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A modified hybrid Izhikevich neuron: modeling, synchronization, and experiments ⋆

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Abstract: The original Izhikevich neuronal model is described by a nonlinear mathematical model with a static reset map. Due to the fact that the reset is applied instantaneously, it is not easy to implement this model with analog circuits. Consequently, this paper presents a modified Izhikevich neuronal model, in which the static and instantaneous reset is replaced by a dynamic reset, which is physically implementable. Furthermore, the resulting system is modeled as a hybrid system with two discrete modes. Additionally, the occurrence of synchronization in a pair of modified Izhikevich neurons is investigated and a comment on the local stability of the synchronous solution is given. Ultimately, the performance of the modified Izhikevich model is experimentally validated using electronic circuits.

Keywords: Izhikevich model, hybrid system, dynamic reset, neuronal dynamics, electronic neuron

1. INTRODUCTION

There exist several mathematical models aiming to describe the dynamic behavior observed in a neuron cell. These models may be classified according to their complexity or according to the type of behavior(s) they can reproduce. On the other hand, some of these models are biologically inspired, as the Hodking-Huxley model (Hodgkin and Huxley, 1990), whereas others are computational efficient, as the integrate-and-fire models (Jolivet et al., 2004).

Among the existing models, the Hodking-Huxley model is considered as one of the most complete models in the sense that it allows explaining, in a qualitative and quantitative manner, the generation of the action potential in the squid giant axon. However, due the complexity of this model, simplified models which mainly focus on the membrane potential, have been developed. This is the case of the Izhikevich model, (Izhikevich, 2003), which is biologically inspired and computationally efficient. The Izhikevich model is a nonlinear model with a static reset map. When the variable describing the membrane potential reaches a certain upper threshold value the state of the system is reset instantaneously to a fixed value. Although the reset map can be easily implementable in software, its physical implementation with standard analog circuits is not an easy task, as mentioned in (Kormaz et al., 2016).

Note that a physical implementation with electronic circuits may be desirable, specially for the cases of large arrays of coupled neurons, where a physical implementation is much faster than the computer simulations, see e.g. (Wagemakers and Sanjuán, 2013).

This paper presents a modified Izhikevich model, in which the original static reset map is replaced by a dynamic reset. This modification allows a relatively easy physical implementation of the Izhikevich model.

The theory of hybrid systems is used in the modeling and a design procedure for tuning the dynamic coupling is provided. Additionally, the modified Izhikevich neuron is implemented with electronic circuits and a controller is designed to induce synchronous behavior in a pair of neurons. Ultimately, the proposed modified model is validated numerically and experimentally.

The outline of this paper is as follows. First, the original Izhikevich model is briefly described in Section 2. Then, the problem statement is given in Section 3. Next, the design of the proposed dynamic reset and the resulting Izhikevich model are presented in Section 4, and Section 5 presents the design of a simple controller to synchronize a pair of modified Izhikevich models. After that, the physical implementation with electronic circuits of the system and some experiments on synchronization are provided in Section 6. Finally, a discussion and some conclusions are provided in Section 7.

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2. IZHIKEVICH MODEL

The Izhikevich model is described by the following set of equations (Izhikevich, 2003)

$$\dot{x}_1 = 0.04x_1^2 + 5x_1 + 140 - x_2 + I, \tag{1}$$

$$\dot{x}_2 = a(bx_1 - x_2), \tag{2}$$

with the auxiliary after-spike resetting

$$\text{if} \quad x_1 \geq r_1, \text{ then} \begin{cases} x_1 & c \\ x_2 & x_2 + d, \end{cases} \tag{3}$$

where x_1 $\mathbb{R}$ and x_2 $\mathbb{R}$ represent the membrane potential of the neuron and the membrane recovery variable, respectively. The parameter a describes the time scale of x_2 and parameter b describes the sensitivity of x_2 to the subthreshold fluctuations of x_1. On the other hand, I denotes the synaptic currents or injected dc-currents.

