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On consensus and collective behavior over heterogeneous temporal networks ⋆

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Abstract: We study the problem of self-coordination of a network of dynamical systems toward a common state, which has a wide range of applications, such as studying the emergence of collective behaviors in social, economical, and biological groups. Most of the literature on this topic focuses on static networks, challenging our mathematical understanding of coordination in temporal networks. Here, we expand the state of the art by studying consensus problems over temporal networks, modeled as activity driven networks. Such a modeling framework allows to include heterogeneity in the network, whereby some nodes are more involved in the process of information sharing than others. Through stochastic stability theory and eigenvalue perturbation techniques, we analyze the French-DeGroot consensus protocol over activity driven networks. We derive closed-form expressions for the expected consensus state and the rate of convergence in a mean-square sense, which points at a detrimental effect of moderate levels of heterogeneity for large-scale networks. Finally, we discuss the scenario in which there is a set of leaders that aim at steering the whole network to their state. Here, we demonstrate that heterogeneity may be beneficial to their objective. Simulations are conducted to support and illustrate our analytical findings.

Keywords: Consensus; Mean-square; Multiagent systems; Stochastic; Time-varying

1. INTRODUCTION

The consensus problem consists of a network of dynamical systems that coordinate toward a common state, executing a distributed algorithm. In view of its broad range of applications, encompassing opinion formation, distributed estimation, and multi-vehicle coordination, the consensus problem has received wide attention in the last decades; see, for instance, Ren and Beard (2008), Olfati-Saber et al. (2007), and Cao et al. (2013). However, most of the literature focuses on static networks, challenging our understanding of the emergence of coordination phenomena over networks of interactions that are inherently time-varying, such as those that characterizes most complex systems. Results on consensus problem on time-varying topologies have been investigated, for instance, in Olfati-Saber and Murray (2004) and Ren and Beard (2005).

Here, we expand the literature by studying the discrete-time consensus problem over time-varying, stochastic networks, modeled through activity driven networks (ADNs). In the framework of ADNs, introduced in Perra et al. (2012), each node is characterized by a fixed parameter, called activity potential, which encapsulates the node's propensity to communicate with its peers, thereby exchanging information with them. Briefly, the activity potential measures the probability that a node is activated in a time unit. Activated nodes generate a fixed number of ephemeral interactions, consistently with numerosity-constraints often observed in real-world complex networks (Dunbar, 1992; Tegeder and Krause, 1995). The distribution of the activity potentials across nodes models heterogeneity in individuals' behavior. ADNs are a powerful tool to study dynamical systems over complex networks. In fact, *i*) they allow for representing networks with a desired level of heterogeneity in the nodes' propensity to generate connections, in contrast with existing models of time-varying, stochastic networks (Abaid and Porfiri, 2011), and *ii*) they beget mathematical models

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that are analytically tractable and amenable to fast simulations (Perra et al., 2012).

Some preliminary endeavors toward a mathematical treatment of consensus problems over ADNs can be found in Buscarino et al. (2018) and in Ogura et al. (2018). Therein, results are mostly based on numerical simulations and on the assumption of a time-scale separation between the evolution of the network and the nodes' dynamics. Here, we build on these first endeavors toward a rigorous treatment of consensus over ADNs by leveraging stochastic stability theory and eigenvalue perturbation techniques. Through the derivation of a closed-form expression for the rate of convergence to consensus, we unveil the role of heterogeneity in the emergence of collective behaviors. Specifically, we prove that its effect is always detrimental for large-scale systems. Then, the explicit characterization of the expected consensus state allows to understand how nodes contribute to the collective behavior depending on their activity, pointing out that low-activity nodes have a preponderant role. Finally, we present the scenario in which some nodes act as leaders, with the purpose of steering the whole network to their state. In this case, our analysis concludes that leaders may benefit from the presence of heterogeneity in the network. Part of the technical proofs of the results presented in this extended abstract can be found in Zino et al. (2019) and in Hasanyan et al. (2020).

