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Stabilization of Equilibrium for Underactuated Mechanical Systems Without Potential Energy ⋆

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Abstract: This paper investigates the equilibrium stabilization problem for a class of underactuated mechanical systems which do not possess potential energy. The dynamics of the system is established under the framework of Rimannian geometry, and differential geometric methods are employed in the design of stabilization controller. The main novelty of this paper is that we stabilize the equilibrium by constructing an artificial potential for the closed-loop system, which is related to the designed configuration feedback. Once the artificial potential satisfy certain requirements with respect to the equilibrium, the stability of the system can be guaranteed. Furthermore, by incorporating dissipative feedback into the control strategy, we successfully obtain the exponential stability of the equilibrium.

Keywords: Stabilization, Underactuated Mechanical Systems, Differential Geometric Methods, Artificial Potential, Exponential Stability

1. INTRODUCTION

Mechanical control systems have attracted much attention due to their extensive application in autonomous vehicles, aircrafts and so on. Generally, the stabilization of an equilibrium for mechanical systems is one of the most challenging and interesting problems. A wide range of control techniques have been employed to tackle the stabilization problem. For example, backstepping (Dixon et al. (2000); Farrell et al. (2009); Mazenc et al. (2019)), feedback linearization (Banaszuk and Hauser (1996); Wang et al. (2007); Reis et al. (2018)), and transverse function (Morin and Samso (2001, 2003); Pazderski (2017)) have been used in the stabilization for nonlinear systems.

A number of works solve the stabilization problem from the perspective of energy shaping. Takegaki and Arimoto (1981) was a pioneering work presenting a linear state feedback to shape the potential of the system. Afterwards, van der Schaft (1986) developed this method and originally used it into the stabilization for underactuated Hamiltonian systems. Stability of underwater vehicles was studied in Leonard (1997), which employed symmetry breaking potentials to shape the energy of closed-loop system. The controlled Lagrangians (CL) approach for stabilization was presented in Bloch et al. (2001), which augmented relevant constructions to include symmetry-breaking modifications to the potential energy of mechanical systems. A recent application of CL method for wheeled mobile robots was shown in Tayefi and Geng (2018). In addition, the passivity-based control (PBC) (Ortega et al. (2001)) could also be used in the stabilization of underactuated mechanical systems. Ortega et al. (2002) and Gómez-Estern et al. (2001) presented a new PBC design methodology known as interconnection and damping assignment (IDA), which made the closed-loop energy related to the choice of desired subsystems interconnections and damping.

Apart from equilibrium, a variety of researches also pay attention to the stabilization of relative equilibrium (Jalnapurkar and Marsden (2000)). Aiming at underactuated systems on Riemannian manifolds, Bullo (2000) presented a control law to stabilize their relative equilibria. Justh and Krishnaprasad (2004) investigates all possible relative equilibria for planar vehicles under arbitrary group invariant curvature controls. A control strategy of task-induced symmetry and reduction for systems on Lie group is proposed in Kallem et al. (2010), which can be applied to relative equilibrium stabilization. Other examples can be found in Wu and Geng (2010); Niu and Geng (2019).

Aforementioned literatures are all excellent works in nonlinear system stabilization, but most of them focus on either fully actuated systems or relative equilibrium stability. Actually, how to stabilize an equilibrium of underactuated systems is a more challenging problem. On one hand, compared to fully actuated systems, the number of independent control inputs for underactuated systems is less than their degrees of freedom. In other words, for certain directions lacking input channels, we cannot straightly stabilize them only by simple feedback like fully actuated systems. Thus, novel methods should be brought up for underactuated systems stabilization. On the other hand, the stabilization of relative equilibrium is relatively convenient to deal with in the sense that the states need to be stabilized is fairly less. By definition, when system

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converges to a relative equilibrium, it is only required that the velocity reaches a constant. In contrast, for equilibrium stabilization, the configuration variables should converge to constants, and meantime the velocity should become zero. Thus, it is tougher to make the equilibrium stable.

In this paper, we consider the stabilization problem for underactuated mechanical systems. The dynamics of systems is established under the framework of Rimannian manifold, so that differential geometric methods are employed in the design of stabilization controller. Basically, due to the existence of orientation, the actual configuration space of a mechanical control system is not a linear space but a nonlinear manifold, loosely speaking, a curved space. Only in exceptional circumstances can the configuration be described by vectors in the Euclidean space (Bullo and Lewis (2005)). The most significant advantage of the manifold description for mechanical systems is globalness and uniqueness. That is to say, it does not rely on local coordinates. Of course, although with such great advantages, the control design and analysis on a manifold is much more sophisticated and evolving than that in the Euclidean space. This is because linear operations are not be applicable on manifolds anymore, which brings more challenges to our design.

