ is denoted by $\rho(F)$. A positive (semi)-definite matrix $S$ is denoted by $S \succ 0$, ($S \succeq 0$). A function $l(\theta)$ is (strongly) convex at $\theta$ if its Hessian matrix satisfies $\nabla^2 l(\theta) \succ 0$, ($\nabla^2 l(\theta) \succeq 0$). The projection operator that measures the distance of point $x$ to set $\Theta$ is $\mathcal{P}_\Theta = \inf_{\theta\in\Theta} \|x-\theta\|$. We highlight the dependence of $A$, $\mu$, $\Sigma$ on $\theta$ to represent estimates of $A$, $A_\theta$, and so forth. By $A_\theta$ we understand matrix $A$ with the $n_A$ unknown elements evaluated on $\theta$. Next, $I_n$ denotes the $n \times n$ identity matrix.

⋆ This material is based upon work supported in part by the Defense Advanced Research Projects Agency (DARPA) Award HR00111890037 Physics of AI (PAI) Program.

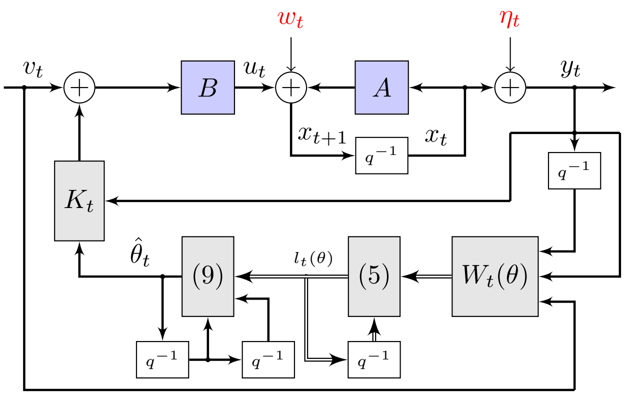


Fig. 1. Block diagram of the proposed scheme for joint identification and control. Double line arrows refer to processing of function signals. Single line arrows refer to vector signals. The $q^{-1}$ modules represent unit time-lag of signal. Alphanumeric modules refer to the enumerated formulas in the text.

We use relation $\propto$ for any variable proportional to a value, e.g. $\varepsilon_t \propto f(t)$ means $\varepsilon_t = R \cdot f(t)$ for constant $R > 0$.

## 3. PROBLEM SETUP

Let $x_t \in \mathbb{R}^n$ be state vector at time $t \geq 0$. The evolution of $x_t$ is governed by linear time invariant stochastic system

$$\begin{aligned} x_{t+1} &= A\, x_t \, + B\, u_t + w_t \\ y_t &= x_t + \xi_t \end{aligned} \tag{1}$$

subject to initial condition $x_0 \in \mathbb{R}^n$. The control signal at time $t$ is $u_t \in \mathbb{R}^q$ and state and control input matrices $A$ and $B$ are of appropriate dimension. Dynamics are subject to external disturbance $w_t \in \mathbb{R}^n \sim \mathcal{N}\big(\mu, \Sigma\big), \ \forall\ t \geq 0$. The observable output $y_t$ is internal state $x_t$ corrupted with measurement noise $\xi_t \in \mathbb{R}^n \sim \mathcal{N}\big(0, \Xi\big), \ \forall\ t \geq 0$, independent of $w_t$.

### *3.1 Identification Vector*

In theory, every parameter of (1) can be considered potentially deficient and it needs to be inferred from data. We refer to (Ljung, 1998) for a general theory of system identification. Our interest in (1) is focused on a problem where only some of system parameters $A$, $\mu$ and $\Sigma$ need to be identified. The objective is to asses these quantities from imposed control inputs and measurements. If $n_A$ elements of $A$, $n_\mu$ elements of $\mu$ and $n_\Sigma$ elements of $\Sigma$ need to be identified we can stack them all in a vector $\theta \in \mathbb{R}^{n_\theta}$ for $n_\theta = n_A + n_\mu + n_\Sigma$ with the true value $\theta_* \in \mathbb{R}^{n_\theta}$. Throughout the paper we assume that $\theta_*$ belongs inside

$$\Theta = \big\{\ \theta \in \mathbb{R}^{n_\theta}\ \ :\ \ \|\theta - \theta_0\| \leq \overline{\vartheta}\ \big\}.$$

Constant $\theta_0 \in \mathbb{R}^{n_\theta}$ and scalar $\overline{\vartheta}$ are assumed known, characterizing the set of acceptable parameter values. We also assume that we can estimate "how far" inside $\Theta$ we should expect to find $\theta_*$, i.e. we know parameter

$$\kappa \in (0,1)\ :\ \|\theta_* - \theta_0\| \leq \kappa\,\overline{\vartheta}. \tag{2}$$

It is remarked that identifiable parameters are restricted in internal system inter-dependencies (i.e. elements of $A$) as well as the effect of exogenous disturbances (i.e. elements of $\mu$ and $\Sigma$). We conclude with a condition on part of $\theta$ that concerns state matrix $A$.