Note that when the membrane potential x_1 reaches the upper threshold value r_1, the state of the system is reset according to (3), where $r_1 = 30mV$.

Depending on the value of the parameter values a, b, c and d, different types of neuronal behavior can be reproduced by model (1)-(3). For example, regular spiking, bursting, fast spiking, low-threshold spiking, among others (Izhikevich, 2004; Nobukawa et al., 2015)

3. PROBLEM STATEMENT

The reset function (3) of the Izhikevich model updates *instantaneously* the value of the variables x_1 and x_2 in Eqs. (1)-(2). Although this reset can be easily implemented in numerical simulations, its physical implementation with analog circuits turns out to be complicated, see e.g. (Kormaz et al., 2016) (**?**) (**?**).

Therefore, the problem addressed in this paper consists in finding a suitable replacement for the reset function (3), such that it can be implemented with analog circuits and without affecting the dynamic behavior observed in the original Izhikevich system (1)-(3).

To achieve this, the theory of modeling hybrid systems is used.

4. MODIFIED IZHIKEVICH MODEL

In this section, a modified Izhikevich system is presented. As a first step, the reset function (3) is replaced by the following dynamic reset

$$\text{if} \quad x_1 \geq r_1, \text{ then} \begin{cases} \dot{x}_1 = -\gamma(x_1 - c), \\ \dot{x}_2 = \beta, \\ \dot{z} = 1. \end{cases} \tag{4}$$

Note that in the dynamic reset we have included a time variable z which play the key role of determining the amount of time that the system trajectories spend in the dynamic reset mode (4). Specifically, while $z < r_2$ for certain suitably chosen $r_2 > 0$, the system evolves according to (4) . After that, at $z = r_2$, the system 'jumps back' to Eqs (1)-(2).

Consequently, the modified Izhikevich system can be modeled by the hybrid automaton given in Figure 1. From the automaton it is clear that the system operates in two discrete modes: the mode q_1, in which the dynamics are exactly the dynamics (1)-(2) of the original Izhikevich model and mode q_2, which is the proposed dynamic reset (4).

Initially, the system starts in mode q_1 and evolves in this mode according to (1)-(2). Then, when the membrane potential x_1 reaches the upper-threshold value r_1, the system jumps to mode q_2, i.e. to the dynamic reset.

In mode q_2 besides the equations describing the time evolution of x_1 and x_2, there is an additional variable, namely z, which is a time variable. Then, the values of γ and β in the dynamic equations describing x_1 and x_2 in mode q_2 are chosen such that when $z = r_2$, it holds that $x_1 \approx c + \delta$, $\delta << 0$ and $x_2 = x_2(0) + d$.

Note that, we are considering the practical case where $x_1 \approx c + \delta$ and not the ideal case where $x_1 = c + \delta$. This is due to the fact that the equality only holds when t . However, the system remains in the reset mode q_2 for a finite and short period of time.

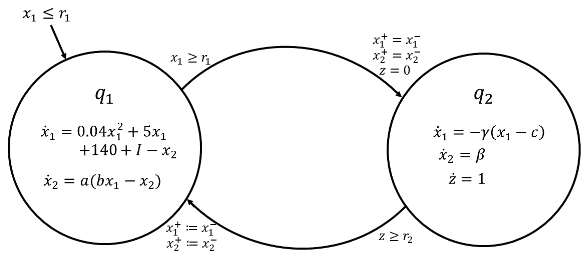

Fig. 1. Hybrid automaton for the modified Izhikevich neuron

4.1 How to determine the values of γ and β in the dynamic reset (4)

This part provides an algorithm for computing the values of γ, β, and r_2 in the dynamic reset (4).