The control strategy we proposed here is targeted for a particular type of underactuated mechanical systems which do not possess potential energy. In fact, such a kind of system is quite common. For example, the vehicle or vessel moving in a plane does not have gravitational potential energy. In addition, regarding submarine whose center of buoyancy coincides with center of mass, it is not endowed with potential energy, either. The main idea of stabilizing the equilibrium is to construct an artificial potential by configuration feedback, which is also known as potential shaping in several literatures. If the artificial potential satisfies certain conditions with respect to the equilibrium, then the stability of the system can be guaranteed.

The contributions of this paper lie on the following three aspects. Firstly, motivated by Bullo (2000), we design a configuration feedback control for the none-potential open loop system, such that the closed-loop system becomes endowed with an artificial potential related to the given feedback. Secondly, based on the designed potential, we propose the condition about how to make the equilibrium of the underactuated system Lyapunov stable. Finally, under the assumption of linear controllability, we obtain the exponential stability of the equilibrium by introducing a dissipative feedback into the control strategy. It should be emphasized that the designed controller can stabilize all of the variables of interest, that is, all of the configuration variables and velocity variables. Therefore, the results in this paper focus on full-state stabilization instead of output stabilization.

The organization of this paper is outlined as follows. In Section 2, we introduce the preliminaries of differential manifolds and mechanical control systems. Main results relevant to artificial potential, Lyapunov stability and exponential stability are presented in Section 3. We conclude this paper and provide discussion about future work in Section 4.

2. PRELIMINARIES

Preliminaries about Riemannian manifold and mechanical control system are provided before the main results. We assume that the audience of this paper has general knowledge about differential manifolds. For more information of geometric control on Riemannian manifolds, please refer to Bullo and Lewis (2005).

2.1 Notions and definitions on manifolds

Let Q denote a smooth manifold and TQ is the tangent bundle of Q. Let q be a point on Q, and v_q be a point on T_qQ, which is the tangent space at q. We use $I \subset \mathbb{R}$ to represent a real interval, and $\gamma : I \to Q$ is a curve on Q. On the manifold, $f(Q) \in \mathbb{R}$ and $X_q \in T_qQ$ represent the smooth functions and vector fields respectively, and more general (r, s) tensor fields are defined as real-valued multi-linear maps on $(T_qQ^*)^r \times (T_qQ)^s$, where T_qQ^* is the cotangent space at q. We employ $C(Q)$ and $\mathfrak{X}(Q)$ to denote the set of functions and vector fields on Q. Lie derivatives of a function f and Lie bracket between two vector fields X and Y are denoted by $\mathscr{L}_X f$ and $\mathscr{L}_X Y$, where $f \in C(Q)$ and $X, Y \in \mathfrak{X}(Q)$.

A Riemannian metric on the manifold Q is a $(0, 2)$ symmetric and positive-definite tensor field $\mathbb{G}_q$, which is a real valued map associating to each $q \in Q$ an inner product $\langle\cdot,\cdot\rangle_q$ on T_qQ. A manifold endowed with a Riemannian metric is named a Riemannian manifold. An affine connection on Q is a smooth map which assigns to a pair of vector fields X, Y a new vector field $\nabla_X Y$ such that

$$\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z,$$
$$\nabla_X (fY + Z) = (\mathscr{L}_X f)Y + f\nabla_X Y = \nabla_X Z,$$

where $f \in C(Q)$ and $X, Y, Z \in \mathfrak{X}(Q)$. We also call $\nabla_X Y$ the covariant derivative of Y with respect to X. For a Riemannian metric $\mathbb{G}_q$ on Q, there exists a unique affine connection named Levi-Civita connection, such that for all $X, Y, Z \in \mathfrak{X}(Q)$ there holds

$$\mathscr{L}_X Y = \nabla_X Y - \nabla_Y X,$$
$$\mathscr{L}_X \langle Y, Z\rangle_q = \langle\nabla_X Y, Z\rangle_q + \langle Y, \nabla_X Z\rangle_q.$$

In the following, we introduce the covariant derivative along a curve. Consider a smooth curve $\gamma(t) \in Q$, and a vector field $v(t) \in T_{\gamma(t)}Q$ which is defined along γ. Let a vector field $X \in \mathfrak{X}(Q)$ satisfy $X(\gamma(t)) = v(t)$, then the covariant derivative of v along γ is defined as

$$\nabla_{\dot\gamma(t)} v(t) = \nabla_{\dot\gamma(t)} X(q)|_{q=\gamma(t)}.$$

In the subsequent sections, for the sake of convenience, we use the notation $\frac{\mathrm{D}}{\mathrm{d}t}$ to represent the covariant derivative along a curve $\nabla_{\dot\gamma(t)}$.

