Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Disruption via Grounding and Countermeasures in Discrete-Time Consensus Networks ⋆

Yamin Yan * **Sonja Stüdli** * **Maria M. Seron** * **Richard H. Middleton** *

* *School of Electrical Engineering and Computing, The University of Newcastle, Callaghan, NSW 2308, Australia, (e-mail: yamin.yan, sonja.stuedli, maria.seron, richard.middleton@newcastle.edu.au).*

Abstract: We investigate the disruption of discrete-time consensus problems via grounding. Loosely speaking, grounding a network occurs if the state of one agent no longer responds to inputs from other agents and/or changes its dynamics. Then, the agent becomes a leader or a so-called stubborn agent. The disruption of the agent can be caused by internal faults, safety protocols or externally due to a malicious attack. In this paper we investigate how the grounding affects the eigenratio of expander graph families that usually exhibit good scaling properties with increasing network size. It is shown that the algebraic connectivity and eigenratio of the network decrease due to the grounding causing the performance and scalability of the network to deteriorate, even to the point of losing consensusability. We then present countermeasures to such disruptions in both passive and active manners. Our findings are supported by numerical simulations given within the paper.

Keywords: Grounding, discrete-time systems, scalability, consensusability, consensus performance

1. INTRODUCTION

The multi-agent consensus problem has been a popular research area over the past few decades (Jadbabaie et al., 2003; Olfati-Saber et al., 2007; Knorn et al., 2016). Among the many studies on consensus, fundamental questions such as whether the network can achieve consensus (consensusability) (You and Xie, 2011), how to achieve consensus and consensus on what (Li et al., 2010), are the major topics of interest. In addition, analysis of consensus performance such as convergence rate is of both theoretical and practical importance (Olfati-Saber et al., 2007; Chen et al., 2015).

Efficient distributed networked control requires desirable scalability and consensus performance where scalability means preservation of stability of the entire network as the network size grows large (addition of agents). Consensus networks with bounded nodal degree usually scale poorly, demonstrating scaling fragility. One type of graph family, called expander family (or expanders), scales well with bounded nodal degree (Pinsker, 1973). These graphs play an important role in designing efficient communication networks. It is known that the algebraic connectivity, the second smallest eigenvalue of the graph Laplacian, is crucial in characterizing scalability and consensus performance. The algebraic connectivity of these expander families with bounded nodal degree is bounded away from zero, thus possessing desirable scalability and consensus performance. Early studies on consensus of expanders can be seen in Li et al. (2009).

Recently, Tegling et al. (2019) revealed the scale fragility of expander families towards grounding in a *continuous-time* setting. To the best of our knowledge, the many important properties such as scalability, convergence rate, consensusability of *discrete-time* consensus in expander graph families towards grounding have yet to be studied.

Grounding means that the grounded node is no longer affected by other agents while still influencing its neighbors and by doing so the complete network. Another way of interpreting this behavior is that the grounded node acts as a leader hence turning the whole network from a leaderless architecture to a leader-following one. The terminology stems from its application in power networks where grounding a node means literally connecting the bus to ground forcing the state to be set to zero. Grounding can be caused by internal faults, safety protocols or externally due to an attack. The latter would be viewed as disruption/deception attacks by either disconnecting the input channel or changing the dynamics (Dibaji et al., 2019).

Once a network has been grounded, its dynamics can be described by a grounded Laplacian (Barooah and Hespanha, 2006). The study of the grounded Laplacian has recently received increasing attention. For example, Pirani and Sundaram (2015); Pirani et al. (2017) extensively study the spectral properties of the grounded Laplacian for undirected graphs, while Xia and Cao (2017) considers directed grounded networks. In discrete-time settings, the factor called eigenratio (You and Xie, 2011), i.e., the

⋆ This work was supported by the Australian Research Council through grant DP190102859.

ratio of the second smallest to the largest eigenvalue of the Laplacian, plays a significant role in characterizing consensusability of undirected graphs. In this paper, we find that, for expander graph networks, while the eigenratio of the nongrounded graphs is bounded away from zero with increasing network size, this no longer holds for the grounded graph. For unstable system dynamics, this reduction of the eigenratio impacts on consensusability, which in the worst case can be lost.

The contributions of this paper are three-fold. Firstly, we summarize different ways of grounding a node to turn it into a leader. Secondly, we investigate the properties of scalability, consensus performance and consensusability of expander networks towards grounding. The fragility of grounding is revealed by showing that the grounded eigenratio decreases with network size. Thirdly, we propose countermeasures to mitigate the undesirable fragility over grounding in both passive and active manners.

Network graph definitions

A graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \cdots, N\}$ is the node set and $\mathcal{E}$ is the edge set. A graph is undirected if the edge set consists of unordered pairs $(i, j) \in \mathcal{E}, i, j = 1, \cdots, N$, if there is communication between nodes i and node j. A graph is called simple if there are no loops ($(i, i) \notin \mathcal{E}\ \forall i \in \mathcal{V}$) and each edge is present only once in $\mathcal{E}$. A graph family $\{\mathcal{G}_N\}$ is a sequence of graphs with increasing number of nodes, $N \to \infty$.

