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State estimation and sliding mode control for non-linear singular systems with time-varying delay

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Abstract: This paper deals with the sliding mode control (SMC) problem for a class of non-linear discrete-time singular systems with external disturbances and time-varying delay. The main contribution of this paper is to design an observer-based sliding mode control scheme for the system under consideration. First, a delay dependent criterion is built to guarantee the closed-loop system to be robustly admissible. Then, a (SMC) law is synthesized to guarantee the state trajectories of the closed-loop systems to be stable, to and ensure the reachability of the specified sliding surface in a short time interval. An illustrative example is given to numerically demonstrate the effectiveness of the proposed control scheme.

Keywords: discrete-time singular systems - time delay - sliding mode - state estimation - LMI

1. INTRODUCTION

Recently, singular systems have received many researchers' interest since they provide the dynamic property, as well as the interconnection property for many practical processes like power systems, robotic systems and economic systems. From mathematical point of view, these properties are represented, respectability, by a differential and algebraic equations. The study of singular systems needs more attention because the aforementioned properties may cause the irregularity, impulses, and the non-causality Dai [1989]. Meanwhile, time-delay constitutes an inherent feature of several dynamic systems and can be the main source of instability and performance degradation Xia et al. [2009], Gu et al. [2003]. Based on singular models with time-delay and using LMI techniques, many important results have been carried out. See for instance Duan [2010], Xu and Lam [2006].

On the other hand, sliding mode control (SMC) has been widely applied to complex and engineering systems. It is considered as useful and effective control technique that achieves robustness, fast response and invariance to matched uncertainties and external disturbances Chang [2012], Ding et al. [2011], Kchaou et al. [2014], Liu et al. [2013], Jiang et al. [2015]. Additionally, it is well known that for many practical applications, it is not always possible to have access to all state variables, and only partial information are available via measured outputs. In this case, the design of observers to estimate the system states is more realistic. The sliding mode observer approach has been adopted to deal with the state estimation problems for linear and non-linear systems Gao et al. [2014], Kchaou [2019]. However, the problem of observer-based SMC for uncertain discrete-time singular systems with time-varying delays and external disturbances has not been fully investigated. This concern motivates us to tackle this problem for such class of systems.

This study is interested to the state estimation and robust SMC for a class of uncertain discrete-time singular systems with time-varying delay and external disturbance. The main contributions of this paper lie in the following: (1) by considering an appropriate Lyapunov-Krasovskii functional, a new delay-dependent criterion is established to prove that the sliding mode dynamics are admissible, (2) by constructing an observer to estimate the system states, a sliding mode surface and a SMC law are designed such that the discrete quasi-sliding mode is reachable, and the states of the closed-loop system evolve around a region near equilibrium point. (3) An example is given to show the advantages of the proposed control strategy.

Notations. The notations in this paper are quite standard except where otherwise stated. $X \in \mathbb{R}^n$ denotes the $n-$dimensional Euclidean space, while $X \in \mathbb{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices; $X > 0$ (respectively, $X \geq 0$) means that the matrix X is real symmetric positive definite (respectively, positive semi-definite); sym(X) stands for $X + X^T$; $\|.\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices or long matrix expressions, we use a star $*$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. SYSTEM DESCRIPTION AND STABILITY ANALYSIS

2.1 System description

Consider a class of discrete-time singular systems with state delay described by:

$$\begin{cases} Ex(k+1) = (A+\Delta A)x(k) + (A_d + \Delta A_d)x(k-d(k)) \\ \qquad\qquad + B\big(u(k) + f(k, x(k))\big) \\ \quad y(k) = C_2 x(k) \\ \quad x(k) = \phi_0(k), \ k \in [-d_M, 0] \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $f(k, x(k))$ is a bounded function satisfying $\|f(k, x(k))\| \le \rho \|x(k)\|$, $d(k)$ is a positive integer representing the time-varying delay that satisfies $0 \le d_m \le d(k) \le d_M$, where d_m and d_M are positive integers. $\phi_0(k)$ is a compatible initial condition. Matrix $E \in \mathbb{R}^{n\times n}$ may be singular with $rank(E) = r < n$. A, A_d, B and C_2 are known real constant matrices with appropriate dimensions. We assume that the output matrix C_2 being of full column rank. By virtue of singular values decomposition, the following relationship can be got:

$$C_2 = U\begin{bmatrix} S & 0 \end{bmatrix} V^T \tag{2}$$

ΔA and ΔA_d are unknown matrices representing the parametric uncertainties, assumed to be of the form

$$[\Delta A \ \Delta A_d] = MF(k)\,[N \ N_d] \tag{3}$$

where M, N and N_d are known real constant matrices with appropriate dimensions, and $F(k)$ is an unknown matrix function satisfying $F^T(k)F(k) \le I$.