First, from the first equation in (4) it follows that

$$x_1(t) = (x_1(0) - c)e^{-\gamma t} + c, \tag{5}$$

where $x_1(0) = r_1$. Ideally, it is desired that $x_1(t) = c$ but this only occurs in the limit when t . Instead, the practical value of interest of $x_1(t)$ is $x_1(t) = \delta + c$. In other words, we are interested in the value of x_1 when the transient part of (5) satisfies

$$(x_1(0) - c)e^{-\gamma t} = \delta, \tag{6}$$

for certain $\delta << 1$.

Then, the time at which (6) is satisfied is given by

$$t_\delta := -\frac{1}{\gamma} ln\left(\frac{\delta}{x_1(0) - c}\right). \tag{7}$$

Alternatively, the value of γ can be obtained from (6), i.e.

$$\gamma = -\frac{1}{t_\delta} ln\left(\frac{\delta}{x_1(0) - c}\right). \quad (8)$$

Finally, note that the argument of the logarithmic function in (7) is very small and consequently $t_\delta > 0$.

On the other hand, from the second equation in (4) it follows that

$$x_2(t) = \beta \int_0^t dt + x_2(t_0), \quad (9)$$
$$= \beta t + x_2(0). \quad (10)$$

From (3) it follows that the desired value of $x_2(t)$ at time t_δ is

$$x_2(t_\delta) = x_2(0) + d. \quad (11)$$

Hence, from Eq. (10) it follows that (11) is satisfied if

$$\beta = \frac{d}{t_\delta}. \quad (12)$$

By replacing t_δ, given in (7) into (12) yields

$$\beta = -\frac{d\gamma}{ln\left(\frac{\delta}{x_1(0)-c}\right)}. \quad (13)$$

Finally, it should be noted that the time the system spends in mode q_2 is t_δ. Hence, the value of the threshold value r_2, see Figure 1 is

$$r_2 = t_\delta = -\frac{1}{\gamma} ln\left(\frac{\delta}{x_1(0) - c}\right). \quad (14)$$

This procedure is schematically depicted in Figure 2, where the evolution of the system dynamics in mode q_2, i.e. during the dynamic reset, is explained. The left plot shows the time evolution of x_1, whereas the top and bottom plots on the right depict the time evolution of x_2 and z, respectively.

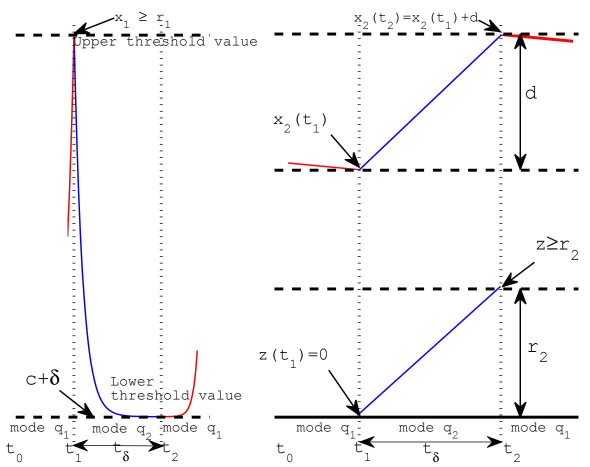

Fig. 2. Time evolution of the system during the dynamic reset (4). The time that the system spends in this mode is denoted by t_δ, see Eq. (7).

Algorithm for tunning the parameters in the dynamic reset:

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(1) Choose a transient time $t_\delta << 1$ and a $\delta << 1$
(2) Find the value of γ from Eq. (8)
(3) Find the value of β from Eq. (13)

4.2 Numerical results

In this part, a numerical comparison between the original Izhikevich model (1)-(3) and the modified model derived here described by Eqs. (1),(2) with dynamic reset (4), is provided. Consequently, Eqs. (1)-(3) and Eqs. (1)-(2) with dynamic reset (4) are numerically integrated by using the following parameter values $a = 0.002$, $b = 0.2$, and $I = 15$. Two cases are considered, namely, the case of regular spiking ($c = -65$ and $d = 6$) and bursting ($c = -50$ and $d = 2$). For the former case the values of the dynamic reset are $t_\delta = 50\mu s$ and $\delta = 0.0043$ and consequently $\gamma = 200000$ and $\beta = 12000$. For the case of bursting the values of the dynamic reset are $t_\delta = 50\mu s$, $\delta = 0.0043$, $\gamma = 1.9662e5$ and $\beta = 40000$.