If $(i, j) \in \mathcal{E}$, node j is called a neighbor of node i. The degree of a node is equal to the number of neighbors the node has. A graph is called d-regular if the degree of each node is equal to d.

Let $\mathcal{A} = [\alpha_{ij}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$ be the adjacency matrix of $\mathcal{G}$ with $i, j = 1, \cdots, N$, $\alpha_{ii} = 0$, and $\alpha_{ij} = 1 \Leftrightarrow (j, i) \in \mathcal{E}$. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1}^N \alpha_{ij}$ and $l_{ij} = -\alpha_{ij}, i \neq j$. The second smallest eigenvalue of the Laplacian matrix is called the algebraic connectivity of the graph.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first present the type of networks we consider and provide background information for the leaderless (nongrounded) and leader-following (grounded) consensus problems. We then summarize the properties and results that this paper analyzes and establishes for these problems.

2.1 General (leaderless) discrete-time consensus network

We consider a discrete-time multi-agent system where N agents communicate among each other to achieve consensus on their states. We denote the set of agents that communicate with agent i as $\mathcal{N}_i$.

Each agent is governed by a discrete-time dynamic system given in the form of

$$x_i(k + 1) = Ax_i(k) + Bu_i(k), k \in \mathbb{Z}^+,\ i = 1, \ldots, N \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ denote the system state and control input of the i^{th} agent, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$. $\mathbb{Z}^+$ denotes the set of nonnegative integers $\mathbb{Z}^+ = \{0, 1, \cdots\}$. The standard consensus algorithm is adopted as follows,

$$u_i(k) = K \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_j(k) - x_i(k)) \quad (2)$$

where $K \in \mathbb{R}^{1\times n}$ is the control gain matrix.

Throughout this paper, we consider undirected, simple, and connected communication graphs. This means that the adjacency matrix and the Laplacian matrix are symmetric. The adjacency matrix of the communication graph associated with (1), (2) is defined in the previous section. L is the Laplacian matrix of the communication system. The eigenvalues of L are denoted by $\lambda_i \in \mathbb{R},\ i = 1, \cdots, N$ and in an ascending order are written as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. As in You and Xie (2011), λ_2/λ_N, is called the eigenratio of an undirected graph.

The closed-loop system composed of (1) and (2) can be put into the following compact form

$$x(k + 1) = (I_N \otimes A - L \otimes BK)x(k) \quad (3)$$

where $x = [x_1^T, \ldots, x_N^T]^T$ and $\otimes$ denotes the Kronecker product.

Throughout the paper we assume the following.

Assumption 1. For $i = 1, \cdots, N$, the pair (A, B) is controllable and

$$\prod_j |\lambda_j^u(A)| < \zeta^{-1} < \frac{1 + \lambda_2/\lambda_N}{1 - \lambda_2/\lambda_N} \quad (4)$$

for a constant $0 < \zeta < 1$, where $\lambda_j^u(A)$ is an unstable eigenvalue of A and the product in (4) is over all such eigenvalues. If A is stable, then $\zeta = 1$. ∎

It is known that a necessary and sufficient condition for consensus of system (3) is that there exists K such that $A - \lambda_i BK$ is Schur (i.e. all its eigenvalues are inside the open unit circle) for $i = 2, \ldots, N$ (You and Xie, 2011). Under Assumption 1, by designing

$$K = \frac{2}{\lambda_2 + \lambda_N} \frac{B^T PA}{B^T PB} \quad (5)$$

where $P = P^T > 0$ is a solution to the modified algebraic Riccati inequality

$$P - A^T PA + (1 - \zeta^2) \frac{A^T PBB^T PA}{B^T PB} > 0, \quad (6)$$

$A - \lambda_i BK$ will be Schur matrices for $i = 2, \ldots, N$. Then, consensus can be achieved with all the states $x_i(k)$ approaching $x^*(k) = (1/N) \sum_{i=1}^N x_i(k)$.

2.2 Leader-following consensus in grounded networks

Grounding a node of a network turns the node into one that influences other nodes but is not affected in return. In a multi-agent context, this grounded node acts as a leader and converts the whole network from a leaderless architecture to a leader-following one. In other contexts, the grounded node can be interpreted as a "stubborn agent"(Ghaderi and Srikant, 2013). As mentioned previously, the terminology stems from its application in power networks where grounding a node means literally connecting the bus to ground forcing the state, in this case the voltage, to be set to 0. To put this concept in a general framework using networked control language, we consider three different ways to ground a node, say node 1. These different ways will have different influences on network consensus.