The nominal unforced discrete singular time-delay system of (1) is as follows:

$$\begin{aligned} Ex(k+1) &= Ax(k) + A_d x(k - d(k)) \\ x(k) &= \phi_0(k), \ k \in [-d_M, 0] \end{aligned} \tag{4}$$

Definition *1.* Dai [1989], Xu and Lam [2006]

(1) Pair (E, A) is said to be regular if $\det(zE - A) \neq 0$.
(2) Pair (E, A) is said to be causal, if it is regular and $deg\Big(det(zE - A)\Big) = rank(E)$.
(3) System (4) is said to be admissible if it is regular, causal and stable.

We end this section by recalling the following lemma:

Lemma *2.1.* Park et al. [2011] For any vectors ψ_1, ψ_2, matrices S, R and real scalars $\alpha_1 \ge 0, \alpha_2 \ge 0$, satisfying

$$\begin{bmatrix} R & S \\ S^T & R \end{bmatrix} \ge 0, \quad \alpha_1 + \alpha_2 = 1, \ \psi_i = 0 \text{ if } \alpha_i = 0 \ (i = 1, 2)$$

we have

$$-\frac{1}{\alpha_1}\psi_1^T R \psi_1 - \frac{1}{\alpha_2}\psi_2^T R \psi_2 \le -\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}^T \begin{bmatrix} R & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

2.2 Stability analysis

In this section, we provide a sufficient condition under which system (4) is admissible.

Theorem *2.1.* System (4) is admissible, if there exist a positive scalar ε and matrices $P > 0$, $Q > 0$, $Q_1 > 0$, $Q_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, S, R, G_i, $i = 1, \cdots 3$, such that the following inequalities hold:

$$\Phi(E, A, A_d) < 0 \tag{5}$$

$$\begin{bmatrix} Z_1 & Y \\ Y^T & Z_1 \end{bmatrix} > 0 \tag{6}$$

where

$$\Phi(E, A, A_d) = \begin{bmatrix} \Phi_{11} & E^T Z_1 E & \Phi_{13} & \Phi_{14} & \Phi_{15} \\ * & \Phi_{22} & \Phi_{23} & E^T R^T E & 0 \\ * & * & \Phi_{33} & \Phi_{34} & \Phi_{35} \\ * & * & * & \Phi_{44} & 0 \\ * & * & * & * & \Phi_{55} \end{bmatrix} \tag{7}$$

$$\begin{aligned} \Phi_{11} =& Q_1 + Q_2 + (d_r + 1)Q_3 + sym(G_1^T(A - E)) \\ & - E^T Z_1 E \\ \Phi_{22} =& - Q_1 - E^T Z_2 E - E^T Z_1 E \\ \Phi_{13} =& G_1^T A_d + (A - E)^T G_2 \\ \Phi_{23} =& - E^T R^T E + E^T Z_3 E \\ \Phi_{33} =& - Q_3 + sym(G_2^T A_d) - 2E^T Z_2 E + \mathrm{sym}(E^T R E) \\ \Phi_{14} =& \\ \Phi_{34} =& - E^T R E + E^T Z_3 E \\ \Phi_{44} =& - Q_2 - E^T Z_2 E \\ \Phi_{15} =& E^T P + S\Omega_0^T - G_1^T + (A - E)^T G_3 \\ \Phi_{35} =& - G_2^T + A_d^T G_3 \\ \Phi_{55} =& d_m^2 Z_1 + d_r^2 Z_2 - sym(G_3) \end{aligned} \tag{8}$$

$\Omega_0 \in \mathbb{R}^{n\times n-r}$ is any matrix with full column rank satisfying $E^T\Omega_0 = 0$ and $d_r = d_M - d_m$.