We conclude this section with the first and second variation of a function. Give a function $f \in C(Q)$, its gradient $\mathrm{grad} f$ is a vector field implicitly defined as

$$\mathscr{L}_X f = \langle \mathrm{grad} f, X\rangle_q. \qquad (1)$$

According to this definition, gradient $\mathrm{grad} f$ can be explicitly expressed as $\mathrm{grad} f = \mathbb{G}^\sharp_q(\mathrm{d}f)$, where $\mathbb{G}^\sharp_q$ is the sharp map, and $\mathrm{d}f$ is the differential of f. The Hessian of f denoted by Hessf is a $(0, 2)$ symmetric tensor field, which is defined as

$$\mathrm{Hess} f(X, Y) = (\mathscr{L}_Y \mathscr{L}_X - \mathscr{L}_{\nabla_Y X}) f, \qquad (2)$$

for all $X, Y \in \mathfrak{X}(Q)$. In local coordinates, when $\text{grad} f(q) = 0$, the Hessian of f can be written as

$$\text{Hess} f\left(X^i \frac{\partial}{\partial q^i}, Y^j \frac{\partial}{\partial q^j}\right)(q) = \frac{\partial^2}{\partial q^i \partial q^j} X^i Y^j(q). \quad (3)$$

Note that $\text{Hess} f$ maps $T_q Q \times T_q Q$ to real space $\mathbb{R}$. We usually investigate whether $\text{Hess} f$ is positive definite over certain sub-bundles of TQ.

2.2 Mechanical control systems

In this section, we introduce mechanical control systems based on aforementioned concepts on manifolds. Generally, a mechanical control system $(Q, \mathbb{G}_q, Y, \mathscr{F})$ is defined by the following objects:

- a manifold Q in n dimensions, describing the configuration space,
- a Riemannian metric $\mathbb{G}_q$ for kinetic energy, usually denoted by $\langle \cdot, \cdot \rangle_q$,
- a function V on Q representing the potential energy,
- a codistribution $\mathscr{F} = \text{span}\{F^1, \cdots, F^m\}$ in m dimensions defining the input forces.

It should be emphasized that we assume the dimension of codistribution $\mathscr{F}$ is less than that of configuration manifold Q, i.e., $m < n$. Thus, such a mechanical control system is underactuated.

Let $q \in Q$ denote the configuration of the system and v_q its velocity. Using the sharp map, we define the input vector fields $Y_i = \mathbb{G}_q^\sharp(F^i)$ where $i = 1, \cdots, m$, so that the input distribution can be denoted by $\mathscr{Y} = \text{span}\{Y_1, \cdots, Y_m\}$. The total energy of the system, or the Hamiltonian $H : TQ \to \mathbb{R}$ is

$$H = \frac{1}{2}\langle v_q, v_q \rangle_q + V(q). \quad (4)$$

The dynamic equation of the system can be written as

$$\frac{\text{D} v_q}{\text{d}t} = -\text{grad} V + Y_i u^i, \quad (5)$$

where u^i is the control input function, and Einstein summation convention is employed herein. Equation (5) is called Euler-Poincaré equation, which is in a coordinate independent form. For a number of mechanical control systems moving in a plane, such as nonholonomic vehicles and underactuated ships, there are generally no conservative forces exerted on them. In other words, these systems are without potential energy and only described by $(Q, \mathbb{G}_q, \mathscr{F})$. In this case, the dynamic equation (5) can be simplified as

$$\frac{\text{D} v_q}{\text{d}t} = Y_i u^i. \quad (6)$$

In this paper, what we investigate is the stabilization for underactuated mechanical control systems without potential energy.

3. MAIN RESULTS

In this Section, we propose the control strategy that stabilize the equilibrium of an underactuated mechanical control system. The main idea is designing an artificial potential for the closed-loop system to make the equilibrium stable. Moreover, the ultimate goal is to achieve exponential convergence for all of variables of the system.

3.1 Artificial Potential

Proportional feedback with respect to configuration has been studied in a large number of researches. From the perspective of the energy, such a control law can shape the potential energy of the system. Similarly, employing this approach, we are able to construct the artificial potential for the closed-loop system, which is illustrated in the following lemma.

Lemma 1. Consider an underactuated mechanical control system $(Q, \mathbb{G}_q, \mathscr{F})$ without potential energy. Assume there exists a function $\psi : Q \to \mathbb{R}$, such that

$$\text{grad}\psi = c^i(q) Y_i, \ i = 1, \cdots, m, \quad (7)$$

in which $c^i : Q \to \mathbb{R}$ is a continuous function. If the control input is designed as

$$u^i = -c^i(q)\psi, \quad (8)$$

then the closed-loop system is a mechanical system $(Q, \mathbb{G}_q, V_a)$, where V_a is the artificial potential formulated as $V_a = \frac{1}{2}\psi^2$.