A first form of grounding consists in fixing node 1's state $x_1(k)$ at some time k_0 to be either its current state or any constant value $\bar{c}$ in the proper dimension. Then, $x_1(k) = \bar{c}$ for all $k \geq k_0$. The closed-loop system for the remaining nodes can be described as

$$\begin{aligned}\bar{x}(k+1) = &(I_{N-1} \otimes A - \bar{L} \otimes BK)\bar{x}(k) \\ &+(\Lambda \otimes BK)(\mathbf{1}_{N-1} \otimes \bar{c})\end{aligned} \tag{7}$$

where $\bar{L}$ is the grounded Laplacian (Barooah and Hespanha, 2006) obtained by deleting the first row and column of L, Λ is a diagonal matrix with diagonal entries equal to α_{i1} and $\bar{x}$ is obtained from x by removing the states of node 1. If $(I_{N-1} \otimes A - \bar{L} \otimes BK)$ is Schur (which will be discussed in more detail in Section 4), then $\bar{x}$ will approach $(I_{n(N-1)} - (I_{N-1} \otimes A - \bar{L} \otimes BK))^{-1}(\Lambda \otimes BK)(1_{N-1} \otimes \bar{c})$.

A second form of grounding consists in cutting the control channel so that $u_1 = 0$, and optionally change the dynamics of node 1. Then, the first node's dynamics are $x_1(k+1) = \bar{A}x_1(k)$. If $\bar{A} = A$, the consensus trajectory will be the same as that of x_1 if grounding happens after the consensus was achieved; the consensus trajectory will be different from that of x_1 if grounding happens before the consensus is achieved.

A third form of grounding consists in taking control of u_1 such that x_1 is steered towards a deliberately designed trajectory, for example, a certain setpoint c_0. Specifically, a stabilizing controller $u_1 = -K_1x_1 + c_1$ makes the closed-loop dynamics of node 1 $x_1(k+1) = (A - BK_1)x_1(k) + Bc_1$ with $(A - BK_1)$ Schur. The closed-loop system for the remaining nodes will be

$$\begin{aligned}\bar{x}(k+1) = &(I_{N-1} \otimes A - \bar{L} \otimes BK)\bar{x}(k) \\ &+(\Lambda \otimes BK)(\mathbf{1}_{N-1} \otimes x_1(k))\end{aligned} \tag{8}$$

If $(I_{N-1} \otimes A - \bar{L} \otimes BK)$ is Schur, all states of the remaining nodes will approach $c_0 = (I - (A - BK_1))^{-1}Bc_1$.

Note that to analyze the consequences of grounding, a key system matrix to be analyzed is $(I_{N-1} \otimes A - \bar{L} \otimes BK)$. Actually, when letting $x_1 = \mathbf{0}$, the closed-loop system for the remaining nodes will be

$$\bar{x}(k+1) = (I_{N-1} \otimes A - \bar{L} \otimes BK)\bar{x}(k). \tag{9}$$

The performance of the grounded network will be directly related to the grounded Laplacian $\bar{L}$. We denote the eigenvalues of $\bar{L}$ as $\bar{\lambda}_i$ and they are numbered as $0 < \bar{\lambda}_1 \leq \cdots \leq \bar{\lambda}_{N-1}$. The smallest eigenvalue $\bar{\lambda}_1$ is known as grounded eigenvalue (Tegling et al. (2019)). We denote the ratio $\bar{\lambda}_1/\bar{\lambda}_{N-1}$ by grounded eigenratio.

2.3 Problem formulation

In the following sections, we present results and discussions on spectral properties of $\bar{L}$ in regard to scalability, consensus performance, and consensusability. Specifically, we address the following problems.

Scalability of consensus: Suppose the network graph has good scaling properties, that is, it is possible to achieve consensus as the network size grows large (addition of agents). Is scalability preserved when a node is grounded.

Consensus performance: Is the convergence rate to consensus affected by grounding.

Consensusability for unstable systems: How is the consensusability condition of Assumption 1 affected by grounding and can consensusability be lost after grounding.

In addition, what are the possible countermeasures that can be taken to correct or minimise the effect of grounding.

3. EXPANDERS AND SCALABILITY OVER GROUNDED NETWORKS

In the following, we will investigate how the grounded network in (9) compares to the nongrounded network in (3) when using expander families as communication networks in regard to the above three properties.

For consensus networks, poor scaling property is usually observed when the network size grows large with bounded nodal degree. An exception is when the network graph belongs to the expander family, which scale well for growing networks with bounded degree. In this section we first review the concepts of connectivity and the expander family with its desirable scaling property and then present its scaling fragility and performance degradation over grounded networks in the discrete-time setting.

3.1 Connectivity measures and the expander families

In relation to the consensus problem the algebraic connectivity of a graph, the second smallest eigenvalue of its Laplacian matrix λ_2, is of great interest. It is well known that the algebraic connectivity is larger than 0 if and only if the graph is connected, hence we find $\lambda_2 > 0$. As a measure of connectivity the algebraic connectivity is closely related to the isoperimetric constant defined below.