The obtained results are shown in Figure 3, where the plots on the left correspond to the original Izhikevich model, whereas the plots on the right correspond to the modified Izhikevich model. Clearly, the obtained results are in good agreement.

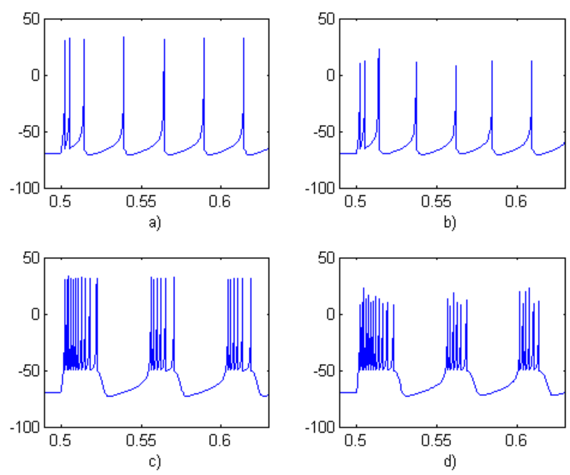

Fig. 3. Panels a) and c): time series obtained with the original Izhikevich model. Panels b) and d): time series for the modified Izhikevich model. x-axis: time. y-axis: x_1.

5. SYNCHRONIZATION OF A PAIR OF MODIFIED IZHIKEVICH MODELS

This section presents the design of a controller for inducing synchronization in a pair of modified Izhikevich models coupled in a master-slave configuration.

The master system is described by

$$\Sigma_M : \begin{cases} \dot{x}_m = F(x_m), \\ y_m = Cx_m, \end{cases} \quad (15)$$

and the slave system

$$\Sigma_S : \begin{cases} \dot{x}_s = F(x_s) + Bu_s, \\ y_s = Cx_s, \end{cases} \quad (16)$$

where $x_i \quad \mathbb{R}^2$ and $y_i \quad \mathbb{R}$, $i = m, s$ are the state and output, of master and slave systems, respectively, and $F(x_i)$ is a nonlinear function containing the intrinsic dynamics of Izhikevich model, and is given by

$$F(x_i) = \begin{bmatrix} 0.04x_{i1}^2 + 5x_{i1} + 140 + I - x_{i2} \\ a\,(bx_{i1} - x_{i2}) \end{bmatrix}, \quad (17)$$

where a, b, and I are positive parameters and $i = m, s$. Furthermore, each system has a dynamic reset (4) and

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [1 \;\, 0]\,. \quad (18)$$

The static feedback control input u_s applied to the slave system is

$$u_s = k(y_m - y_s) = kC(x_m - x_s), \quad (19)$$

where $k \quad \mathbb{R}_+$ is the coupling strength.

The synchronization problem reduces to find a suitable value of coupling strength k, such that the master and slave system synchronize, at least locally, such that the following is satisfied

$$\lim_{x} \,(x_m(t) - x_s(t)) = 0. \quad (20)$$

5.1 *A comment on the local stability of the synchronous solution*

The (global) stability analysis of the synchronous solution (20) in the coupled systems (15)-(16) is not a trivial task, in part because of the hybrid nature of the system. However, under some assumptions it is possible to get some insight into this issue.

Therefore, let assume that the master and slave modified Izhikevich neurons start in mode q_1, see Figure 1, and furthermore, let assume that the systems synchronize before the reset takes place.