Proof. Substituting (8) into (6), it can be easily obtained that

$$\frac{\text{D} v_q}{\text{d}t} = Y_i(-c^i(q)\psi).$$

Due to the fact that $c^i(q)$ is a real-valued function, we can regard $c^i(q)$ as a scalar. Thus, there holds

$$\frac{\text{D} v_q}{\text{d}t} = -c^i(q) Y_i \psi.$$

According to (7), $c^i(q) Y_i$ is the gradient of ψ. Therefore, we have

$$\frac{\text{D} v_q}{\text{d}t} = -\psi \text{grad}\psi = -\text{grad}(\frac{1}{2}\psi^2).$$

Define $V_a = \frac{1}{2}\psi^2$, then there holds

$$\frac{\text{D} v_q}{\text{d}t} = -\text{grad} V_a,$$

which describes a mechanical system with potential V_a.

Remark 2. According to Lemma 1, we construct the potential by configuration feedback, which makes the original potential-free system transformed into the closed-loop system with artificial potential V_a. Therefore, motivated by Lemma 1, we will look for the function $\psi : Q \to \mathbb{R}$, such that

$$\text{grad}\psi \in \mathscr{Y} = \text{span}\{Y_1, \cdots, Y_m\}.$$

Once obtaining ψ, we can use feedback to construct closed-loop system and design the artificial potential V_a. With respect to a point $q_0 \in Q$, if V_a satisfies

$$\text{d}V_a(q_0) = 0, \quad (9)$$

$$\text{Hess} V_a(X, X)(q_0) > 0, \quad (10)$$

for all $X \in TQ$, then it can be proved that q_0 is a Lyapunov stable equilibrium of the closed-loop system.

Remark 3. Actually, property (7) is the key point to deal with the underactuation of the system. This requires $\text{grad}\psi$ should lie in the distribution spanned by input vector fields Y_i. Otherwise, we cannot use feedback (8) to construct artificial potential V_a for the closed-loop system. For fully-actuated system, (7) is no longer necessary because in such a case the input vector fields can always span the whole tangent bundle. This also implies the fully-actuated systems are easier to handle compared with underactuated systems.

Now, our mission becomes to find function ψ for artificial potential construction. The following Lemma illustrates how to define such functions.

Lemma 4. Given a input distribution $\mathscr{Y}$, we define its orthogonal complement as

$$\mathscr{Y}^{\perp} = \{X \in \mathfrak{X}(Q),\ \langle X, Y_i\rangle = 0,\ i = 1, \cdots, m\}, \quad (11)$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in Euclidean space. Then, $\mathscr{Y}^{\perp}$ has m integral functions $\psi_1, \cdots, \psi_m$ satisfying

$$\mathrm{grad}\psi_i \in \mathscr{Y}\ \ i = 1, \cdots, m.$$

Furthermore, given any point $q_0 \in Q$, these m functions can be chosen such that $\psi_i(q_0) = 0$.

Proof. For function ψ, it is required $\mathrm{grad}\psi \in \mathscr{Y}$, so that there holds $\mathrm{grad}\psi \perp \mathscr{Y}^{\perp}$. In other words, for $\forall X \in \mathscr{Y}^{\perp}$, we have $\mathscr{L}_X\psi = 0$. Thus, we can arbitrarily choose two vector fields X_1 and X_2 in $\mathscr{Y}^{\perp}$ such that $\mathscr{L}_{X_1}\psi = 0$ and $\mathscr{L}_{X_2}\psi = 0$. Define the Lie bracket of X_1 and X_2 as X_3, i.e.,$X_3 = [X_1, X_2]$, and compute $\mathscr{L}_{X_3}\psi$ as follows

$$\mathscr{L}_{X_3}\psi = \mathscr{L}_{[X_1,X_2]}\psi = \mathscr{L}_{X_1}\mathscr{L}_{X_2}\psi - \mathscr{L}_{X_2}\mathscr{L}_{X_1}\psi = 0.$$

This implies that $\mathrm{grad}\psi \perp X_3$, i.e., $X_3 \in \mathscr{Y}^{\perp}$. Note that X_3 is the Lie bracket of vector fields in $\mathscr{Y}^{\perp}$, so that the distribution $\mathscr{Y}^{\perp}$ is involutive. Furthermore, according to Frobenius Theorem, $\mathscr{Y}^{\perp}$ is completely integrable, which means there exist m integral functions $\psi_1, \cdots, \psi_m$ such that

$$\mathrm{span}\{\mathrm{grad}\psi_1, \cdots, \mathrm{grad}\psi_m\} = \mathscr{Y}.$$