Definition 3.1. (Krebs and Shaheen, 2011) The *isoperimetric constant* or *Cheeger constant* of a graph $\mathcal{G}$ with vertex set $\mathcal{V}$ is defined as

$$h(\mathcal{G}) = \min\left\{\frac{|\partial X|}{|X|}\,\Big|\; X \subset \mathcal{V} \text{ and } |X| \leq \frac{|\mathcal{V}|}{2}\right\} \tag{10}$$

where the boundary ∂X of X is the set of edges with one vertex in X and the other in $\mathcal{V} - X$. ∎

The isoperimetric constant is another measure of connectivity and robustness that captures how many edges need to be removed from a graph to disconnect a somewhat large number of nodes from the rest of the graph. A small number indicates the presence of a bottleneck, which is a subset of nodes connected by only few edges to the remainder of the graph. In turn this indicates a low algebraic connectivity, in fact the Cheeger inequality states (Lountzi, 2015)

$$\frac{h(\mathcal{G})^2}{2d} \leq \lambda_2 \leq 2h(\mathcal{G}) \tag{11}$$

for d-regular graphs. Similarly, the isoperimetric constant allows us to find a lower bound on the eigenratio

$$\frac{\lambda_2}{\lambda_N} \geq \frac{h(\mathcal{G})^2}{4d^2}. \tag{12}$$

Further, we define

Definition 3.2. (Davidoff et al., 2003) Let c be a positive constant. A d-regular graph $\mathcal{G}$ is called a c-expander, if $h(\mathcal{G}) \geq c$. ∎

The above concept becomes important when looking at a graph family $\{\mathcal{G}_N\}$ with $N \to \infty$. Generally, as the number of nodes increases while keeping a constant nodal degree the isoperimetric constant and the algebraic connectivity tend towards zero, which means that with increasing N the graph loses its connectivity and the performance of the consensus algorithm deteriorates. An expander graph family does not exhibit this decrease in connectivity while maintaining a bounded nodal degree.

Definition 3.3. (Krebs and Shaheen, 2011) (Expander family) Let $\{\mathcal{G}_N\}$ be a graph family in which $N \to \infty$. If the sequence $h(\mathcal{G}_N)$ is bounded away from 0, $\{\mathcal{G}_N\}$ is an expander family. ∎

Note that since in an expander family $h(\mathcal{G}_N)$ is bounded away from 0, so is the algebraic connectivity λ_2 and the eigenratio $\frac{\lambda_2}{\lambda_N}$ due to the Cheeger inequality, see (11) and (12).

3.2 Scaling fragility and performance degradation over grounded networks

The advantages of using expander graphs for the consensus algorithm are clear from the previous section. These advantages are not perserved when grounding the network.

First and foremost the scalability of the grounded network is limited. While λ_2 is bounded away from 0 due to the property of the expander family, the larger λ_2 the better the system scales. However, in the grounded network, $\bar{\lambda}_1$ approaches zero as N grows. This means that the scalability is limited.

Secondly the performace (convergence rate) of the grounded network degrades. The convergence rate directly depends on the algebraic connectivity (Olfati-Saber and Murray, 2004; Olfati-Saber et al., 2007). A lower algebraic connectivity indicates a slower convergence of the consensus algorithm. The following result presents the performance degradation by showing $\lambda_2 > \bar{\lambda}_1$ for large enough N.

Lemma 3.1. *Consider a Laplacian matrix L and its grounded Laplacian matrix $\bar{L}$, of an undirected connected d-regular c-expander graph family $\{\mathcal{G}_N\}$, then, there exists a network size $\bar{N}$ such that $\lambda_2 > \bar{\lambda}_1$, for $N > \bar{N}$.* ∎

Proof: By the eigenvalue interlacing theorem (Haemers, 1995), we have $\lambda_2 \geq \bar{\lambda}_1$. Now our purpose is to prove that λ_2 is strictly greater than $\bar{\lambda}_1$ for expander families $\{\mathcal{G}_N\}$. By Corollary 2.3 of Berman and Zhang (2000), we have

$$\lambda_2 \geq d - \sqrt{d^2 - c^2}, \tag{13}$$

which does not relate to the network size N.

If the graph is grounded, from Tegling et al. (2019),

$$\bar{\lambda}_1 \leq \frac{d}{N-1}. \tag{14}$$

Then, $\bar{\lambda}_1 \to 0$ for $N \to \infty$. Therefore, from (13) and (14), there exists a network size $\bar{N} = 1 + \frac{d}{d-\sqrt{d^2-c^2}}$ such that $\lambda_2 > \bar{\lambda}_1$, for $N > \bar{N}$. □

4. LOSS OF CONSENSUSABILITY OVER GROUNDED NETWORKS

It is known that the eigenratio $\frac{\lambda_2}{\lambda_N}$ is an important factor in discrete-time networks. A larger eigenratio corresponds to better consensusability of the communication graph. Grounding causes degradation of consensus and can even disrupt consensusability. We investigate in this section when this occurs. Our argument is based on the observation that the eigenratio of the grounded network is smaller than that of the nongrounded network for large N. We then have the following result.