Notation

We gather here the notational convention used throughout the extended abstract. We denote as $\mathbb{R}$ and $\mathbb{Z}^+$ the sets of real and nonnegative integer numbers, respectively. The vector of all ones is denoted as $\mathbb{1}$. Given a vector x, x^T denotes its transpose and $||x||$ its Euclidean norm. I is the identity matrix. Matrices and vectors' dimensions are omitted when not necessary. Expected values of random variables are denoted as $\mathbb{E}[\cdot]$. We use Landau's symbol $O(x^k)$ to denote a generic function $f(x)$ such that $\limsup_{x\to 0} |f(x)/x^k| < \infty$.

2. ACTIVITY DRIVEN NETWORKS

We consider a set of n nodes connected through a temporal directed graph that evolves along a discrete-time index $k \in \mathbb{Z}^+$. Each node has activity potential $a_i \in (0, 1]$. The graph $\mathcal{G}_k$ is generated according to the following procedure, from $k = 0$, as illustrated in Fig. 1.

(1) At each unit time-step, every node i is activated with probability equal to a_i, independent of the others and of the past history of the process.
(2) If node i is activated at time k, then it generates $m \le n - 1$ directed links, connecting it with an m-tuple of nodes, selected uniformly at random among the remaining $n - 1$ nodes. Links are oriented from the activated node toward the selected nodes.
(3) The time index is increased by 1, all connections are deleted, and the whole process resumes to step (1).

We define the average activity potential and the standard deviation of the activity potential as

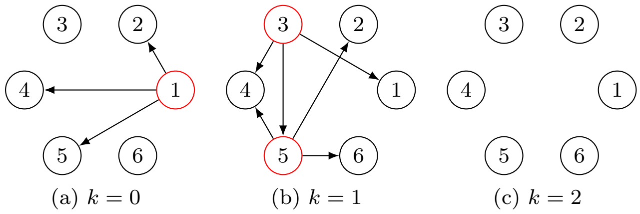


Fig. 1. Exemplary evolution of 3 time-steps of an ADN.

$$\bar{a} := \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad \sigma := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2}. \tag{1}$$

The standard deviation σ measures the heterogeneity in the nodes' activity potentials. When $\sigma = 0$, the ADN reduces to the model of conspecific agents proposed in Abaid et al. (2012). An alternative way of describing such a heterogeneity consists of separating the average activity, as follows:

$$a_i = \bar{a} + \sigma h_i, \tag{2}$$

where $h \in \mathbb{R}^n$ measures the deviation of each node from the average. Note that, by definition, $\mathbb{1}^T h = 0$ and $||h|| = \sqrt{n}$.

For any discrete time $k \in \mathbb{Z}^+$, we define the adjacency matrix of the time-varying network as $A_k \in \{0, 1\}^{n\times n}$, where $(A_k)_{ij} = 1 \iff (i, j) \in \mathcal{E}_k$, and the Laplacian matrix as $L_k := \mathrm{diag}(A_k\mathbb{1}) - A_k$.

3. CONSENSUS OVER ADNS

Each node i has a continuous state $x_i(k) \in \mathbb{R}$, which evolves according to a discrete-time consensus protocol starting from an initial condition $x_0 \in \mathbb{R}^n$. At each time-step, every node updates its state by averaging with the nodes with which it is temporarily connected, yielding the following time-varying linear system:

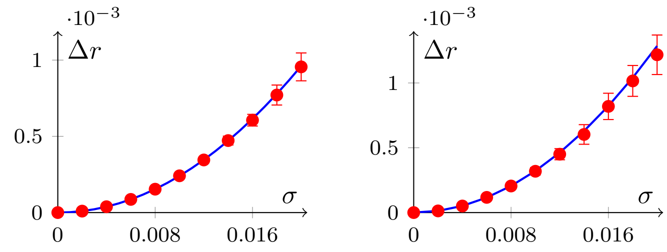
$$x(k + 1) = (I - \varepsilon L_k)x(k) := P_k x(k), \tag{3}$$

where the parameter $\varepsilon > 0$ is used to capture the nodes' tendency to compromise: the larger is ε, the more a node will favor the average state of the neighbors against its own during the updating process. We say that the consensus protocol converges to a consensus state $\bar{x}$ if $\lim_{k\to\infty} x(k) = \bar{x}\mathbb{1}$, that is, all nodes asymptotically attain the same state. The symbol $\mathbb{1}$ denotes the n-dimensional all-1 vector. Given that (3) is a stochastic system, convergence must be defined in a stochastic sense. Specifically, defining the disagreement vector as the difference between each network state and the average state of the network, that is, $\xi(k) := x(k) - \frac{1}{n}\mathbb{1}^T x(k)\mathbb{1}$, we quantify the speed of convergence of the protocol toward consensus by means of the asymptotic convergence factor of the disagreement dynamics (Zhou and Wang, 2009), that is,

$$r := \sup_{||\xi_0||\neq 0} \lim_{k\to\infty} \left(\frac{\mathbb{E}[||\xi(k)||^2]}{||\xi_0||^2}\right)^{1/k}. \tag{4}$$