Proof: First we prove that the nominal case of (4) is admissible

Since $\mathrm{rank}(E) = r < n$, there always exist two nonsingular matrices $\hat{M}$ and $\hat{N} \in \mathbb{R}^{n\times n}$ such that

$$\hat{E} = \hat{M}E\hat{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

Then, Ω_0 can be characterised as $\Omega_0 = \hat{M}^T \begin{bmatrix} 0 \\ \hat{\Phi} \end{bmatrix}$, where $\hat{\Phi} \in \mathbb{R}^{(n-r)\times(n-r)}$ is any nonsingular matrix.

We also define

$$\hat{A} = \hat{M}A\hat{N} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{A}_d = \hat{M}A_d\hat{N} = \begin{bmatrix} \hat{A}_{d11} & \hat{A}_{d12} \\ \hat{A}_{d21} & \hat{A}_{d22} \end{bmatrix},$$

$$\hat{S} = \hat{N}^T S = \begin{bmatrix} \hat{S}_{11} \\ \hat{S}_{21} \end{bmatrix}. \tag{10}$$

It follows from (5) that

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0 \tag{11}$$

where

$$\begin{aligned} \Psi_{11} =& sym(G_1^T(A - E)) - E^T Z_1 E \\ \Psi_{12} =& E^T P + S\Omega_0^T - G_1^T + (A - E)^T G_3 \\ \Psi_{22} =& - sym(G_3) \end{aligned}$$

Pre- and post-multiplying (11) by $\Big[I, A^T\Big]$ and its transpose, respectively, we obtain

$$sym\Big(E^T(P - G_3 - G_1)A - G_1E + S\Omega_0^T A\Big) < 0 \quad (12)$$

Pre- and post-multiplying (12) by $\hat{N}^T$ and $\hat{N}$, respectively, and then using expression (9) and (10), yields

$$sym(\hat{S}_{21}\hat{\Phi}^T\hat{A}_{22}) < 0 \quad (13)$$

and thus $\hat{A}_{22}$ is nonsingular.

Next, we show that system (4) is stable. To this end, we select a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned}
V(k) =& V_1(k) + V_2(k) + V_3(k)\\
V_1(k) =& x^T(k)E^TPEx(k)\\
V_2(k) =& \sum_{s=k-d_m}^{k-1} x^T(s)Q_1x(s) + \sum_{s=k-d_M}^{k-1} x^T(s)Q_2x(s)\\
&+ \sum_{\theta=-d_M}^{-d_m}\sum_{s=k+\theta}^{k-1} x^T(s)Q_3x(s)\\
V_3(k) =& d_m \sum_{\theta=-d_m}^{-1}\sum_{s=k+\theta}^{k-1} \eta^T(s)E^TZ_1E\eta(s)\\
&+ d_r \sum_{\theta=-d_M}^{-d_m-1}\sum_{s=k+\theta}^{k-1} \eta^T(s)E^TZ_2E\eta(s)
\end{aligned} \quad (14)$$

where $\eta(k) = x(k+1) - x(k)$.

By calculating the difference $\Delta V(k) = V(k+1) - V(k)$ along the trajectory of system (4), we get

$$\Delta V_1(k) = \eta^T(k)E^TPE\eta(k) + 2x^T(k)E^TPE\eta(k) \quad (15)$$

$$\begin{aligned}
\Delta V_2(k) \le& x^T(k)\big(Q_1 + Q_2 + d_rQ_3\big)x(k)\\
&- x^T(k-d_m)Q_1x(k-d_m)\\
&- x^T(k-d_M)Q_2x(k-d_M)\\
&- x^T(k-d(k))Qx(k-d(k))
\end{aligned} \quad (16)$$

$$\begin{aligned}
\Delta V_3(k) =& \eta^T(k)E^T\Big(d_m^2Z_1 + d_r^2Z_2\Big)E\eta(k)\\
&- d_m \sum_{s=k-d_m}^{k-1} \eta^T(s)E^TZ_2E\eta(s)\\
&- d_r \sum_{s=k-d_M}^{k-d_m-1} \eta^T(s)E^TZ_2E\eta(s)
\end{aligned} \quad (17)$$