Lemma 4.1. *Consider a Laplacian matrix L and its grounded Laplacian matrix $\bar{L}$, of an undirected connected d-regular c-expander graph family $\{\mathcal{G}_N\}$, then, there exists a network size $\bar{N}$ such that $\frac{\lambda_2}{\lambda_N} > \frac{\bar{\lambda}_1}{\bar{\lambda}_{N-1}}$, for $N > \bar{N}$.* ∎

Proof: By Theorem 3 of Pirani et al. (2017), we have

$$\bar{\lambda}_{N-1} \geq d. \tag{15}$$

Then $\frac{\bar{\lambda}_{N-1}}{\lambda_N} \geq \frac{1}{2}$ for some $N \geq \bar{N}$ noting $\lambda_N \leq 2d$. By (13) and (14), $\frac{\bar{\lambda}_1}{\lambda_2} \leq \frac{d}{(N-1)(d-\sqrt{d^2-c^2})}$. Thus there exists $\bar{N} = 1 + \frac{2d}{d-\sqrt{d^2-c^2}}$ such that $\frac{\lambda_2}{\lambda_N} > \frac{\bar{\lambda}_1}{\bar{\lambda}_{N-1}}$, for $N > \bar{N}$.

□

As seen from (4) in Assumption 1, the eigenratio characterizes the upper bound of allowable unstable margin for discrete-time system dynamics. Grounding a network leads to a smaller eigenratio, then "less unstable" system dynamics will be allowed. In the case that the unstable system dynamics exceed the consensusability upper bound after grounding, the consensusability of the whole network is lost.

Fig. 1 shows the nongrounded and grounded eigenratios as a function of the network size for an example of expander graph of nodal degree 4. It can be seen that Lemma 4.1 holds. In fact, the nongrounded eigenratio is greater than the grounded eigenratio for all N.

Next, a condition is given for achieving consensus for a grounded network using a controller designed for the nongrounded network.

Lemma 4.2. *Under Assumption 1, the grounded network* (9) *can achieve consensus using the nongrounded controller* (5) *if all the eigenvalues of $\bar{L}$ satisfy,*

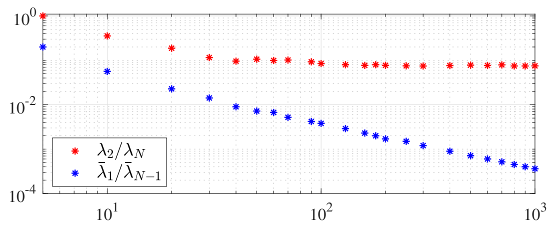

Fig. 1. Eigenratio versus network size.

$$\frac{(1-\zeta)(\lambda_2+\lambda_N)}{2} \leq \bar{\lambda}_i \leq \frac{(1+\zeta)(\lambda_2+\lambda_N)}{2} \quad (16)$$

for $i = 1, \dots, N-1$. ∎

Proof: To prove that K in (5) also stabilizes the matrix $(I_{N-1} \otimes A - \bar{L} \otimes BK)$, it is equivalent to prove $A - \bar{\lambda}_i BK$ Schur for $i = 1, \dots, N-1$.

Direct calculation gives

$$\begin{aligned}&(A-\bar{\lambda}_i BK)^\intercal P(A-\bar{\lambda}_i BK) - P\\ =& -P + A^\intercal PA\\ &- \left(\frac{4\bar{\lambda}_i}{\lambda_2+\lambda_N} - \frac{4\bar{\lambda}_i^2}{(\lambda_2+\lambda_N)^2}\right)\frac{A^\intercal PBB^\intercal PA}{B^\intercal PB}. \quad (17)\end{aligned}$$

If (16) holds, then

$$\frac{4(\lambda_2+\lambda_N)\bar{\lambda}_i - 4\bar{\lambda}_i^2}{(\lambda_2+\lambda_N)^2} \geq 1-\zeta^2,$$

and using (6),

$$-P+A^\intercal PA-\left(\frac{4\bar{\lambda}_i}{\lambda_2+\lambda_N} - \frac{4\bar{\lambda}_i^2}{(\lambda_2+\lambda_N)^2}\right)\frac{A^\intercal PBB^\intercal PA}{B^\intercal PB} < 0$$

Thus, it is proved that $A - \bar{\lambda}_i BK$, $i = 1, \cdots, N-1$, is Schur, which implies that the grounded network (9) can achieve consensus. The proof is completed. □

5. DISCUSSION ON POSSIBLE COUNTERMEASURES

In the previous sections we analyzed the undesirable impacts that grounding has on the network consensus, in particular, on scalability, consensus performance as well as consensusability relating to algebraic connectivity and eigenratio. In this section, we propose both passive and active countermeasures to recover from the effect of grounding.

(1) *Passive countermeasure:* design the controller beforehand to be resilient to grounding. At the stage of controller design, we select K such that $(I_{N-1} \otimes A - \bar{L} \otimes BK)$ is Schur for every $\bar{L}$ resulting from grounding the i^{th} node, for $i = 1, \dots, N$. Then, check that K also stabilizes $(A - \lambda_i BK)$ for $i = 2, \dots, N$. Such a K is expected to exist by noting $\bar{\lambda}_1 < \lambda_2 \leq \cdots \leq \bar{\lambda}_{N-1} \leq \lambda_N$, the interlacing relationship of λ_i and $\bar{\lambda}_i$. This technique may be only practical for networks of small size.