The smaller r, the faster the convergence of the dynamics is. In Costa and Fragoso (2004), it is proved that $r < 1$ is a necessary and sufficient condition for convergence to consensus in a mean-square sense.

Building on the claims in Abaid and Porfiri (2011), in which the asymptotic convergence factor is written as the spectral radius of a $n^2 \times n^2$ matrix constructed from



(a) 30 nodes (b) 200 nodes

Fig. 2. Variation of the convergence factor with respect to the homogeneous case (Δr), for increasing values of σ, for two different choices of the model parameters with increasing network sizes. The numerical estimations performed over 100 independent runs (red circles) confirm our analytical prediction (blue curve). The average activity is equal to $\bar{a} = 0.1$ and vector h is generated uniformly at randomly under the constraints $\mathbb{1}^T h = 0$ and $||h|| = \sqrt{n}$. Consequently, the activity potential of node i is $a_i = 0.1 + \sigma h_i$.

the Laplacian of the ADNs L_k, we use a second-order eigenvalue perturbation argument to derive closed-form results for the rate of convergence to consensus as a function of the model parameters. The details of the analytical derivation and the explicit expression of r can be found in Zino et al. (2019). The derivation of a closed-form expression for r finds particular interest in large-scale applications, for which the numerical computation of the spectral radius of a $n^2 \times n^2$ matrix becomes unfeasible. For $n \to \infty$, our result reduces to

$$\begin{aligned} r = {} & 1 - 2\varepsilon \bar{a} m + \varepsilon^2 \bar{a} m(m+1) \\ & + \sigma^2 \frac{m(2 - \varepsilon(m+1))(2 - \varepsilon m)}{\bar{a}} + O(\sigma^3). \end{aligned} \tag{5}$$

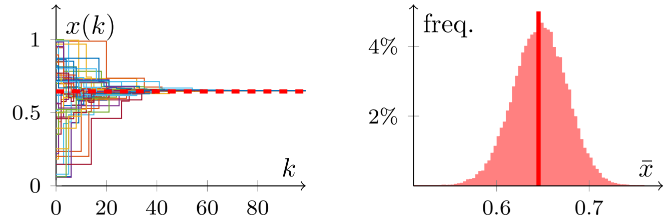
Since the coefficient of σ^2 is strictly positive for any choice of parameters for which $r < 1 + O(\sigma^3)$, (5) establishes that the convergence factor increases with the square of the standard deviation of the activity distribution, suggesting that the speed of convergence is hindered by the heterogeneity of the nodes' activities, at least for moderate levels of heterogeneity. Figure 2 illustrates the results of a campaign of Monte Carlo numerical simulations, which confirms our analytical predictions [1] .

Then, we use stochastic stability theory to characterize the expected value of the consensus state reached by the network nodes. Different from homogeneous systems, where the expected consensus state coincides with the average of the initial conditions, our analytical findings lead us to conclude that the consensus state is dominated by low-activity nodes. Specifically, we establish that the expected consensus state is equal to

$$\mathbb{E}[\bar{x}] = \pi^T x_0, \text{ with } \pi_i = \frac{a_i^{-1}}{\sum_{j=1}^n a_j^{-1}}. \tag{6}$$

Figure 3 shows numerical simulations of the evolution of a network of 50 dynamical systems, supporting our analytical results.

[1] The simulations presented in Figs. 2 and 4, where our analytical result is compared with numerical computations, are performed on small to medium systems, due to the numerical complexity of performing the numerical computations of the $n \times n$ matrix on large-scale systems.