*Assumption 1.* The pair $\big(A_\theta, B\big)$ is controllable, $\forall\ \theta\ \in\ \Theta$.

Controllability of candidate models is a both mild and reasonable requirement. If we accept that the true system ought to be controllable, every surrogate model $(A_\theta, B)$ we consider it is expected to satisfy such a condition.

### *3.2 Output Feedback Control*

The control to be implemented at time $t$ is of the form

$$u_t = -K_t\, y_t + v_t, \tag{3}$$

where $K_t$ is a time-varying feedback gain matrix and $v_t$ is a control signal of appropriate dimension to assist the identification process. Our overall approach is illustrated in Figure 1.

## 4. PRELIMINARY RESULTS

In this section we lay the groundwork of our analysis with a collection of results that characterize the statistics of the output and the form of the metric function for our estimates.

### *4.1 Output Statistics*

We begin with the statistics of $\{y_t\}_{t\geq 0}$ where the linearity of our reference model yields explicit stochastic behavior.

*Proposition 2.* Let the dynamics of (1) with control input (3). Given $y_t$ and $v_t$, the event $\big\{y_{t+1} | y_t, v_t\big\}$ satisfies

$$\big\{y_{t+1}\ \big|\ y_t, v_t\big\}\ \sim\ \mathcal{N}\big(m_t\ ,\ \Pi\big)$$

with $m_t = [(A - B\ K_t)\ \ B] \begin{bmatrix} y_t \\ v_t \end{bmatrix} + \mu$, $\Pi = A\, \Xi\, A^T + \Xi + \Sigma$.

The essence of this result relies on the form of (1). Indeed, $y_t$ yields the aforementioned result on the distribution of measurements that is particularly elegant. It bypasses the need to estimate the state $x_t$ allowing us to work directly with input/output signals.

### *4.2 The t-MLE function*

Evidently, by time $t$, measurement and input data $Y_t$ and $V_t$ are available and one can, in theory, consider the log-likelihood function

$$L_t(\theta) = \log p_\theta\big(Y_t | V_t\big)$$

that is is among the most reliable metrics for statistical inference. As (Ljung, 1998) explains the $\theta$-maximization of $L_t(\theta)$ yields asymptotically unbiased estimators that converge to $\theta_*$. Maximization of $L_t(\theta)$ is equivalent to finding $\theta_t$ such that $\theta_t = \text{argmin}_{\theta\in\Theta}\ l_t(\theta)$ for

$$l_t(\theta) := \frac{1}{t}\Big[ - 2 L_t(\theta) + \text{const.}\Big] \tag{4}$$

where "const." stands for terms independent of $\theta$. The advantage of $l_t(\theta)$ is that it can be computed in a recursive fashion, economizing on computational cost.

*Lemma 3.* The MLE at time $t$ of $\theta_*$, $\theta_t$, satisfies

$$\theta_t = \text{argmin}_{\theta\in\Theta}\ l_t(\theta)$$

where

$$l_t(\theta) = \left(1 - \frac{1}{t}\right) l_{t-1}(\theta) + \frac{1}{t}\, W_t(\theta) \tag{5}$$

with

$$W_t(\theta) = \log\left|\Pi_\theta\right| + \mathrm{Tr}\big(\Pi_\theta^{-1}\, C_\theta\big),$$

and $C_\theta = \big(y_t - m_{t-1}(\theta)\big)\big(y_t - m_{t-1}(\theta)\big)^T$, and $\Pi_\theta = \Pi$, $m_{t-1}(\theta) = m_{t-1}$ are as in Proposition 2.