Having obtained these m functions $\psi_1, \cdots, \psi_m$ from Lemma 4, we can implicitly design the control input u_i by the equality

$$\sum_{i=1}^{m} Y_i u^i = -\sum_{i=1}^{m} k_i \psi_i \mathrm{grad}\psi_i, \quad (12)$$

where $k_1, \cdots, k_m$ are positive scalars. Then, based on Lemma 1, in this case the closed-loop system is still mechanical system with artificial potential

$$\tilde{V}_a = \frac{1}{2}\sum_{i=1}^{m} k_i \psi_i^2,$$

and the Hamiltonian is

$$\tilde{H} = \frac{1}{2}\langle v_q, v_q\rangle_q + \frac{1}{2}\sum_{i=1}^{m} k_i \psi_i^2.$$

3.2 Lyapunov stability

We have designed the artificial potential for the closed-loop system. Once it satisfies certain conditions, the stability of the system can be guaranteed. In the following, we provide the control strategy which can make an equilibrium Lyapunov stable.

Theorem 5. Consider a mechanical control system $(Q, \mathbb{G}_q, \mathscr{Y})$, whose equilibrium is denoted by q_0. Let $\psi_1, \cdots, \psi_m$ be m functions obtained from Lemma 4. With out of generality, we can set

$$Y_i = \mathrm{grad}\psi_i,\ i = 1, \cdots, m.$$

If artificial $\tilde{V}_a = \frac{1}{2}\sum_{i=1}^{m} k_i\psi_i^2$ satisfies

$$\mathrm{d}\tilde{V}_a(q_0) = 0, \quad (13)$$

$$\mathrm{Hess}\tilde{V}_a(X, X)(q_0) > 0, \quad (14)$$

for $\forall X \in TQ$, then there exist such feedback control inputs

$$u^i = -k_i\psi_i, \quad (15)$$

that make the equilibrium q_0 Lyapunov stable.

Proof. The Hamiltonian of closed-loop system is $\tilde{H} = \frac{1}{2}\langle v_q, v_q\rangle_q + \tilde{V}_a(q)$, and we choose it as the Lyapunov function. For the simplicity of illustration, we define a new variable $x = (q, v_q)$ with the characteristic $x_0 = (q_0, 0)$. Then, the differential of $\tilde{H}$ can be computed as

$$\mathrm{d}\tilde{H} = \frac{1}{2}\mathrm{d}\langle v_q, v_q\rangle_q + \mathrm{d}\tilde{V}_a = \mathbb{G}_q^{\flat}(v_q) + \mathrm{d}\tilde{V}_a(q),$$

where $\mathbb{G}_q^{\flat}$ is the flat map. Because $\mathrm{d}\tilde{V}_a(q_0) = 0$ and $\mathbb{G}_q^{\flat}(0) = 0$, we can obtain that

$$\mathrm{d}\tilde{H}(x_0) = 0. \quad (16)$$

In addition, the Hessian of $\tilde{H}$ is $\mathrm{Hess}\tilde{H} = \mathbb{G}_q + \mathrm{Hess}\tilde{V}_a$. Due to the fact that $\mathbb{G}_q$ is positive definite and $\mathrm{Hess}\tilde{V}_a(X, X)(q_0) > 0$, there holds

$$\mathrm{Hess}\tilde{H}(X, X)(x_0) > 0, \quad (17)$$

for $\forall X \in TQ$. Based on conditions (16) and (17), it is indicated that $x_0 = (q_0, 0)$ is the minimum of the Lyapunov function $\tilde{H}$. Therefore, the equilibrium q_0 is Lyapunov stable.

3.3 Exponential stability

In this section, the dissipative feedback is introduced to the control law in order to achieve the exponential stability of the equilibrium. At first, we provide the following lemma which shows stabilization techniques for nonlinear systems.

Lemma 6. (Lemma 2.1, Bullo (2000)). Let Q be a smooth manifold, and consider the affine control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad (18)$$

where f, g_i are smooth vector fields and u_i is bounded measurable function. Let x_0 be an equilibrium of the system, and let $W : Q \to \mathbb{R}$ be the Lyapunov function. For $x \in B(x_0)$, where $B(x_0)$ is a neighborhood of x_0, the following stability results hold.

(i) If the time derivative of W along f is 0, and u_i is dissipative input, in other words, if

$$\mathscr{L}_f W = 0, \quad (19)$$

$$u_i = -\mathscr{L}_{g_i} W, \quad (20)$$

then the point x_0 is Lyapunov stable in the sense that $W(x(t)) \le W(x(0))$. If the system satisfies the linear controllability rank condition for $\forall x \in B(x_0)$, that is, if

$$\mathrm{rank}\{g_i, \mathrm{ad}_f g_i, \cdots, \mathrm{ad}_f^n g_i\}(x) = n,\ i = 1, \cdots, m \quad (21)$$

then the point x_0 is asymptotically stable in the sense that $\lim_{t\to\infty} x(t) = x_0$.

