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Analysis of Quantized Consensus With Subtractive Dither ⋆

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Abstract: This paper presents a performance analysis of the average consensus for multi-agent systems, where the information exchange between agents in the system is quantized by subtractive dither method. The performance is evaluated by the distance from average. The estimated value depends on quantization noises and the correlation between separated edges.

Keywords: Multi-agent systems, Consensus, Quantization noise, Dither,

1. INTRODUCTION

Multi-agent systems consist of multiple subsystems called "agents," which form a network. The agents exchange their information with each other through the network, and they achieve a common task. Such systems are controlled distributedly, so it is convenient to extend the systems by additional agents. On the other hand, when the number of agents increases, the number of communication channels between the agents in a system increases more than the agents. Because of this reason, distributed control systems tend to consist of an enormous amount of communication traffic. To avoid such a situation, we consider signal quantization for communication between agents. If the information exchange in the system is quantized, the increase of the network traffic will be gentle.

For consensus problems, which is one of the fundamental problems of multi-agent system control, some previous results are found (Aysal et al. (2008); Cai et al. (2011); Kashyap et al. (2007)). Following these results, we consider the case that information exchange is quantized. In particular, this paper provides an analysis of the system where communications between agents are quantized by a subtractive dither method. It is one of signal quantization methods where an artificial noise-like signal is added before quantization, and the same signal is subtracted after quantization. The diagram of this structure is shown in Fig. 1. If $w_i \neq 0$ and $w_j = 0$, it is called "non-subtractive dither," and it is more widely used in various fields than subtractive dither ($w_i = w_j$). However, the subtractive dither method provides better performance for control systems if it can be used (Morita (2016)).

In addition, the structure in Fig. 1 is similar to a kind of encryption mechanism by regarding dither signals w_i and w_j as encryption keys. To consider the system like in such a way, we assume that agents in a system have their own "dither signals," which are created by the identity of the agents like hash functions of computer file systems. If all

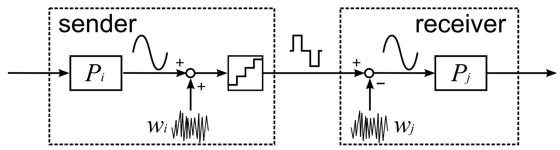

Fig. 1. Subtractive dither method.

of the agents in a system have the same key, the agents in the system are validated, and the system is expected to achieve consensus well because the information exchange of agents will work with a usual subtractive dither method. In the case where some agents are broken or intruded by external attackers, the dither signal of the agent will be an incorrect one, and then the subtractive dither method will not work well. As a result, the system will not achieve consensus.

According to this inspection, the author has evaluated the performance of quantized consensus with subtractive dither to use it for error detection (Morita et al. (2019)). In this previous work, we find that some correlation between separated edges contributes to the performance of consensus, but it has not been evaluated quantitatively. In this paper, we explicitly evaluate this influence. The effects by the correlation of multiple edges are not apparent when the communication noises have the same stochastic property, *e.g.,* information exchange is quantized by a non-subtractive dither method. In this paper, a subtractive dither method is adopted for quantization, and we assume to be not always $w_i = w_j$ of Fig. 1. This paper also considers the case $w_i \neq w_j$ and simulates different stochastic properties of quantization noises.

2. PROBLEM FORMULATION

A multi-agent system M with $N \in \mathbb{N}$ nodes is considered. The topology of M is given by the graph $G(V, E)$ where $V = \{1, 2, \dots, N\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. The graph G is assumed to be undirected and connected. The dynamics of the agent $i \in V$ of M is given by

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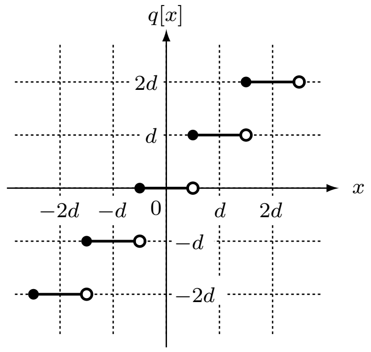