According to Jensen Lemma, we get

$$-d_m \sum_{s=k-d_m}^{k-1} \eta^T(s)E^TZ_1E\eta(s) \le -\gamma_2(k)E^TZ_1E\gamma_2(k) \quad (18)$$

and

$$\begin{aligned}
&- d_r \sum_{s=k-d_M}^{k-d_m-1} \eta^T(s)E^TZ_2E\eta(s)\\
=& - d_r \sum_{s=k-d_M}^{k-d(k)-1} \eta^T(s)E^TZ_2E\eta(s)\\
&- d_r \sum_{s=k-d(k)}^{k-d_m-1} \eta^T(s)E^TZ_2E\eta(s)\\
\le& -\frac{1}{\alpha_1}\psi_1^T(k)E^TZ_2E\psi_1(k) - \frac{1}{\alpha_2}\psi_2^T(k)E^TZ_2E\psi_2(k)
\end{aligned} \quad (19)$$

where $\alpha_1 = \frac{d_M - d(k)}{d_M - d_m}$, $\alpha_2 = \frac{d(k) - d_m}{d_M - d_m}$, $\gamma_1(k) = x(k) - x(k - d_M)$, $\gamma_2(k) = x(k) - x(k - d_m)$.

$$\begin{aligned}
\psi_1(k) &= x(k-d(k)) - x(k-d_M),\\
\psi_2(k) &= x(k-d_m) - x(k-d(k))
\end{aligned} \quad (20)$$

According to Lemma 2.1, we conclude

$$\begin{aligned}
&- d_r \sum_{s=k-d_M}^{k-d_m-1} \eta^T(s)E^TZ_2E\eta(s)\\
&\le -\begin{bmatrix}\psi_1(k)\\ \psi_2(k)\end{bmatrix}^T \begin{bmatrix}E^TZ_2E & E^TRE\\ E^TR^TE & E^TZ_2E\end{bmatrix}\begin{bmatrix}\psi_1(k)\\ \psi_2(k)\end{bmatrix}
\end{aligned} \quad (21)$$

Note that when $d(k) = d_m$ or $d(k) = d_M$, we have $\psi_1(k) = 0$ or $\psi_2(k) = 0$, respectively. Thus, (21) still holds.

Let $\xi(k) = \text{col}\Big\{x(k)\ \ x(k-d_m)\ \ x(k-d(k))\ \ x(k-d_M)\ \ E\eta(k)\Big\}$. From (4), it is easy to see that the following equation holds for any matrices G_1, G_2 and G_3 with appropriate dimensions

$$\begin{aligned}
&2\Big[x^T(k)G_1^T + x^T(k-d(k))G_2^T + \eta^T(k)E^TG_3^T\Big]\\
&\Big[(A-E)x(k) + A_dx(k-d(k)) - E\eta(k)\Big] = 0
\end{aligned} \quad (22)$$

On the other hand, it is clear that

$$2x^T(k)S\Omega_0^TE\eta(k) = 0 \quad (23)$$

From (15)-(23), we have

$$\Delta V(k) \le \xi^T(k)\Phi(E, A, A_d)\xi(k) \quad (24)$$

According to Lyapunov stability theory, then there exists a scalar $\alpha > 0$ such that

$$\Delta V(k) \le -\alpha\|x(k)\|^2 \quad (25)$$

Therefore, we have

$$\sum_{i=0}^{k}\|x(i)\|^2 \le \frac{1}{\alpha}V(0) < \infty \quad (26)$$

that is, the series $\sum_{i=0}^{k}\|x(i)\|^2$ converges, which implies that $\lim_{k\to\infty} x(k) = 0$. Thus, according to Definition 2, system (4) is stable.

□

3. SLIDING MODE OBSERVER DESIGN

Based on the sliding mode approach, we aim in this study to design an observer-based controller for the addressed singular system (1) such that the closed-loop system will be robustly admissible.

The observer is defined by the following model:

$$\begin{cases} E\hat{x}(k+1) = A\hat{x}(k) + A_d\hat{x}(k-d(k)) + B\big(u(k) + f(k,\hat{x}(k))\big) \\ \qquad\qquad + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C_2\hat{x}(k) \\ \hat{x}(k) = 0,\ k \in [-d_M, 0] \end{cases} \tag{27}$$

where $\hat{x}(k)$ is the estimated state and L is the observer gain. Let $e(k) = y(k) - \hat{y}(k)$ and $\Delta f = f(k,x(k)) - f(k,\hat{x}(k))$, the observer error is defined as

$$\begin{aligned} Ee(k+1) =& (A - LC_2 + \Delta A)e(k) + (A_d + \Delta A_d)e(k-d(k)) \\ &+ \Delta A\hat{x}(k) + \Delta A_d\hat{x}(k-d(k)) + B\Delta f \end{aligned} \tag{28}$$

3.1 Sliding surface design

Based on the estimated state, we construct the following sliding surface

$$S(k) = \mathbf{S}E\hat{x}(k) - \sum_{s=0}^{k-1}\mathbf{S}\big(A + BK - E\big)\hat{x}(s) \tag{29}$$

where $\mathbf{S} \in \mathbb{R}^{m\times n}$ is a matrix such that $\mathbf{S}B$ is invertible. K is a gain matrix to be designed.