Then, by defining the synchronization error as $e = x_m - x_s$, yields the following error dynamics for the coupled systems (15)-(16), with function F defined in (17)

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 5-k & -1 \\ ab & -a \end{bmatrix}}_{A} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + 0.04 \underbrace{\begin{bmatrix} x_{1m}^2 - x_{1s}^2 \\ 0 \end{bmatrix}}_{g(t,e)} \quad (21)$$

Due to the reset, the trajectories of the master and slave systems are *always* bounded and therefore, the term $g(t, e)$ can be considered as a perturbation 'vanishing on e', satisfying

$$g(e) \;\; _2 \leq \gamma \;\; e \;\; _2\,, \quad (22)$$

where γ is a nonnegative constant, and $\;\cdot\;\; _2$ represents the Euclidean norm. Therefore, from the results presented in (Khalil, 2002), it follows that a sufficient condition for the local stability of the error dynamics (21), is that matrix A should be Hurwitz. This condition can be easily verified by looking at the characteristic polynomial of matrix A, which is given by

$$det(\lambda I - A) = \lambda^2 + (a + k - 5)\lambda + a(b + k - 5) = 0. \quad (23)$$

Then, the value of k in the control input (19) should be chosen as follows to ensure that A is Hurwitz

$$k \geq max \;\; 5 - a, 5 - b \;\; . \quad (24)$$

6. EXPERIMENTAL RESULTS

In this section a pair of modified Izhikevich models (1),(2) with dynamic reset (4) is implemented with analog circuits. Furthermore, the circuits are synchronized in a master-slave configuration, as given in (15)-(16), with input and output vectors as given in (18) and using the control input (19) .

The system (1)-(2) with dynamic reset (4) is scaled in amplitude and time, introducing the following change of variables $t = \tau\eta$, $x_1 = \alpha\bar{x}_1$, $x_2 = \theta\bar{x}_2$, $z = \Gamma\bar{z}$. Then, the scaled equations for the system Σ_i in (15)-(16) are given by

$$\begin{aligned} \bar{x}_{i1} &= \left[0.04\alpha\bar{x}_{i1}^2 + 5\bar{x}_{i1} + \frac{1}{\alpha}\,(140 + I - \theta\bar{x}_{i2} + u_i)\right]\eta, \\ \bar{x}_{i2} &= a\left(\frac{\alpha}{\theta}b\bar{x}_{i1} - \bar{x}_{i2}\right)\eta, \end{aligned} \quad (25)$$

where   denotes differentiation with respect to τ, and the dynamic reset is

$$\text{if} \quad \bar{x}_{i1} \geq \frac{r_1}{\alpha}, \text{ then} \begin{cases} \dot{\bar{x}}_{i1} \;=\; -\gamma(\bar{x}_{i1} - \dfrac{c}{\alpha})\eta, \\ \dot{\bar{x}}_{i2} \;=\; \dfrac{1}{\theta}\beta\eta, \\ \dot{\bar{z}} \;\;\;=\; \eta/\Gamma, \end{cases} \quad (26)$$

where $i = m, s$.

Likewise, the static feedback control input u_s, see (19), is scaled as follows

$$u_s = k\alpha(x_{m1} - x_{s1}), \quad (27)$$

and the parameter values are $a = 0.002$, $b = 0.2$, $k = 5$, and the scaling parameters are defined as $\alpha = 10$, $\theta = 100$, $\Gamma = 1$, $\eta = 1000$.

The electronic implementation was constructed using the scaled equations (25)(26) (27) and the corresponding schematic diagram is given in Figure A.1. The constructed experimental setup used to validate the numerical simulations is shown in Figure 4.

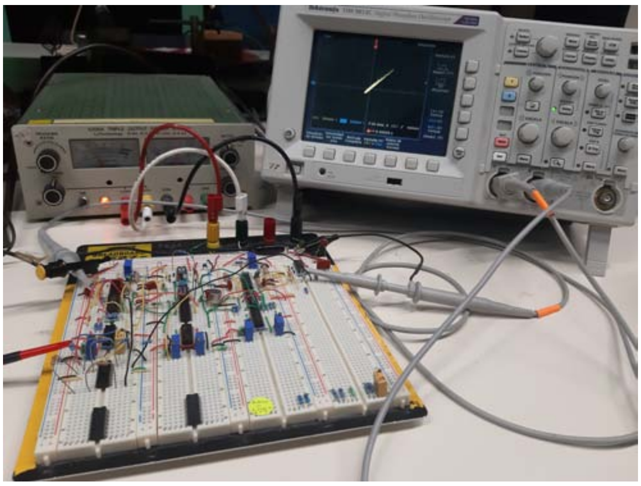

Fig. 4. Experimental setup at CICESE.