Fig. 2. Static uniform quantizer with quantization interval d

$$x_i(k+1) = x_i(k) + rv_i(k) \tag{1}$$

where $k \in \{0, 1, \ldots\}$ is the discrete time, $x_i(k) \in \mathbb{R}$ is the state of the agent i, $v_i(k) \in \mathbb{R}$ is the input for the agent i, and $r \in \mathbb{R}$ is a constant value. The input $v_i(k)$ is given by

$$v_i(k) = \sum_{j \in \mathcal{N}_i} v_{ij}(k), \tag{2}$$

$$v_{ij}(k) = q[x_j(k) - x_i(k) + w_j(k)] - w_i(k), \tag{3}$$

where $\mathcal{N}_i$ denotes the set of neighboring agents of i and the symbol $q[\cdot]$ denotes the static uniform quantization with interval $d \in \mathbb{R}$ which is shown in Fig. 2. In (3), $w_i(k) \in \mathbb{R}$ and $w_j(k) \in \mathbb{R}$ are the dither signals which are specific ones for agents i and j respectively. They are random variables with uniform distribution in $(-d/2,\ d/2]$ and i.i.d. with respect to k. When $w_i(k) = w_j(k)$, the input v_{ij} is said to be quantized with "subtractive dither" and when $w_i(k) = 0,\ w_j(k) \neq 0$, it is called "non-subtractive dither." We consider two situations for subtractive dither signal in this paper. The first is normal subtractive dither, that is, the case $w_i(k) = w^*(k), \forall i, k$. The second case is that only one agent of M, we call it "agent z", has a different dither signal w', and the other agent has the same dither signal w^*. In other words, $w_z(k) = w'(k)$, and $w_i(k) = w^*(k), i \in V \setminus z, \forall k$.

By this formulation, the inputs v_{ij} contain the unquantized information of the neighboring agents. This assumption is not appropriate from the viewpoint of quantized communication. If all of the state values are exchanged as discrete values, the quantization will be nested, and the system will be hard to analyze. Thus, we relax it to the situation that the unquantized relative states are available. This relaxation simulates vehicle formation control by agents with physical distance sensors. The quantization, in this case, works just a kind of encoding.

To evaluate the performance of the multi-agent system M, we introduce the average $\mu(k) \in \mathbb{R}$ of the states and the difference $\delta(k) \in \mathbb{R}^N$ from the average, that is,

$$\mu(k) := \frac{1}{N}\sum_{i=1}^{N} x_i(k), \qquad \delta(k) := \begin{bmatrix} x_1(k) - \mu(k) \\ x_2(k) - \mu(k) \\ \vdots \\ x_N(k) - \mu(k) \end{bmatrix}.$$

If $\lim_{k\to\infty}(x_i(k) - x_j(k)) = 0$ for any $i, j \in V$, system M reaches consensus and the consensus value will be the average of the states, that is, $x_i(k) \to \mu(k)$ if $k \to \infty$. In usual consensus problems, $\|\delta(k)\|_2$ is often used for a performance index but it is difficult to derive this directory in our formulation. Thus, we introduce another variable $\Delta(k) \in \mathbb{R}^N$ which is given by

$$\Delta(k) := \delta(k) - \delta'(k) \tag{4}$$

where $\delta'(k) \in \mathbb{R}^N$ is a difference from average state in a virtual system with the input which is given by $v_{ij}(k) = x_j(k) - x_i(k)$. Then, we define the performance index J as

$$J := \mathsf{E}[\|\Delta(k)\|_2^2] = \mathsf{E}[\|\delta(k) - \delta'(k)\|_2^2] \tag{5}$$

where $\mathsf{E}[\cdot]$ denotes the expectation value.

It is clear that $\lim_{k\to 0}\Delta(k) = \delta(k)$ so it is easy to compare the theoretical value of J and actual value of $\|\delta(k)\|_2$. Therefore, the purpose of this paper is to obtain the value of J. Note that the performance is evaluated by an expectation value because the system contains random value.