(2) *Active countermeasures:* It is possible to take the following actions:

a. Suppose the grounded network is consensusable with respect to the agent dynamics. Then, redesign the controller after grounding such that $(I_{N-1} \otimes A - \bar{L} \otimes BK)$ is Schur. Then, consensus will be achieved.

b. Suppose the grounded network is unconsensusable with respect to the agent dynamics. If the system dynamics are unstable, and consensusability is lost, then there does not exist a K to stabilize $(I_{N-1} \otimes A - \bar{L} \otimes BK)$. Neither redesign nor predesign the controller for grounding will work in this case. We then propose a possible approach to regain the consensusability by deliberately grounding more nodes to increase the upper bounds for the allowable unstable dynamics. This may sound counter-intuitive, but it will be seen from the following Corollary 5.1 that by grounding more nodes, the grounded eigenvalue increases (or does not decrease), and the spectral radius of the grounded Laplacian decreases (or does not increase), thus leading to a potentially increased eigenratio and larger allowable region for the dynamics to be unstable.

Corollary 5.1. *Let* $\bar{L}^{(m)}$ *be obtained by removing the first* m *rows and* m *columns of* L, $m \in \mathcal{Z}^+$, $0 < m < N$, $\bar{\lambda}_1^{(m)}$ *be the smallest eigenvalue of* $\bar{L}^{(m)}$, $\bar{\lambda}_{N-m}^{(m)}$ *the largest eigenvalue of* $\bar{L}^{(m)}$. *Let* $q \in Z^+$ *be such that* $0 < m < q < N$. *Then,* $\bar{\lambda}_1^{(m)} \leq \bar{\lambda}_1^{(q)}$, $\bar{\lambda}_{N-m}^{(m)} \geq \bar{\lambda}_{N-q}^{(q)}$. ∎

Remark 5.1. The proof of Corollary 5.1 can be completed noticing the eigenvalue interlacing theorem and is thus omitted. Also, simulation studies suggest that the strict inequalities generally hold, that is, $\bar{\lambda}_1^{(m)} < \bar{\lambda}_1^{(q)}$, $\bar{\lambda}_{N-m}^{(m)} > \bar{\lambda}_{N-q}^{(q)}$. ∎

It is always possible to recover consensusability by grounding a sufficiently large number of additional nodes for regular expanders. To see this, note that Assumption 1 restricts the system dynamics to be $\prod_j |\lambda_j^u(A)| < \frac{1+\lambda_2/\lambda_N}{1-\lambda_2/\lambda_N}$. If there is an m such that $\frac{1+\bar{\lambda}_1^{(m)}/\bar{\lambda}_{N-m}^{(m)}}{1-\bar{\lambda}_1^{(m)}/\bar{\lambda}_{N-m}^{(m)}} > \prod_j |\lambda_j^u(A)|$, the network will be consensusable again.

Consider expanders with $d \geq 3$, when altogether we ground $N-2$ nodes in an extreme case, $\frac{\bar{\lambda}^{(m)}}{\bar{\lambda}_N^{(m)}} \geq \frac{1}{2} > \frac{\lambda_2}{\lambda_N}$. Then, $\prod_j |\lambda_j^u(A)| < \frac{1+\bar{\lambda}_1^{(m)}/\bar{\lambda}_{N-m}^{(m)}}{1-\bar{\lambda}_1^{(m)}/\bar{\lambda}_{N-m}^{(m)}}$ which implies that the consensusability is recovered.

Note that the above considers a worst case estimation and in our simulations grounding only a few additional nodes (< 5) recovered consensusability. How many nodes to ground for regaining consensusability will be based on how unstable the system dynamics are compared to the upper bound in terms of the eigenratio. The closer $\prod_j |\lambda_j^u(A)|$ is to $\frac{1+\lambda_2/\lambda_N}{1-\lambda_2/\lambda_N}$, the more nodes will be needed to ground. It is possible that proper selection of the nodes to be grounded can reduce the necessary number. Detailed analysis of these matters is part of our future work.

6. NUMERICAL SIMULATIONS

6.1 Lack of scalability over grounding

Consider a leaderless vehicle platoon where the dynamics of each vehicle is modeled as a discrete-time double integrator,

$$\begin{aligned} x_{i1}(k+1) &= x_{i1}(k) + x_{i2}(k) \\ x_{i2}(k+1) &= x_{i2}(k) + u_i(k),\ i = 1, \dots, N \end{aligned} \quad (18)$$

where in (18), x_{i1} denotes the position from a desired setpoint, x_{i2} the velocity, u_i the control input of the i^{th} vehicle. The system takes the form of (1) with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The cooperative control objective is to have the string of vehicles travel while maintaining a certain formation, e.g., a constant target inter vehicle spacing in this example. The reference trajectory can be described as $x_1^*(k) = x_2^* \cdot k + \delta_i(k)$ with a constant speed $x_2^* = 1$ and constant spacing $\delta_i(k) = 5$.