Therefore, the equivalent control law is obtained as follows:

$$\begin{aligned} u(k) =& K\hat{x}(k) - (\mathbf{S}B)^{-1}\mathbf{S}\big(A_d\hat{x}(k-d(k)) + LC_2e(k)\big) \\ &- f(k,\hat{x}(k)) \end{aligned} \tag{30}$$

Substituting (30) into the system (27) and denoting $\overline{\mathbf{S}} = I - B(\mathbf{S}B)^{-1}\mathbf{S}$, the sliding mode dynamics can be formulated as

$$\begin{aligned} E\hat{x}(k+1) =& (A + BK)\hat{x}(k) + \overline{\mathbf{S}}(A_d\hat{x}(k-d(k)) \\ &+ LC_2e(k)) \end{aligned} \tag{31}$$

From (28) and (31), the augmented closed-loop system is written as

$$\bar{E}\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{A}_d(k)\bar{x}(k-d(k)) + \bar{B}\Delta f \tag{32}$$

with $\bar{A}(k) = \bar{A} + \bar{M}\bar{F}(k)\bar{N}$, $\bar{A}_d(k) = \bar{A}_d + \bar{M}F(k)\bar{N}_d$, $\bar{F}(k) = diag\Big(F(k), F(k)\Big)$,

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix},\ \bar{A} = \begin{bmatrix} A + BK & \overline{\mathbf{S}}LC_2 \\ 0 & A - LC_2 \end{bmatrix},\ \bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & A_d \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix},\quad \bar{M} = \begin{bmatrix} 0 \\ M \end{bmatrix},\quad \bar{N} = [N\ N],\quad \bar{N}_d = [N_d\ N_d]$$

3.2 Sliding mode dynamics synthesis

In the sequel, we shall restrict our attention to the design of K and L of the sliding surface such that the sliding mode dynamics (32) is robustly admissible.

Theorem *3.1.* Given positive integers d_m and d_M and tuning parameters λ_i, $(i = 1, 2, 3)$. If there exist positive scalars ε and ε_1, and appropriate dimensions matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, S, R_1, R_{21}, R_{22}, W_1, W_2, Y, $\widehat{\mathbf{G}}_{11} \in \mathbb{R}^{q\times q}$, $\widehat{\mathbf{G}}_{22} \in \mathbb{R}^{(n-q)\times(n-q)}$, $\widehat{\mathbf{G}}_{21} \in \mathbb{R}^{(n-q)\times n}$, $\mathcal{G} \in \mathbb{R}^{n\times n}$, $\mathbf{K} \in \mathbb{R}^{m\times n}$, $\mathbf{L} \in \mathbb{R}^{n\times q}$ such that (6) and the following LMI hold

$$\begin{bmatrix} \Phi_{11} & \Phi_{12}^T & \Phi_{13}^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0 \tag{33}$$

where

$$\begin{aligned} \Phi_{11} =& \Phi(\bar{E}^T, \mathbf{A}, \mathbf{A}_d) + \varepsilon e_1^T\bar{M}\bar{M}^Te_1 + \varepsilon e_3^T\bar{M}\bar{M}^Te_3 \\ &+ \varepsilon_1 e_1^T diag(0, \rho^2 I)e_1 \\ \Phi_{12} =& \begin{bmatrix} \lambda_1\mathbf{G}^T\bar{N}^T & 0 & \lambda_2\mathbf{G}^T\bar{N}^T & 0 & \lambda_3\mathbf{G}^T\bar{N}^T & 0 & 0 & 0 \\ \lambda_1\mathbf{G}^T\bar{N}_d & 0 & \lambda_2\mathbf{G}^T\bar{N}_d & 0 & \lambda_3\mathbf{G}^T\bar{N}_d & 0 & 0 & 0 \end{bmatrix} \\ \Phi_{13} =& \begin{bmatrix} \lambda_1\mathbf{G}^T\bar{B} & 0 & \lambda_2\mathbf{G}^T\bar{B} & 0 & \lambda_3\mathbf{G}^T\bar{B} & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} A\mathcal{G} + B_2Y & \overline{\mathbf{S}}FC_2 \\ 0 & A\mathbf{G} - FC_2 \end{bmatrix},\ \mathbf{A}_d = \begin{bmatrix} \overline{\mathbf{S}}A_d\mathcal{G} & 0 \\ 0 & A_d\mathbf{G} \end{bmatrix},$$