Note that $l_t(\theta)$ is sufficiently smooth. As we show in Theorem 5, provided that input signals and controls are bounded, then limit function $\lim_t l_t(\theta)$ exists almost surely and the convergence occurs uniformly in $\theta \in \Theta$. MLE estimators $\theta_t$ is that they converge to the true value $\theta_*$ [1] almost surely and that

$$\sqrt{t}\,\big(\theta_t - \theta_*\big) \;\sim\; \mathcal{N}(0, S_*), \qquad t >> 1, \tag{6}$$

for covaraiance matrix $S_*$ is the Fisher information matrix and by the Cramér - Rao bound, the covariance of $\theta_t$ cannot improve below this bound in the sense that

$$\mathbb{E}\big[[\sqrt{t}(\theta_t - \theta_*)][\sqrt{t}(\theta_t - \theta_*)]^T\big] \succeq S_*\ , \quad \forall\, t > 0. \tag{7}$$

While $S^*$ is, in general, not easy to compute, in our analysis we will rely on more conservative estimates based on (7) when necessary. Another remark that is useful to us is that sequence $\{\theta_t\}$ satisfies

$$\theta_t \xrightarrow{a.s.} \Big\{\theta \;\in\; \Theta \;\mid \theta = \mathrm{argmax}\,\mathbb{E}\big[\log p_\theta\big(Y \,|\, V\big)\big]\Big\}$$

for $Y = \lim_t Y_t$ and $V = \lim_t V_t$, respectively. In other words, maximum likelihood estimators will converge to the best possible approximation of the system that is available in the model set $\Theta$, see also §8.3 in (Ljung, 1998).

## 5. AUGMENTED GRADIENT DESCENT

Stationary points of $l_t(\theta)$, $\theta_t$, satisfy $\nabla_\theta l_t(\theta_t) = 0$. Unfortunately, the form of $l_t(\theta)$ is often too complex for a brute force calculation of $\theta_t$. This seems to be the case with (5). Moreover, new data constantly change the shape of $l_t(\theta)$. Consequently, the true value of $\theta_t$ can only be estimated simultaneously with data getting integrated in $l_t(\theta)$. We conclude that non-stationary types of optimization algorithms are a suitable candidate for estimating $\theta_t$. In this work we consider gradient descent schemes of type

$$\hat\theta_{t+1} = \hat\theta_t - \alpha_t\,\nabla_\theta\, l_t(\hat\theta_t) \tag{8}$$

with step $\alpha_t > 0$ to satisfy smallness criteria to be highlighted in the following. In order to meet safety and robustness requirements for identification and control we enhance (8) along three lines:

i. We derive estimates of $\theta_*$ to define a controllable candidate model in accordance to Assumption 1.
ii. While we expect that $l_*(\theta) = \lim_t l_t(\theta)$ will satisfy smoothness and convexity criteria sufficient enough to uniquely identify $\theta_*$, it is not clear that $l_t(\theta)$ are equally elegant and this may steer and perhaps trap estimates $\hat\theta_t$ in undesirable regions of $\Theta$.
iii. It is generally desirable that the overall identification process needs to be fast and efficient. A one-step scheme such as (8) is typically not the best choice.

### 5.1 *Projected Heavy-Ball with Vanishing Regularization*

The iterative scheme we propose to implement is

$$\begin{aligned}\hat\theta_{t+1} = \mathcal{P}_\Theta\big[\hat\theta_t - \alpha_t\nabla_\theta\big(\,l_t(\hat\theta_t) + \frac{\varepsilon_t}{2}\|\hat\theta_t - \theta_0\|^2\big) +\\ + \beta_t\,\big(\hat\theta_t - \hat\theta_{t-1}\big)\big]\end{aligned} \tag{9}$$

for parameters $\alpha_t$, $\varepsilon_t$ and $\beta_t$ to satisfy conditions specified in the convergence analysis section. Scheme (9) augments (8) to meet conditions i. - iii. mentioned above as follows: Projection operator $\mathcal{P}_\Theta$ guarantees controllability of candidate models, quadratic term takes care of possible ill-formed $l_t(\theta)$ for early $t$, as a result of insufficient data. Regularization compensates such case by steering estimates in the heart of $\Theta$. The last term, also known as Heavy Ball, is strengthens the performance of gradient descent, that is very desirable for our setup.

### 5.2 *Convergence Analysis for Simple Convexity*

The first result of the paper discusses weak convergence. Assume that we possess a complete realization of closed loop system (1) with (3), together with $Y$ and $V$ we introduce the corresponding implemented control policies $K := \{K_t\}_{t\geq 0}$. Also, for any finite truncation $Y_t$, $V_t, , K_t$ of the sets we the associated $t$-MLE function $l_t(\theta)$ attains a number of minimizing points in $\Theta$. Let $\Theta_t \subset \Theta$ denote the set of minimizers of $l_t(\theta)$.