The communication graph is generated randomly using the algorithm in Kim and Vu (2003) with two cases, $N = 20$ and $N = 100$, both with degree $d = 6$. See for example in Fig. 2 with 20 nodes.

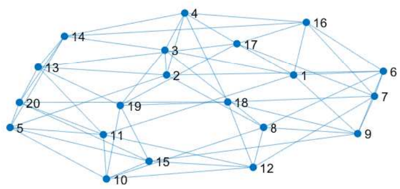

Fig. 2. The random 6-regular graph with $N = 20$.

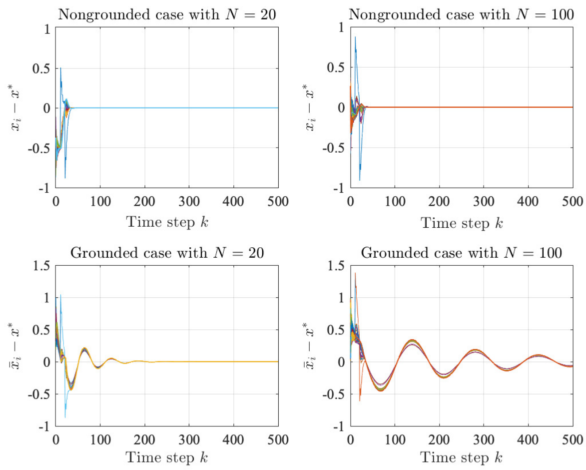

Fig. 3. Profiles of velocity state deviations in nongrounded (leaderless) and grounded case (leader-following) with small and large network, perturbed by a sudden acceleration of one vehicle.

The control gain is chosen as $K = [0.0157\ 0.1826]$. The results of the simulation are shown in Fig. 3 comparing the nongrounded case (leaderless) and the grounded case (leader-following) when relating the state to an independent reference system with dynamics $x^*(k+1) = (A - BK)x^*(k) + Bc_1$, $c_1 = 1$, for two network sizes $N = 20$ and $N = 100$. During the time steps $10 - 20$, one of the vehicles accelerates at a doubled speed. This disturbance is attenuated by the network within a short period of time for the nongrounded network. In contrast, for the grounded networks, the disturbance is attenuated with long settling time for the smaller network and even longer for the larger one. This verifies the scalability limitation over the grounded networks.

6.2 Loss of consensusability over grounding

Consider the consensus network (1) with unstable dynamics $A = \begin{bmatrix} 1.07 & 1 \\ 0 & 1 \end{bmatrix}$, input matrix $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The communication graph is assumed to be the same as in Section 6.1 with $N = 20$. By properly designing the controller, the consensus is achieved before $k = 40$. At $k = 40$, one of the agents is grounded, as illustrated in Fig. 4. Then, the network becomes unconsensusable since the consensusability condition (4) is not satisfied after grounding. More specifically, $\bar{\delta}_A = \frac{1+\bar{\lambda}_1/\bar{\lambda}_{N-1}}{1-\lambda_1/\lambda_{N-1}} = 1.0596 < \prod_j |\lambda_j^u(A)| = 1.07 < \delta_A = \frac{1+\lambda_2/\lambda_N}{1-\lambda_2/\lambda_N} = 1.6935$, which means that the grounded case allows a less unstable A.

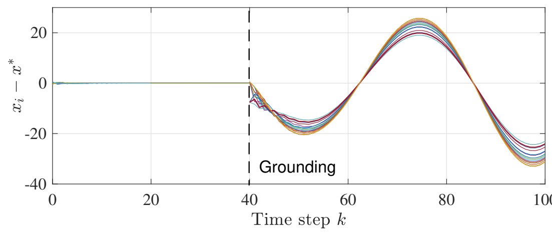

Fig. 4. Profiles of state deviations: loss of consensusability after grounding.

6.3 Countermeasure through grounding more nodes

Consider the same unstable network subject to the same communication graph as in Section 6.2. At $k = 50$, node 1 is grounded, then the same as in Section 6.2, the consensusability is lost and the system states diverge. At $k = 150$, we deliberately ground one more properly chosen node, node 2. Then, the whole network will gradually achieve consensus again since the consensusability condition (4) can be satisfied with $\prod_j |\lambda_j^u(A)| = 1.07 < \frac{1+\bar{\lambda}_1^{(2)}/\bar{\lambda}_{N-2}^{(2)}}{1-\bar{\lambda}_1^{(2)}/\bar{\lambda}_{N-2}^{(2)}} = 1.1266$.

In an additional test, at $k = 150$, instead of grounding only one more node, we deliberately ground two more nodes, nodes 2 and 3. It is observed that the consensus is achieved faster than the previous case, since $\bar{\lambda}_1^{(3)} = 0.7026 > \bar{\lambda}_1^{(2)} = 0.5654$.

These findings are illustrated in Fig. 5.