(ii) In addition, if the second variation of W at x_0 is positive definite, i.e., if

$$\delta^2 W(x_0) = \left.\frac{\partial^2 W}{\partial x^i \partial x^j}\right|_{x=x_0} \delta x_i \delta x_j > 0, \quad (22)$$

then, the point x_0 is exponentially stable in the sense that $W(x(t)) \le cW(x(0))\mathrm{e}^{-\lambda t}$, for some positive scalars c and λ.

Proof. (i) The time derivative of Lyapunov function W is

$$\begin{aligned}\dot{W} &= \frac{\partial W}{\partial x}\dot{x} = \frac{\partial W}{\partial x}f + \sum_{i=1}^{m}\frac{\partial W}{\partial x}g_i u_i \\ &= \mathscr{L}_f W + \sum_{i=1}^{m}(\mathscr{L}_{g_i}W)u_i. \end{aligned} \quad (23)$$

Substituting (19) and (20) into (23), we can obtain

$$\dot{W} = -\sum_{i=1}^{m}\|\mathscr{L}_{g_i}W\|^2,$$

where $\|\cdot\|$ is the l_2-norm. Furthermore, if the system satisfies the linear controllability rank condition (21), then there holds

$$\sum_{i=1}^{m}\|\mathscr{L}_{g_i}W\|^2 > 0.$$

If $\sum_{i=1}^{m}\|\mathscr{L}_{g_i}W\|^2 = 0$ holds as well, there will be a contradiction to the premise, which is deduced in the following. From $\mathscr{L}_{g_i}W = 0$ and $\mathscr{L}_f W = 0$, we have

$$f, g_i \in \mathrm{Ker}\left(\frac{\partial W}{\partial x}\right),$$

where Ker(·) means the kernel space. For the Lie bracket $\mathrm{ad}_f g_i = [f, g_i]$, there holds

$$\frac{\partial W}{\partial x}[f, g_i] = \mathscr{L}_{[f,g_i]}W = \mathscr{L}_f\mathscr{L}_{g_i}W - \mathscr{L}_{g_i}\mathscr{L}_f W = 0,$$

which implies

$$\mathrm{ad}_f g_i \in \mathrm{Ker}\left(\frac{\partial W}{\partial x}\right).$$

Similarly, we can prove

$$\mathrm{ad}_f^s g_i \in \mathrm{Ker}\left(\frac{\partial W}{\partial x}\right),\ s = 2, \cdots, m.$$

Thus, it can be obtained

$$\begin{aligned}\mathrm{rank}\{g_i, \mathrm{ad}_f g_i, \cdots, \mathrm{ad}_f^n g_i\}(x) &= \dim\left(\mathrm{Ker}\left(\frac{\partial W}{\partial x}\right)\right) \\ &= n-1,\end{aligned}$$

which is contradictory to the linear controllability rank condition (21). Therefore, when condition (21) holds, there has $\sum_{i=1}^{m}\|\mathscr{L}_{g_i}W\|^2 > 0$, and we can further obtain

$$\dot{W} = -\sum_{i=1}^{m}\|\mathscr{L}_{g_i}W\|^2 < 0,$$

which guarantees the asymptotical stability of the equilibrium.

(ii) Exponential stability can be proven by noting two facts (Bullo (2000)): firstly, the results in (i) can be applied to the linearized closed-loop system with $\delta^2 W(x_0)$ as a Lyapunov function, and secondly, the asymptotical stability of the linearized system indicates the exponential stability of the nonlinear system. Please refer to Corollary 5.30 in Sepulchre et al. (1997) for a similar discussion.

Eventually, we state the exponential stability results in the following theorem.

Theorem 7. Consider a mechanical control system $(Q, \mathbb{G}_q, \mathscr{Y})$, whose equilibrium is denoted by q_0. Let $\psi_1, \cdots, \psi_m$ be m functions obtained from Lemma 4. With out of generality, we can set

$$Y_i = \mathrm{grad}\psi_i,\ i = 1, \cdots, m.$$

If the following two requirements hold, that is, if

(i) for $q \in B(q_0)$, system satisfies the linear controllability rank condition (21);

(ii) artificial $\tilde{V}_a = \frac{1}{2}\sum_{i=1}^{m}k_i\psi_i^2$ satisfies

$$\mathrm{d}\tilde{V}_a(q_0) = 0,$$
$$\mathrm{Hess}\tilde{V}_a(X, X)(q_0) > 0,$$

for $\forall X \in TQ$, then there exist such feedback control inputs

$$u^i = -k_i\psi_i - d\dot{\psi}_i, \quad (24)$$

that make the equilibrium q_0 exponentially stable.