$$\widehat{\mathbf{G}} = \begin{bmatrix} \widehat{\mathbf{G}}_{11} & 0 \\ \widehat{\mathbf{G}}_{21} & \widehat{\mathbf{G}}_{22} \end{bmatrix},\ \mathbf{G} = V\widehat{\mathbf{G}}V^T,\ \bar{G} = \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbf{G} \end{bmatrix}.$$

Then the closed-loop singular system (32) is robustly admissible; the gain matrices are given as $K = Y\mathcal{G}^{-1}$ and $L = FUS\widehat{\mathbf{G}}_{11}^{-1}S^{-1}U^T$, where U, V and S come from (2).

Proof: Now consider the following singular delay system

$$\bar{E}^T\zeta(k+1) = \big(\bar{A}^T\zeta(k) + \bar{A}_d^T\zeta(k-d(k))\big) \tag{34}$$

Note that $\det\big(z\bar{E} - \bar{A}\big) = \det\big(z\bar{E}^T - \bar{A}^T\big)$, then the pair $\big(\bar{E}, \bar{A}\big)$ is admissible if and only if the pair $\big(\bar{E}^T, \bar{A}^T\big)$ is admissible. Moreover, $\det\big(z\bar{E} - \bar{A} - z^{-d_M}\bar{A}_d\big) = 0$ and $\det\big(z\bar{E}^T - \bar{A}^T - z^{-d_M}\bar{A}_d^T\big) = 0$ have the same solution.

As long as the regularity, being impulse-free and stability are concerned, we can consider system (34) instead of (32).

Set $G_1 = \lambda_1\bar{G}$, $G_2 = \lambda_2\bar{G}$ and $G_3 = \lambda_3\bar{G}$. From Theorem (2.1) yields

$$\Phi(\bar{E}^T, \bar{A}(k), \bar{A}_d(k)) < 0 \tag{35}$$

Considering condition 2, we can verify that

$$\begin{aligned} C_2\mathbf{G} =& U\Big[S \quad 0\Big]V^TV\begin{bmatrix} \widehat{\mathbf{G}}_{11} & 0 \\ \widehat{\mathbf{G}}_{21} & \widehat{\mathbf{G}}_{22} \end{bmatrix}V^T \\ =& U\Big[S\widehat{\mathbf{G}}_{11} \quad 0\Big]V^T = US\widehat{\mathbf{G}}_{11}S^{-1}U^TU\Big[S \quad 0\Big]V^T = \widehat{\mathbf{G}}C_2 \end{aligned}$$

and then $\bar{A}\bar{G} = \mathbf{A}$. Considering 3, with some algebraic manipulations we get

$$\begin{aligned} &\Phi(\bar{E}^T, \mathbf{A}, \mathbf{A}_d) + \varepsilon e_1^T\bar{M}\bar{M}^Te_1 + \varepsilon e_3^T\bar{M}\bar{M}^Te_3 \\ &+ \varepsilon^{-1}\Phi_{12}^T\Phi_{12} < 0 \end{aligned} \tag{36}$$

and by introducing a scalar $\varepsilon_1 > 0$, the following inequality holds

$$2\xi^T(k)\breve{G}^T\bar{B}\Delta f \leq \varepsilon_1^{-1}\xi^T(k)\breve{G}^T\bar{B}\bar{B}^T\breve{G}\xi(k) + \varepsilon_1\rho^2e^T(t)e(t) \tag{37}$$

Adding (37) to (36), condition (33) is equivalently obtained using the well-known Schur complement.

□

3.3 Sliding Mode control law synthesis

Once the sliding surface is appropriately designed, we will carry out the procedure of synthesizing the SMC law such that the estimated $\hat{x}(k)$ reaches the sliding surface even though uncertainties are present.

Theorem *3.2.* Consider the discrete-time non-linear singular system in (1) with all admissible uncertainties.