*Proposition 4.* Assume that data-sets $Y$, $V, K$, are time-series formed by elements that are uniformly bounded in time. Denote by $L_t := \sup_{\theta\in\Theta}\|\nabla^2_\theta l_t(\theta)\|$ and assume that it is also uniformly bounded with respect to $t$. Then $l_*(\theta) := \lim_{t\to\infty} l_t(\theta)$ is twice differentiable in $\Theta$.

The assumed boundedness on $Y, V$ and $K$ is crucial for our result to hold. The uncertainty in the way input/output data is integrated with time does not allow reasonable speculation on the form of $l_t(\theta)$. Nevertheless, by virtue of (6) we expect that $l_*(\theta)$ to be locally convex in $\Theta$. This extra condition, together with the ones in Proposition 4 seem to suffice for convergence.

*Theorem 5.* Assume that conditions of Proposition 4 hold true. Also let for $t$-MLE function (5) to satisfy

$$\ell_t\, I_{n_\theta} \preceq \nabla^2 l_t(\theta) \preceq L_t\, I_{n_\theta}, \quad \forall\,\theta\in\Theta.$$

If $\ell_t \to \ell_* \geq 0$, then iterations of (9) with

$$\varepsilon_t \propto \begin{cases} t^{-\zeta}, & \ell_t \geq 0 \\ t^{-\zeta} - \delta\,\ell_t, & \ell_t < 0\end{cases} \quad , \quad 0 < \alpha_t < \frac{2}{L_t + \varepsilon_t}$$

that is uniformly bounded away from zero, $\zeta \in (0, 1/2)$, $\delta > 1$ and $\beta_t \equiv 0$ guarantee

$$\hat\theta_t \to \theta_{**} \in \Theta_*, \quad \theta_{**} = \mathrm{argmin}_{\theta\in\Theta}\|\theta - \theta_0\|,$$

where $\Theta_* = \lim_t \Theta_t \subset \Theta$ the set of optimizers of $l_*(\theta)$.

The second round of results assume that limit function $l_*(\theta)$ is strongly convex in $\Theta$.

### 5.3 *Convergence Analysis for Strong Convexity*

Assume that there exists a unique minimizer of $l_*(\theta)$. In such case Heavy Ball approach can presumably improve the performance of our scheme.

*Theorem 6.* Assume that conditions of Theorem 5 hold true and, in addition, the smallest eigenvalue of the Hessian $\ell_t \to \ell_* > 0$. The scheme (9) with

$$\varepsilon_t \propto \begin{cases} 0, & \ell_t \geq 0 \\ -2\,\ell_t, & \ell_t < 0\end{cases}, \quad \beta_t \in \left(0, \frac{\ell_t + \varepsilon_t}{L_t + \varepsilon_t}\right)$$

[1] Whenever such $\theta_*$ uniquely exists.

and $\alpha_t \in \left(\frac{2\beta_t}{\ell_t+\varepsilon_t}, \frac{2}{L_t+\varepsilon_t}\right)$ guarantees convergence to $\Theta_*$.

The assumption of strong convexity in $l_*(\theta)$ is essential to characterize the convergence rate to minimum. The overall performance of the scheme is, however, determined by the rate with which $l_t(\theta)$ will converge to $l_*(\theta)$. Although $\alpha_t$ and $\beta_t$ can be carefully tuned to maximize the performance (see a similar argument in (Polyak, 1987)) the ultimate speed of convergence is expected to be dominated by data integration in $t$-MLE. Condition (6) and bound (7) can provide a somewhat stronger result for large $t$.

When $l_*(\theta)$ is strongly convex in $\Theta$ with its minimal point to be in the interior of $\Theta$, then gradient descent schemes will asymptotically return estimates almost surely in $\Theta$. In other words, for almost every bounded realization $Y$, $V$ and $K$, there is a finite instance $\tilde{t}$ after which if we neglect regularization we have

$$\hat{\theta}_t - \alpha_t \nabla g_t(\hat{\theta}_t) + \beta_t\,(\hat{\theta}_t - \hat{\theta}_{t-1}) \in \Theta, \quad \forall \;\; t > \tilde{t}.$$

Consequently, projection operator becomes obsolete in (9) resulting in simplifications and into more explicit convergence rates. For the exposition of the next result we define, for brevity, $g_t(\theta) := l_t(\theta) + \frac{\varepsilon_t}{2}\|\theta - \theta_0\|^2$ together with its Hessian

$$H_t = \int_0^1 \nabla^2 g_t\big(\omega\,\hat{\theta}_t + (1-\omega)\theta_{\varepsilon_t;t}\big)\, d\omega. \tag{10}$$