The obtained experimental results are shown in Figures 5 to 7. The experimental data have been obtained by using the TDS3012C Digital Oscilloscope from Tektronix.

Figure 5 shows the numerical simulations and the measurements from the circuit, using different parameters.

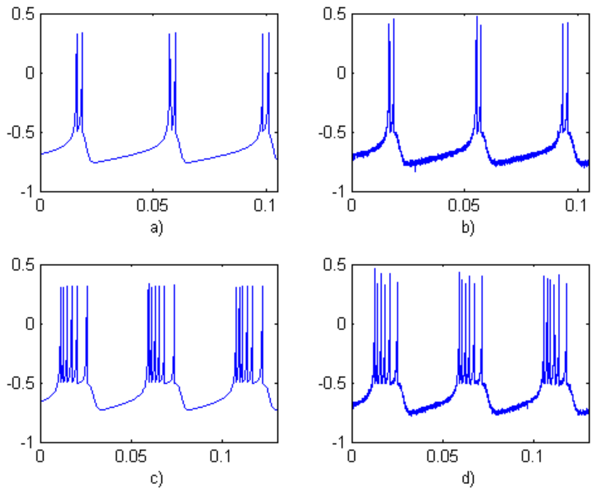

Fig. 5. Comparison between numerical and experimental results with parameters $a = 0.02$, $b = 0.2$, $I = 15$; a) Numerical results for $c = -50$, $d = 6$; b) Circuit measurements for $c = -50$, $d = 6$; c) Numerical results for $c = -50$, $d = 2$; d) Circuit measurements for $c = -50$, $d = 2$. x-axis: time. y-axis: x_1

Figure 6 shows the synchronization of the master and slave Izhikevich electronic neurons.

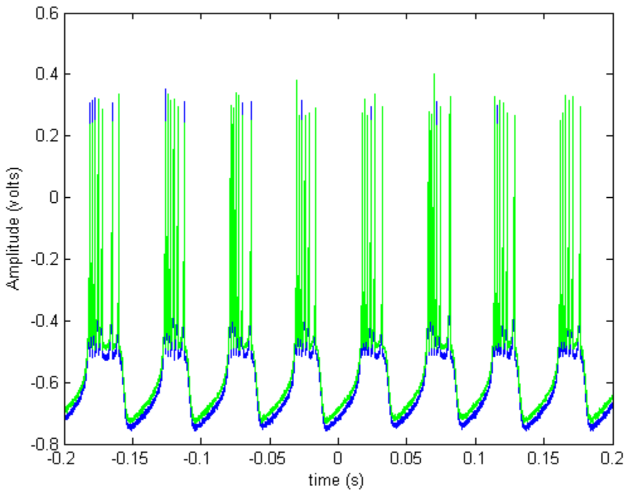

Fig. 6. Synchronized bursting behavior in the coupled Izhikevich electronics neurons

The phase plane projection of the outputs of the systems is shown in Figure 7. These experimental results confirm that the modified Izhikevich model has been successfully implemented with analog circuits.