7. CONCLUSION

In this paper, we have analyzed the scaling fragility of expanders over grounding in a discrete-time context. As in a continuous-time setting, grounded expanders do not scale well. We give a proof that the eigenratio of the grounded network will approach zero, while the one of the nongrounded expander family is bounded away from zero. This shows that the consensus performance is deteriorated

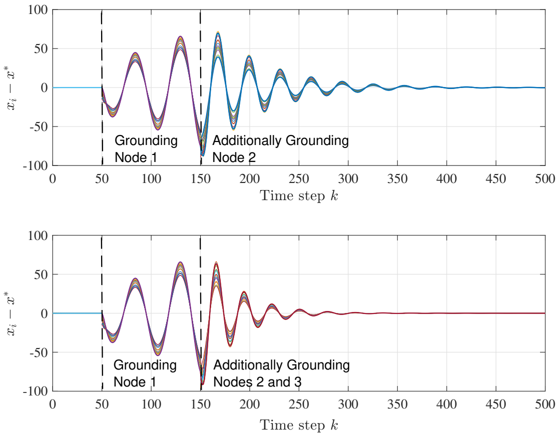

Fig. 5. Above: Loss (at k=50) and regaining (after k=150) of consensusability after grounding; bottom: grounding two more nodes (Nodes 2 and 3) improves the performance (convergence rate) of consensus compared with grounding one more node.

and in extreme cases can even lose consensusability. We give a condition under which the grounded network is able to achieve consensus. In addition, we have discussed possible countermeasures for avoiding the loss of or regaining consensusability. The three methods discussed are to design the initial controller such that the grounded network remains stable, redesign the controller once grounding occurred, or deliberately ground additional nodes. How to detect the grounded node and then either adjust the controller or select additional nodes to ground will be investigated in future work.

REFERENCES

Anderson Jr, W.N. and Morley, T.D. (1985). Eigenvalues of the laplacian of a graph. *Linear and multilinear algebra*, 18(2), 141–145.

Barooah, P. and Hespanha, J.P. (2006). Graph effective resistance and distributed control: Spectral properties and applications. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 3479–3485.

Berman, A. and Zhang, X.D. (2000). Lower bounds for the eigenvalues of laplacian matrices. *Linear Algebra and its Applications*, 316(1-3), 13–20.

Chen, Y., Ho, D.W., Lü, J., and Lin, Z. (2015). Convergence rate for discrete-time multiagent systems with time-varying delays and general coupling coefficients. *IEEE transactions on neural networks and learning systems*, 27(1), 178–189.

Davidoff, G., Sarnak, P., and Valette, A. (2003). *Elementary number theory, group theory and Ramanujan graphs*, volume 55. Cambridge University Press.

Dibaji, S.M., Pirani, M., Flamholz, D.B., Annaswamy, A.M., Johansson, K.H., and Chakrabortty, A. (2019). A systems and control perspective of cps security. *Annual Reviews in Control*.

Ghaderi, J. and Srikant, R. (2013). Opinion dynamics in social networks: A local interaction game with stubborn agents. In *2013 American Control Conference*, 1982–1987. IEEE.

Haemers, W.H. (1995). Interlacing eigenvalues and graphs. *Linear Algebra and its applications*, 226, 593–616.

Jadbabaie, A., Lin, J., and Morse, A.S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on automatic control*, 48(6), 988–1001. doi:10.1109/TAC.2003.812781.

Kim, J.H. and Vu, V.H. (2003). Generating random regular graphs. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, 213–222. ACM.

Knorn, S., Chen, Z., and Middleton, R.H. (2016). Overview: Collective control of multiagent systems. *IEEE Transactions on Control of Network Systems*, 3(4), 334–347.

Krebs, M. and Shaheen, A. (2011). *Expander families and Cayley graphs: a beginner's guide*. Oxford University Press.

Li, T., Fu, M., Xie, L., and Zhang, J.F. (2009). Quantized consensus over expander networks and communication energy minimization. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, 5809–5814.

Li, Z., Duan, Z., Chen, G., and Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1), 213–224.

Lountzi, A. (2015). Expander graphs and explicit construction. *Master's thesis, Uppsala University.*

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215–233.

Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on automatic control*, 49(9), 1520–1533.

Pinsker, M.S. (1973). On the complexity of a concentrator. In *7th International Teletraffic Conference.*

Pirani, M., Shahrivar, E.M., Fidan, B., and Sundaram, S. (2017). Robustness of leader–follower networked dynamical systems. *IEEE Transactions on Control of Network Systems*, 5(4), 1752–1763.

Pirani, M. and Sundaram, S. (2015). On the smallest eigenvalue of grounded laplacian matrices. *IEEE Transactions on Automatic Control*, 61(2), 509–514.

Tegling, E., Middleton, R.H., and Seron, M.M. (2019). Scalability and fragility in bounded-degree consensus networks. In *8th IFAC Workshop on Distributed Estimation and Control in Networked Systems.*

Xia, W. and Cao, M. (2017). Analysis and applications of spectral properties of grounded laplacian matrices for directed networks. *Automatica*, 80, 10–16.

You, K. and Xie, L. (2011). Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *IEEE Transactions on Automatic Control*, 56(10), 2262–2275.