*Theorem 7.* Assume that $Y$, $V$ and $K$ amount to a stable system and that $l_*(\theta)$ is strongly convex in $\Theta$. Then for $\alpha_t$ and $\beta_t$ taken from Theorem 6, as $\tilde{t} >> 1$ the following estimate is true with probability $1-\gamma$:

$$\|\hat{\theta}_t - \theta_*\| \leq \overline{q}_{\tilde{t}}^t\,\|\hat{\theta}_{\tilde{t}} - \theta_*\| + 6\frac{n_\theta}{\gamma}\sum_{s=\tilde{t}}^{t-1} \overline{q}_s^t \frac{(1+\kappa)\overline{\vartheta}}{\sqrt{s}}$$

with $\overline{q}_{\tilde{t}}^t = \prod_{s=\tilde{t}}^t \rho(\mathcal{W}_s) < 1$ where $\mathcal{W}_s$ is $2n_\theta \times 2n_\theta$ complex valued matrix

$$\mathcal{W}_s := \begin{bmatrix} 0 & i\sqrt{\beta_s}\, I_{n_\theta} \\ i\sqrt{\beta_s}\, I_{n_\theta} & I_{n_\theta} - \alpha_s\, H_s + \beta_s \end{bmatrix} \tag{11}$$

for $i = \sqrt{-1}$, $H_t$ is the $n_\theta \times n_\theta$ Hessian matrix (10) with $\varepsilon_t \equiv 0$.

The validity of the above estimate relies on (6) and this is the best estimate we can get for now. The manner with which $l_t(\theta) \to l_*(\theta)$ is a very crucial and relevant matter to this work that is part of on-going research. This is a fundamental limitation that characterizes every data-driven process in this or similar works, and it seems, in fact, quite obvious: We cannot do better than the best we can do with the information that is currently available.

## 6. PROBABILISTIC ASYMPTOTIC STABILIZATION

Our discussion so far relies on the hypothesis that closed loop system (1) with control (3) is internally stable. It is only then that $t$-MLE is "well"-behaved and scheme (9) can produce meaningful results. Had we been granted exact knowledge of state matrix $A$, a quite desirable property of (3) would be $\epsilon$-stability. For given margin $\epsilon < 1$, we would like to design $K_t$ such that

$$\rho(A - B\,K_t) \leq \epsilon, \quad \forall\; t \geq 1 \tag{12}$$

where $\rho(\cdot)$ denotes the spectral radius. Evidently, the stabilization problem is that part of $A$ needs to be inferred from data. This is because the output feedback gain would only be based on an estimate that may harm, in our setup destabilize the system. A sufficient condition between $K_t$ and $A_{\hat{\theta}_t}$ is provided below to guarantee internal stability for $t >> 1$.

*Theorem 8.* Assume conditions of Theorem 7 hold true. Fix some $\epsilon \in (0,1)$. If, at every time step $t$, we select $K_t$ such that

$$\rho\big(A_{\hat{\theta}_t} - B\,K_t\big) \leq \frac{\epsilon}{3}$$

then

$$\rho(A - B\,K_t) < \epsilon \tag{13}$$

for all $t$ such that

$$\|\theta_t - \theta_*\| \leq \frac{1}{\sqrt{n_a}}\left(\frac{\epsilon}{3}\right)^n.$$

Moreover, Eq. (13) will hold true with probability $1-\gamma$ for all $t > \tilde{t} >> 1$ such that

$$\overline{q}_{\tilde{t}}^t + \frac{6\,n_\theta}{\gamma}\sum_{s=\tilde{t}}^{t} \overline{q}_s^t \frac{1}{\sqrt{s}} < \frac{\epsilon^n}{3^n\sqrt{n_a}(1+\kappa)\overline{\theta}}.$$

where $\overline{q}_s^t$ are as in Theorem 7.

This result provides a design condition under which data-driven controllers can guarantee, at high level of confidence, closed-loop systems to get eventually internally stabilized. At this point, we feel obligated to raise the concern on the possibility of a circular argument: If non-stationary $t$-MLE function $l_t(\theta)$ can uniformly converge to some $l_*(\theta)$ only if associated dynamics are stable, then the control policy $K_t$ at every instance may not be sufficient to keep the nominal system stable long enough so true value $\theta_*$ is sufficiently approximated. Numerical explorations suggest that if design condition is strong enough, i.e. $\epsilon \leq 1$ is not so close to 1, then identification process is conducted in safety. More investigation is necessary to clarify this vital interplay.