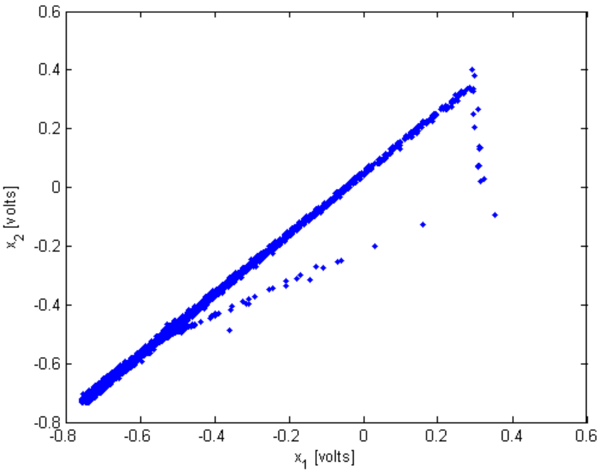

Fig. 7. Projection of the experimental solution of the system onto the (x_1,x_2) plane

7. DISCUSSION AND CONCLUSIONS

A modified Izhikevich model has been presented, in which the static reset function of the original Izhikevich model has been replaced by an ad hoc designed dynamic reset. The resulting system has been modelled as a hybrid system with two discrete modes. It has been demonstrated that the modified Izhikevich model presented here can be physically implemented with analog circuits.

It is our hope that the derived model and circuit will be useful in the study of neural dynamics, together with other analog implementations of neuronal models, see e.g. (Vromen et al., 2016; Savino and Formigli, 2009)

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Appendix A. ELECTRONIC CIRCUIT

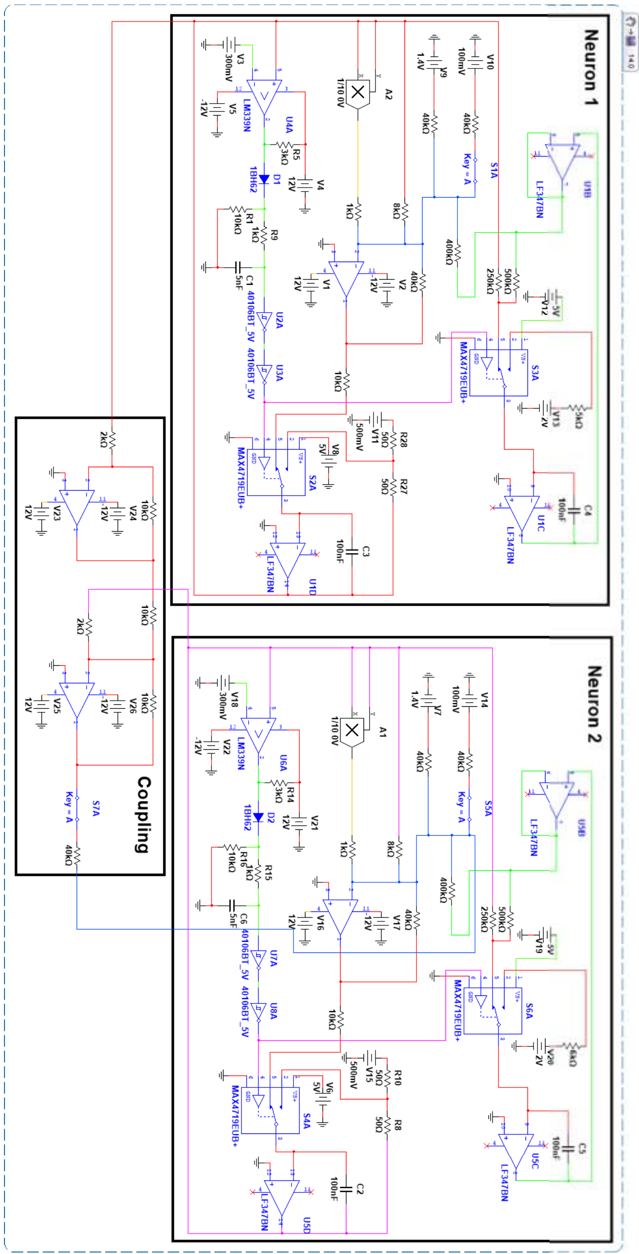

Fig. A.1. Schematic implementation of the modified Izhikevich model with analog circuits.