Proof. Let u^i consist of two components, i.e.,

$$u^i = u_s^i + u_d^i,$$

where u_s^i is potential shaping control and u_d^i is dissipative control. We set $u_s^i = -k_i\psi_i$ and substitute it into (5). Then, the dynamics of the closed-loop system is

$$\begin{cases}\dot{q} = v_q \\ \dfrac{\mathrm{D}v_q}{\mathrm{d}t} = -\mathrm{grad}\tilde{V}_a + Y_i u^i\end{cases} \quad (25)$$

where $\tilde{V}_a = \frac{1}{2}\sum_{i=1}^{m}k_i\psi_i^2$. Define the following variable and vector fields

$$x = \begin{bmatrix} q \\ v_q \end{bmatrix},\ f = \begin{bmatrix} v_q \\ -\mathrm{grad}\tilde{V}_a \end{bmatrix},\ g_i = \begin{bmatrix} 0 \\ Y_i \end{bmatrix},$$

then the dynamic equation (25) can be expressed in the affine form of (18). The Hamiltonian of closed-loop system is $\tilde{H} = \frac{1}{2}\langle v_q, v_q\rangle + \tilde{V}_a(q)$, and we choose it as the Lyapunov function. Next, we compute the Lie derivative of $\tilde{H}$ along f and g_i respectively, i.e.,

$$\mathscr{L}_f\tilde{H} = \langle \mathrm{grad}\tilde{H}, f\rangle_q = \left\langle \begin{bmatrix} \mathrm{grad}\tilde{V}_a \\ v_q \end{bmatrix}, \begin{bmatrix} v_q \\ -\mathrm{grad}\tilde{V}_a \end{bmatrix} \right\rangle_q = 0,$$

$$\mathscr{L}_{g_i}\tilde{H} = \langle \mathrm{grad}\tilde{H}, g_i\rangle_q = \left\langle \begin{bmatrix} \mathrm{grad}\tilde{V}_a \\ v_q \end{bmatrix}, \begin{bmatrix} 0 \\ Y_i \end{bmatrix} \right\rangle_q = \langle v_q, Y_i\rangle_q.$$

Define the dissipative control

$$u_d^i = -d_i\mathscr{L}_{g_i}\tilde{H} = -d_i\langle v_q, Y_i\rangle_q,$$

where d_i is a positive scalar. Due to $Y_i = \mathrm{grad}\psi_i$, we can further obtain

$$u_d^i = -d_i\langle v_q, \mathrm{grad}\psi_i\rangle_q = -d_i\mathscr{L}_{v_q}\psi_i = -d_i\dot{\psi}_i.$$

According to the premise, system satisfies the linear controllability rank condition, and in addition, it has been proven in Theorem 5 that the Hamiltonian $\tilde{H}$ has positive definite second variation. Therefore, based on Lemma 6, the equilibrium q_0 is exponentially stable.

Remark 8. In the design of stabilization controller, we do not linearize the mechanical system, but still keep its essential nonlinearity on manifolds. Of course, we could obtain the linearization system at the equilibrium and design the controller for the derived linear system, which is much simpler than the proposed controller in this paper. However, such a controller from linearization is only effective in a small neighborhood of the equilibrium, while the description on differential manifolds is global. Moreover, for several underactuated systems, their linearized systems at certain equilibrium are not controllable anymore. In Theorem 7, although the linear controllability rank condition (21) is introduced, it has no relationship with the linearization. This is an assumption for the drift and control vector fields of the mechanical system. These vector fields lie in the tangent space, which is a linear space, so that (21) looks like the following controllability rank condition for linear system

$$\mathrm{rank}\{B, AB, \cdots, A^{n-1}B\} = n, \tag{26}$$

where A and B are system matrix and input matrix of a linear system respectively. Actually, condition (26) is a particular form of the linear controllability rank condition (21). In other words, (21) will degenerate to (26) for linear systems.

4. CONCLUSION

In this paper, we study the equilibrium stabilization problem for underactuated mechanical systems without potential energy. The system is established on the Rimannian manifold, and differential geometric methods are employed in the design of stabilization controller. The novelty lies in constructing an artificial potential for the closed-loop system by configuration feedback control. The stability of the system can be realized once the artificial potential certain requirements with respect to the equilibrium, Furthermore, by incorporating dissipative feedback into the control strategy, we successfully obtain the exponential stability of the equilibrium.

Of course, there still exist challenges for future research. The crucial point in the proposed approach is to find a series of functions, which in fact are integral functions for an involutive distribution. Whether such functions can be obtained is significant to the construction of artificial potential. Unfortunately, computing integral functions for involutive distribution of arbitrary dimension and codimension is generally as difficult a providing explicit solutions to a set of ordinary differential equations (Bullo (2000)). Furthermore, these functions not only can be obtained but also should be proper, in the sense that they are supposed to satisfy certain requirements guaranteeing the stability of the equilibrium. Thus, the existence of such functions is an open problem worth studying.