Suppose that the sliding function (29) is appropriately designed. Then, the discrete-time sliding mode reaching condition can be guaranteed with the following control law:

$$u(k) = K\hat{x}(k) - \mathbf{S}A_d\hat{x}(k-d(k)) - \kappa s(k) - \alpha(k)\,\mathrm{sat}(s(k)) \tag{38}$$

where

$$\alpha(k) = \rho\|\hat{x}(k)\| + \|\mathbf{S}\|\|L\|\|C_2\|\|e(k)\|$$
$$\mathrm{sat}(s(k)) = \begin{cases} \dfrac{s(k)}{\sigma}, & \|s(k)\| \le \sigma \\ \mathrm{sign}(s(k)), & \|s(k)\| > \sigma \end{cases} \tag{39}$$

κ and ε are positive given scalars,

Proof: Construct the Lyapunov function candidate for system (32) as

$$V_s(k) = \frac{1}{2}s^T(k)s(k) \tag{40}$$

Without loss of generality, we can choose $\mathbf{S} = B^+$. Thus, $\mathbf{S}B$ is non-singular. From (29), it follows that

$$\begin{aligned} s(k+1) =& u(k) - K\hat{x}(k) + \mathbf{S}LC_e e(k) + \mathbf{S}A_d\hat{x}(k-d(k)) \\ &+ f(k,\hat{x}(k)) + s(k) \end{aligned} \tag{41}$$

Then, we have

$$\begin{aligned} \Delta V_s(k) &= s^T(k)\Delta s(k) + \frac{1}{2}\Delta s^T(k)\Delta s(k) \\ &= s^T(k)\Big(u(k) - K\hat{x}(k) + \mathbf{S}LC_e e(k) \\ &+ \mathbf{S}A_d\hat{x}(k-d(k)) \\ &+ f(k,\hat{x}(k))\Big) + \frac{1}{2}\Delta s^T(k)\Delta s(k) \\ &\le -\kappa s^T(k)s(k) \\ &+ \|s(k)\|\|f(k,\hat{x}(k))\| + \|s(k)\|\|\mathbf{S}LC_e\| \\ &+ s^T(k)\alpha(k)\,\mathrm{sat}(s(k)) + \frac{1}{2}\Delta s^T(k)\Delta s(k) \end{aligned} \tag{42}$$

If $\|s(k)\| > \sigma$, with the controller in (39), we have $\Delta V_s(k) \le -\kappa s^T(k)s(k) + \frac{1}{2}\Delta s^T(k)\Delta s(k)$

If $\|s(k)\| < \sigma$, with the controller in (39), we have $\Delta V_s(k) \le -\kappa s^T(k)s(k) + \frac{\rho(k)}{\sigma}\Big(\sigma\|s(k)\| - \|s(k)\|^2\Big) + \frac{1}{2}\Delta s^T(k)\Delta s(k)$.

If $s(k)$ is in a bounded region with an equilibrium point, κ can be tuned appropriately to guarantee $\Delta V_s(k) < 0$. Correspondingly, $\Delta V_s(k)$ is bounded. Then, the reach motion of system (32) can be achieved by the observer-based SMC law in (39). □

4. A NUMERICAL EXAMPLE

To demonstrate the applicability of the proposed control scheme, we apply it to the following nonlinear bio-economic system

$$\begin{cases} \dot{x}_1(t) = -1.25x_1(t) + 0.15x_2(t) - 50x_3(t) - 0.01x_1^2(t) \\ -x_1(t)x_3(t) + u_1(t) \\ \dot{x}_2(t) = 0.3x_1(t) - 0.1x_2(t) \\ 0 = -0.75x_1(t) - 0.1x_1(t-d(t)) + x_1(t)x_3(t) + u_2(t) \end{cases} \tag{43}$$

Let the sampling period be $T_e = 0.05s$. The discrete-time descriptor model is obtained with the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9375 & 0.0075 & -2.5 \\ 0.025 & 0.995 & 0 \\ -0.0375 & 0 & 0 \end{bmatrix},$$
$$A_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.00375 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.05 & 0 \\ 0 & 0 \\ 0 & 0.05 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = [0\ 0\ 0.1]^T$$
$$N = [0.1\ 0.1\ 0]\,, \quad N_d = [0.1\ 0\ 0]\,.$$