(a) Sample path (b) Estimated consensus state

Fig. 3. Numerical simulations of the consensus dynamics over a heterogeneous network with 50 nodes. Panel (a) compares a sample path of the process with the predicted consensus state (red dashed line). Panel (b) illustrates the empirical distribution of the consensus values for set of Monte Carlo simulations over 50,000 independent runs from the same initial condition of the state variables, which is centered about the analytical prediction (red line) computed using (6). Initial conditions and activities are sampled independently one from the others from a uniform distribution over $[0, 1]$.

4. LEADER-FOLLOWER CONSENSUS OVER ADNS

Finally, we introduce a leader-follower dynamics in the collective behavior, which is typical of many human and animal collective phenomena (Dyer et al., 2009). We partition the network nodes into two sets, where $\ell \geq 1$ of them act as leaders: they are initialized with a common initial condition $s \in \mathbb{R}$ and never update it when interacting with other nodes. Leaders' goal is to steer the state of the whole network to their own state. The state of the followers, instead, evolves as the standard consensus dynamics in (3).

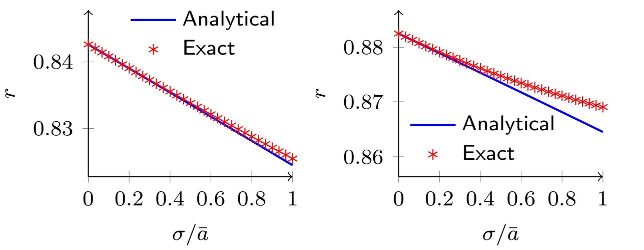
The literature on leader-follower consensus problems is vast, although the majority of existing studies assumes that the communication network among agents is fixed, or that it evolves according to a deterministic process (Cao et al., 2015; Hong et al., 2006). Here, we investigate on time-varying stochastic networks by studying the asymptotic behavior of leader-follower consensus over ADNs.

Following a procedure similar to the one adopted in the analysis of the standard consensus, we study convergence in a mean-square sense by utilizing a first-order eigenvalue perturbation argument. Despite the slightly different definition of the error dynamics, which is here defined as the difference of nodes' state with respect leaders' states as $\xi(k) = x(k) - s\mathbb{1}$, also for the leader-follower consensus it is possible to characterize its asymptotic convergence factor in (4) in terms of the spectral radius of a matrix, which depends on the Laplacian of the ADNs. Through its explicit study, performed in Hasanyan et al. (2020), we demonstrate that, in the presence of leaders, moderate level of heterogeneity among nodes could be beneficial to group decision-making, speeding up the convergence of the whole network to the leaders' state. Figure 4 validates our analytical predictions.

In the limit of large scale networks, that is $n \to \infty$, the asymptotic convergence factor r approaches

$$r = (1 - \varepsilon m \bar{a}(1 - \kappa))^2, \tag{7}$$

if



(a) $n = 18$, $\ell = 6$, $m = 3$ (b) $n = 60$, $\ell = 15$, $m = 4$

Fig. 4. Comparison between the analytical prediction of the asymptotic convergence factor (blue solid line) and its (exact) numerical computation (red stars), for different levels of heterogeneity and parameters settings. The average activity is equal to $\bar{a} = 0.1$ and vector h is generated uniformly at randomly under the constraints $\mathbb{1}^T h = 0$ and $||h|| = \sqrt{n}$. Consequently, the activity potential of node i is $a_i = 0.1 + \sigma h_i$.

$$\varepsilon \leq \frac{2\kappa}{\kappa + m - m\bar{a}(1-\kappa)^2}, \tag{8}$$

and

$$r = 1 - 2\varepsilon m\bar{a} + \varepsilon^2 m\bar{a}(m+\kappa) - 2\sigma\alpha\varepsilon^2 m^2\bar{a}\frac{(1-\kappa)^2(2-\kappa)}{\kappa}, \tag{9}$$

otherwise, where $\kappa = \ell/n$ is the fraction of leaders in the network. This result shows that the effect of the heterogeneity is nonnegligible when ε is sufficiently large. Moreover, if the parameter α, (which measures the overall followers' deviations from the average activity) is positive, then the heterogeneity becomes beneficial to the convergence process. Predictably, we observe that the favorable effect of heterogeneity increases as the number of leaders in the group and the average activity increase.

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