## 7. PRACTICAL IMPLEMENTATION

In order for our framework to be applicable, we see fit to propose guidelines along which an algorithmic process can be developed for potential application scenarios. With the aid of pre-tuned time-stamp indexes we consider a data-driven decision making process that, although fundamentally heuristic, it aims to cover most of the challenges reported so far. As any real-time simulation, $T_{\max} < \infty$ is the maximum number of iterations to be executed. Then we introduce a first checkpoint $T_1$ with the following property: For $t \in [0, T_1]$ regularization term $\frac{1}{2}\varepsilon_t\|\hat{\theta}_t - \theta_0\|^2$ in (9) is always enabled. The idea is that in the initial step of data collection we assume that $t$-MLE is ill-behaved thus we want to leverage the regularization function as the main driving force of our estimates. Then for $[T_1, T_2]$, $T_2 < T_{\max}$ a second checkpoint, we enable regularization conditionally along the lines of Theorem 6. Instead of (10) we evaluate $\nabla^2 l_t(\hat{\theta}_t)$ . The regularization term is triggered if smallest eigenvalue of $\nabla^2 l_t(\hat{\theta}_t)$ falls below a given cut-off $\underline{\lambda} > 0$. When the process reaches iteration $T_2$ the algorithm make a final decision based on the form of the most recent MLE function.

- Smallest eigenvalue of $\nabla^2 l_{T_2}(\hat{\theta}_{T_2})$ is greater than $\underline{\lambda}$. Function $l_*(\theta)$ is likely strongly convex. Regularization term is canceled for $t > T_2$. The algorithm proceeds along the lines of the Theorem 7.

- Smallest eigenvalue of $\nabla^2 l_{T_2}(\hat{\theta}_{T_2})$ is in $(0, \underline{\lambda})$. Function $l_*(\theta)$ is likely convex. The algorithm proceeds along the lines of the Theorem 6.
- $\nabla^2 l_{T_2}(\hat{\theta}_{T_2})$ is negative definite. The identification process has failed and the algorithm aborts.

## 8. IDENTIFICATION AND CONTROL IN QUAD-COPTER DYNAMICS

We apply our method on a case that involves a planar quadrotor, the dynamics of which evolve along the $x$ and $y$ axes and yaw angle, $\phi$. The drone is supposed to hover at a point, $(x_d, y_d, 0)$, with the use of a local stabilizer. Exogenous environment perturbs the quad-copter and it leads the stabilizer at fault state. For the drone recovery, both nature of disturbance and faulted stabilizer require to be diagnosed. The diagnosis is expected to happen online as drone needs to be stabilized immediately. The example's objective is to use input/output data from a motion-capture system and try to simultaneously identify parameters of the faulted controller as well as parameters of the environment. The estimates of the identification vector will be used online to ensure that the copter will maintain an internally locally stable profile. Schematics of the problem is provided in Figure 2.

### *8.1 Derivation of Model*

The dynamics of a quadcopter restricted to move on the horizontal / vertical subspace and turn according to pitch angle are

$$\begin{aligned} m\ddot{\boldsymbol{x}} &= \boldsymbol{u}_1 \sin(\boldsymbol{\phi}) \\ m\ddot{\boldsymbol{y}} &= \boldsymbol{u}_1 \cos(\boldsymbol{\phi}) - m\,g \\ I\ddot{\boldsymbol{\phi}} &= l\boldsymbol{u}_2 \end{aligned}$$

where $m$ is the mass of the drone, $I$ is the moment of inertia of rotors and $l$ the distance of rotors from the center of mass of the quadcopter, and $g \approx 9.81\,m/s^2$ the gravitational acceleration. Let us introduce the state and control vectors

$$\begin{aligned} \mathcal{X} &= \left(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{y}, \dot{\boldsymbol{y}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}\right)^T =: (x_1, x_2, x_3, x_4, x_5, x_6)^T \\ \mathcal{U} &= (\boldsymbol{u}_1, \boldsymbol{u}_2), \end{aligned}$$

respectively so that system reads

$$\frac{d}{dt}\mathcal{X}_t = \boldsymbol{f}(\mathcal{X}_t, \mathcal{U}_t), \tag{14}$$

for appropriate mapping $\boldsymbol{f}$. Next we consider the case that dynamics evolve in a noisy environment where sources of disturbance are modeled via Gaussian noise generators that affect copter's rules of acceleration. This yields the system of nonlinear stochastic differential equations

$$d\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \dfrac{1}{m}\boldsymbol{u}_1(t)\,\sin\big(x_5(t)\big) \\ x_4(t) \\ \dfrac{1}{m}\boldsymbol{u}_1(t)\,\cos\big(x_5(t)\big) - g \\ x_6(t) \\ \dfrac{l}{I}\boldsymbol{u}_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ dw_t^{(2)} \\ 0 \\ dw_t^{(4)} \\ 0 \\ dw_t^{(6)} \end{bmatrix} \tag{15}$$