It is pointed that the pair (E, A) is not causal since the $(3,3)th$ entry of A is 0. Set $d_m = 1$, $d_M = 4$, $\lambda_1 = 0.91$, $\lambda_2 = 0.1$, $\lambda_3 = 0.78$ and $\rho = 0.2$. Theorem 3.1 produces a feasible solution with the following gains:

$$K = \begin{bmatrix} -8.0676 & -0.90369 & 49.409 \\ 2.5606 & 0.18324 & -10.485 \end{bmatrix},$$
$$L = \begin{bmatrix} 1.8468 & -2.6945 \\ 0.11487 & 0.0021635 \\ -0.44792 & 0.50227 \end{bmatrix}$$

For simulation purpose, we set $x_0 = \begin{bmatrix} 0.5 & -0.3 & 0 \end{bmatrix}$ and $f(k, x(k)) = \begin{bmatrix} -0.01x_1^2(k) - x_1(k)x_3(k) \\ x_1(k)x_3(k) \end{bmatrix}$.

Let $\kappa = 0.25$ and $\sigma = 0.1$. Then, the reaching control law can be designed from Theorem 3.2. Figures 1-5 illustrate the simulation results. Figures 1 and 3 depict the responses of the system state and the observer, while Figures 4 and 5 demonstrate, respectively, the convergence behaviours of the control input $u(k)$ and the sliding function $s(k)$.

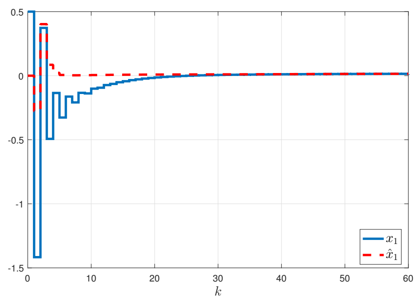

Fig. 1. State and estimation responses $x_1(t)$ and $\hat{x}_1(t)$

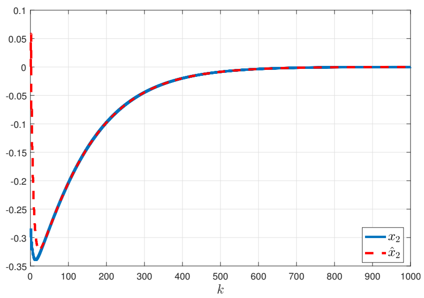

Fig. 2. State and estimation responses $x_2(k)$ and $\hat{x}_2(t)$

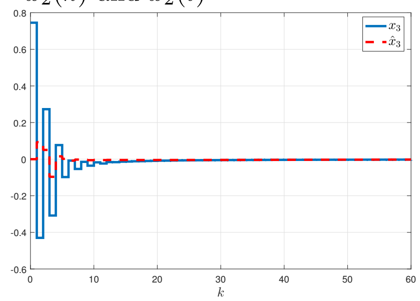

Fig. 3. State and estimation responses $x_3(k)$ and $\hat{x}_3(t)$

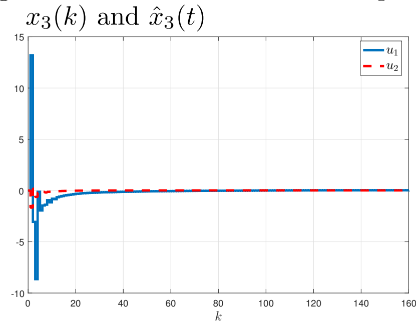

Fig. 4. Input response

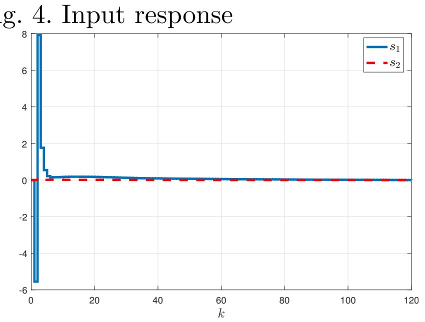

Fig. 5. Surface response

It can concluded that the developed observer-based sliding mode control law stabilizes the non-linear system with a good performance in spite of the time-varying delay and external disturbance, which explains the effectiveness of the proposed SMC law.

5. CONCLUSION

This study is a contribution to design an observer-based sliding mode controller for non-linear singular systems with time-varying delays. Based on a new discrete-time sliding surface, the question of robust admissibility for the sliding mode dynamics is considered and a sliding mode control law is synthesized to guarantee the closed-loop state trajectories to be stable, and to ensure the reachability of the specified sliding surface. The ensuing results have been verified by a numerical example.

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