3. PERFORMANCE ANALYSYS

To evaluate the effects of quantization noise, we introduce "quantization error" $\xi_{ij} \in \mathbb{R}$ which satisfies

$$q[x_j(k) - x_i(k) + w_j(k)] - w_i(k) = x_j(k) - x_i(k) + \xi_{ij}(k). \tag{6}$$

In addition, we define

$$\xi_i(k) = \sum_{j \in \mathcal{N}_i} \xi_{ij}(k), \tag{7}$$

$$\Xi(k) = [\xi_1(k)\ \xi_2(k)\ \cdots\ \xi_N(k)]^\top. \tag{8}$$

By using these notation, the dynamics of system M is written in

$$x(k+1) = (I - rL)x(k) + r\Xi(k), \tag{9}$$

where $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is the graph Laplacian of $G(V, E)$, which is given by

$$l_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E \\ |\mathcal{N}_i| & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

By using (9), the difference $\delta(k)$ is expressed by

$$\delta(k+1) = (I - rL)\delta(k) + \frac{r}{N}L_C\Xi(k), \tag{11}$$

where $L_C \in \mathbb{R}^{N\times N}$ is a graph Laplacian of a complete graph with N nodes. Thus, we obtain

$$\delta(k) = (I-rL)^k\delta(0) + \frac{r}{N}\sum_{n=0}^{k-1}(I-rL)^{k-1-n}L_C\Xi(k). \tag{12}$$

On the other hand, $\delta'(k)$ satisfies

$$\delta'(k) = (I - rL)^k\delta(0). \tag{13}$$

By substituting (12) and (13) to (4), we obtain

$$\Delta(k) = \frac{r}{N}\sum_{n=0}^{k}(I - rL)^{k-1-n}L_C\Xi(k). \tag{14}$$

Next, to derive the value of the performance index J, we compute the value of $\mathsf{E}[\|\Delta(k)\|_2^2]$ and then we obtain

$$
\begin{aligned}
&\mathsf{E}[\|\Delta(k)\|_2^2] \\
&= \mathsf{E}\left[\left(\frac{r}{N}\sum_{n=0}^{k}(I-rL)^{k-1-n}L_C\Xi(k)\right)^{\top}\right.\\
&\qquad\qquad\left.\left(\frac{r}{N}\sum_{n=0}^{k}(I-rL)^{k-1-n}L_C\Xi(k)\right)\right]\\
&= \frac{r^2}{N^2}\mathsf{T}r\left(\mathsf{E}\left[\left(\sum_{n=0}^{k}(I-rL)^{k-1-n}L_C\Xi(k)\right)\right.\right.\\
&\qquad\qquad\left.\left.\left(\sum_{n=0}^{k}(I-rL)^{k-1-n}L_C\Xi(k)\right)^{\top}\right]\right)\\
&= \frac{r^2}{N^2}\mathsf{T}r\left(\sum_{n=0}^{k}(I-rL)^{k-1-n}L_C\mathsf{E}[\Xi(k)\Xi(k)^{\top}]\right.\\
&\qquad\qquad\left. L_C(I-rL)^{k-1-n}\right). \qquad (15)
\end{aligned}
$$

The above equation contains a random variable $\mathsf{E}[\Xi(k)\Xi(k)^{\top}]$ and the other parts are constants values determined by the graph of the system M.

In the situation that $w_i(k) = w^*(k),\ \forall i,k$ is satisfied, $\Xi(k)\Xi(k)^{\top}$ yields

$$
\begin{aligned}
\mathsf{E}[\Xi(k)\Xi(k)^{\top}] &= D\mathsf{E}[\xi_{ij}(k)^2]\\
&\quad + (D\mathbf{1}\mathbf{1}^{\top}D - D)\mathsf{E}[\xi_{ij}(k)\xi_{mn}(k)] \qquad (16)
\end{aligned}
$$

where D is the degree matrix of the system M. Note that there is a correlation between the edge (i,j) and (m,n).