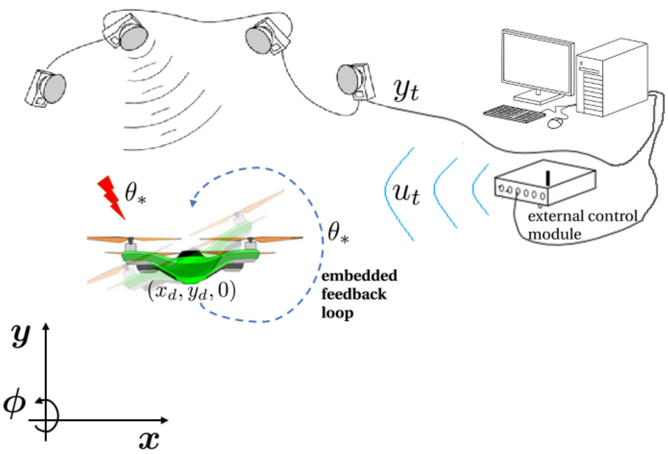


Fig. 2. A case study of joint identification and control. We set values $m = 0.05$, $l = 0.1$ and $I = 1$.

for $dw_t^{(i)}|_{i=1,2,3}$ independent Gaussian measures each of which is supported by $\mathcal{N}(\mu_i, \sigma_i^2)$. In an attitude/altitude control problem the fixed points of type $(x_d, 0, y_d, 0, 0)$ are of interest. Now it is easy to see that a fixed point of (unperturbed) system (15) is

$$\mathcal{X}_* = \left(x_d, 0, y_d, 0, 0, 0\right)^T \quad , \quad \mathcal{U}_* = \left(mg, 0\right)^T$$

First order approximation of (11) around fixed point yields

$$d\,\delta\mathcal{X} = \left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]\delta\mathcal{X}\,dt + \left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{U}}\right]\delta\mathcal{U}\,dt + d\boldsymbol{w}_t$$

where matrices $\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]$, $\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{U}}\right]$ are evaluated at $(\mathcal{X}_*, \mathcal{U}_*)$ and

$$\delta\mathcal{X} := \mathcal{X} - \mathcal{X}_* \quad , \quad \delta\mathcal{U} := \mathcal{U} - \mathcal{U}_*.$$

A nominal (embedded) controller is applied on the drone to guarantee its hovering around $\mathcal{X}_*$. The controller samples system dynamics with sampling period of 1 unit time. Evidently, the controller is leveraging information of the discretization of (11) that reads:

$$\delta\mathcal{X}_{t+1} = \mathcal{A}\,\delta\mathcal{X}_t + \mathcal{B}\,\delta\mathcal{U}_t + w_t \tag{16}$$

with

$$\mathcal{A} = e^{\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]}, \quad \mathcal{B} = \int_{-1}^{0} e^{\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]s} ds \left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{U}}\right]$$

and noise

$$w_t \sim \mathcal{N}\left(\int_{-1}^{0} e^{\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]s}\,ds \begin{bmatrix}\mu_1 \\ \mu_2 \\ \mu_3\end{bmatrix}, \int_{-1}^{0} e^{\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]s} S S^T e^{\left[\frac{\partial \boldsymbol{f}}{\partial \mathcal{X}}\right]^T s}\,ds\right)$$

for $S$ a zero matrix except elements $S_{2,2} = 0.25$, $S_{4,4} = 0.12$ and $S_{6,6} = 0.2$. Now if the embedded controller applies a state feedback law of type $\delta\mathcal{U}_t = -\mathcal{K}\delta\mathcal{X}_t$ with gain matrix

$$\begin{aligned} \mathcal{K}_* &= \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \end{bmatrix} \\ &= \begin{bmatrix} -0.0067 & -0.0280 & 0.0202 & 0.0753 & -0.4588 & -0.4443 \\ 0.1751 & 1.0568 & -0.0013 & -0.0023 & 26.6588 & 40.2878 \end{bmatrix} \end{aligned}$$

stabilizes internally (16) with $\rho(\mathcal{A} - \mathcal{B}\mathcal{K}_*) = 0.6$. The effect of disturbance and/or other mechanical failures lead the controller to the default state