REFERENCES

Banaszuk, A. and Hauser, J. (1996). Approximate feedback linearization: a homotopy operator approach. *SIAM J. Control Optim.*, 34, 1533–1554.

Bloch, A.M., Chang, D.E., Leonard, N.E., and Marsden, J.E. (2001). Controlled Lagrangians and the stabilization of mechanical systems ii: potential shaping. *IEEE Trans. Autom. Control*, 46, 1556–1571.

Bullo, F. (2000). Stabilization of relative equilibria for underactuated systems on Riemannian manifolds. *Automatica*, 36, 1819–1834.

Bullo, F. and Lewis, A.D. (2005). *Geometric control of mechanical systems: modeling, analysis, and design for simple mechanical control systems.* Springer, New York.

Dixon, W., Jiang, Z.P., and Dawson, D. (2000). Global exponential setpoint control of wheeled mobile robots: a lyapunov approach. *Automatica*, 36, 1741–1746.

Farrell, J., Polycarpou, M., Sharma, M., and Dong, W. (2009). Command filtered backstepping. *IEEE Trans. Autom. Control*, 54, 1391–1395.

Gómez-Estern, F., Ortega, R., and Spong, M.W. (2001). Total energy shaping for underactuated mechanical systems. In *Proceedings of 5th IFAC Symposium on Nonlinear Control Systems*, 1135–1140. St Petersburg, Russia.

Jalnapurkar, S.M. and Marsden, J.E. (2000). Stabilization of relative equilibria. *IEEE Trans. Autom. Control*, 45, 1483–1491.

Justh, E. and Krishnaprasad, P. (2004). Equilibria and steering laws for planar formations. *Syst. Control Lett.*, 52, 25–38.

Kallem, V., Chang, D.E., and Cowan, N.J. (2010). Task-induced symmetry and reduction with application to needle steering. *IEEE Trans. Autom. Control*, 55, 664–673.

Leonard, N.E. (1997). Stabilization of underwater vehicle dynamics with symmetry-breaking potentials. *Syst. Control Lett.*, 32, 35–42.

Mazenc, F., Burlio, L., and Malisoff, M. (2019). Backstepping design for output feedback stabilization for a class of uncertain systems. *Syst. Control Lett.*, 123, 134–143.

Morin, P. and Samso, C. (2001). A characterization of the lie algebra rank condition by transverse periodic functions. *SIAM J. Control Optim.*, 40, 1227–1249.

Morin, P. and Samso, C. (2003). Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Trans. Autom. Control*, 48, 1496–1508.

Niu, H. and Geng, Z. (2019). Stabilisation of a relative equilibrium of an underactuated AUV on SE(3). *Int. J. Control*, 92, 1883–1902.

Ortega, R., Spong, M.W., Gómez-Estern, F., and Blankenstein, G. (2002). Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment. *IEEE Trans. Autom. Control*, 47, 1218–1233.

Ortega, R., van der Schaft, A.J., Mareels, I., and Maschke, B. (2001). Putting energy back in control. *IEEE Control Syst. Mag.*, 21, 18–33.

Pazderski, D. (2017). Waypoint following for differentially driven wheeled robots with limited velocity perturbations. *J. Intell. Robot. Syst.*, 85, 553–575.

Reis, M.F., Carvalho, G.P.S., Neves, A.F., and Peixoto, A.J. (2018). Dynamic model and line of sight control of a 3-dof inertial stabilization platform via feedback linearization. In *Proceedings of 2018 American Control Conference*, 1313–1318. Milwaukee, USA.

Sepulchre, R., Janković, M., and Kokotović, P. (1997). *Constructive nonlinear control.* Springer, New York.

Takegaki, M. and Arimoto, S. (1981). A new feedback method for dynamic control of manipulators. *J. Dyn. Syst. Meas. Control*, 103, 119–125.

Tayefi, M. and Geng, Z. (2018). Self-balancing controlled Lagrangian and geometric control of unmanned mobile robots. *J. Intell. Robot. Syst.*, 90, 253–265.

van der Schaft, A.J. (1986). Stabilization of hamiltonian systems. *Nonlinear Anal. Theory Methods Appl.*, 10, 1021–1035.

Wang, J., Qu, Z., Hull, R.A., and Martind, J. (2007). Cascaded feedback linearization and its application to stabilization of nonholonomic systems. *Syst. Control Lett.*, 56, 285–295.

Wu, F. and Geng, Z. (2010). Stabilization of relative equilibria for coordinated underwater vehicles. In *Proceedings of 29th Chinese Control Conference*, 395–400. Beijing, China.