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{25} & 0 \end{bmatrix}$$

Where $k_{25}$ needs to be re-identified. The closed loop system will serve as our starting model to test our identification algorithms. In particular,

$$x_{t+1} = A\,x_t + B\,u_t + w_t \tag{17}$$

with $A = \mathcal{A} - \mathcal{B}\mathcal{K}$ and $B = \mathcal{B}$ and $w_t$ normally distributed as above. The output measurements are obtained through

$$y_t = x_t + \eta_t, \quad \text{for} \quad \eta_t \sim \mathcal{N}(0, I_6).$$

*Identification vector* Together with $k_{25}$ we are also interested in identifying part of the exogenous perturbations that in our example are variance $S_{2,2}$ and expectation $\mu_1$. In conclusion, we have $\theta \in \mathbb{R}^3$ with

$$\theta_1 \leftrightarrow k_{25} \quad \theta_2 \leftrightarrow \mu_1 \quad \theta_3 \leftrightarrow S_{2,2}$$

i.e. $n_a = 1, n_\mu = 1, n_\sigma = 1$. Furthermore we are given $\Theta$ via $\theta_0 = (25, 6, 4)^T$, $\overline{\vartheta} = 3$ and $\kappa = \frac{2.8}{3}$. It is easy to verify that $\Theta$ satisfies Assumption 1.

### 8.2 Simulation Results

In Fig. 3 we present simulation of scheme 9 implemented via Algorithm 1. Following this procedure we run the simulation for 2200 iterations. The first part ends at $T_1 = 300$ iterations and last part begins at $T_2 = 1840$. It appears that function $l_{T_2}(\theta)$ is strictly convex and remains ever after. The cut-off value for the smallest eigenvalue was set at $\underline{\lambda} = 0.01$. Also $\zeta = 0.48$, $C = 3$ and $\delta = 1.2$. We observe that the algorithm works as expected in these three stages. At first the effort was of the scheme was to follow the regularization term. As more data are implemented, the second phase, balances between weakening regularization and more reliable $t$-MLE. Finally at the last step, strong convexity conditions are verified, the algorithm proceeds in full confidence of on the $T_2$-MLE function to stay on the average constant. In the last plot we present the magnitude of feedback controller $K_t$. We also set $\epsilon = 0.6$ and condition $\rho(A_{\hat{\theta}_t} - B\,K_t) < \epsilon$ is implemented by imposing pole placement in the surrogate model $(A_{\hat{\theta}_t}, B)$ at the initial ones. As Figure 3 suggests the output design $\{K_t\}_t$ appears to be finite. This, validates the boundedness assumption on which the convergence results rely. It is remarkable to note that $K_t \to K_*$ as a result of the successful identification of $\theta_*^{(1)} = 26.6588$.

## 9. DISCUSSION

Integration of artificial intelligence and control theory is becoming a necessary step for formal analysis and synthesis of modern autonomous agents that are expected to operate in unpredictable and challenging environments. In this paper, we investigate aspects of this research thrust by inspired by well-known techniques from theory of system identification. Our results, although at initial stage, open up a venue towards a theory of system identification and control for autonomous systems that enforced with for real-time decision-making capabilities from their interaction with the environment. There appears several open issues left as a future work. Perhaps the most notable one is an inherent connection between output feedback control of the unknown system and learning rate of identification vector $\theta_*$. While asymptotic normality of $\theta_t$ is very helpful, it remains an open question for us to state explicit

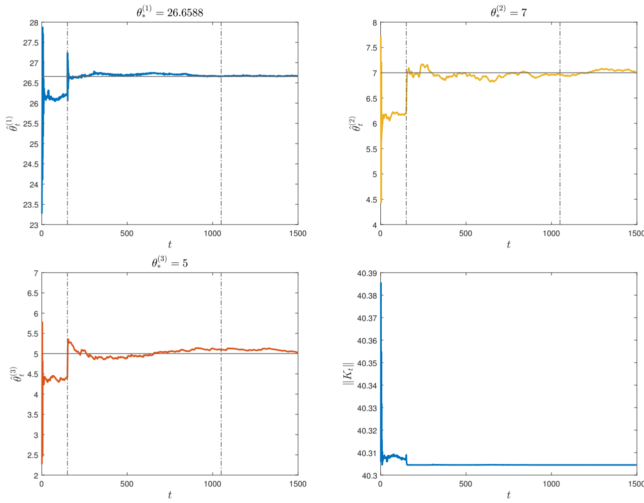


Fig. 3. Simulation for identification of $\theta_*$ (sub-plots 1-3) and norm of control policy $K_t$ vs time (sub-plot 4).

conditions on control policies that guarantee safety certificates. This would require exploration of limit theorems and convergence rates of random processes along the lines of (6).

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