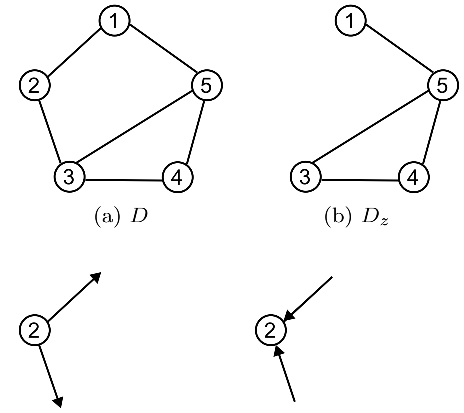
Next, we assume that only one agent of M has a different dither signal w' and the other agent has the same dither signal w^*. In addition, we introduce a special symbol $\xi_{i^*j'}(k)$ for a communication between agents with w^* and w', that is,

$$
\begin{aligned}
\xi_{i^*j'}(k) &= q[x_j(k) - x_i(k) + w'(k)]\\
&\quad - w^*(k) - (x_j(k) - x_i(k)). \qquad (17)
\end{aligned}
$$

This means $w_i(k) = w^*(k)$ and $w_j(k) = w'(k)$. In this situation, $\Xi(k)\Xi(k)^{\top}$ yields

$$
\begin{aligned}
&\mathsf{E}[\Xi(k)\Xi(k)^{\top}]\\
&= D_z\mathsf{E}[\xi_{i^*j^*}(k)^2] + D_{in}\mathsf{E}[\xi_{i'j^*}(k)^2] + D_{out}\mathsf{E}[\xi_{i^*j'}(k)^2]\\
&+ (D_z\mathbf{1}\mathbf{1}^{\top}D_z - D_z)\mathsf{E}[\xi_{i^*j^*}(k)\xi_{m^*n^*}(k)]\\
&+ (D_{in}\mathbf{1}\mathbf{1}^{\top}D_{in} - D_{in})\mathsf{E}[\xi_{i^*j'}(k)\xi_{s^*j'}(k)]\\
&+ (D_{out}\mathbf{1}\mathbf{1}^{\top}D_{out} - D_{out})\mathsf{E}[\xi_{i'j^*}(k)\xi_{i't^*}(k)]\\
&+ (D_{in}\mathbf{1}\mathbf{1}^{\top}D_z + D_z\mathbf{1}\mathbf{1}^{\top}D_{out})\mathsf{E}[\xi_{i^*j^*}(k)\xi_{m^*n'}(k)]\\
&+ (D_{out}\mathbf{1}\mathbf{1}^{\top}D_z + D_z\mathbf{1}\mathbf{1}^{\top}D_{out})\mathsf{E}[\xi_{i^*j^*}(k)\xi_{m'n^*}(k)]\\
&+ (D_{in}\mathbf{1}\mathbf{1}^{\top}D_{out} + D_{out}\mathbf{1}\mathbf{1}^{\top}D_{in})\mathsf{E}[\xi_{i'j^*}(k)\xi_{m^*i'}(k)], \quad (18)
\end{aligned}
$$

where D_z, D_{out}, D_{in} are degree matrices of some special graph defined by follows: when the agent z has the different dither signal w', D_z is for the graph where the agent z is omitted from the system M, D_{out} and D_{in} are is composed of only out-degree and in-degree of agent z, respectively. For example, if the system M is given by Fig. 3 (a), the degree matrices are



(a) D (b) D_z

(c) D_{out} (d) D_{in}

Fig. 3. Corresponding graphs of degree matrices D, D_z, D_{out}, and D_{in}

$$
D = \begin{bmatrix} 2&0&0&0&0\\ 0&2&0&0&0\\ 0&0&3&0&0\\ 0&0&0&2&0\\ 0&0&0&0&3 \end{bmatrix}, \qquad
D_z = \begin{bmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&2&0&0\\ 0&0&0&2&0\\ 0&0&0&0&3 \end{bmatrix},
$$

$$
D_{out} = \begin{bmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}, \qquad
D_{in} = \begin{bmatrix} 0&0&0&0&0\\ 0&2&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix},
$$

and the corresponding graphs are also shown in Fig. 3.

The products of the quantization errors in (18) are calculated as

$$
\begin{aligned}
\mathsf{E}[\xi_{i^*j^*}(k)^2] &= \frac{d^2}{12}, & (19)\\
-\frac{23d^2}{12} < \mathsf{E}[\xi_{i'j^*}(k)^2] &< \frac{d^2}{3}, & (20)\\
-\frac{23d^2}{12} < \mathsf{E}[\xi_{i^*j'}(k)^2] &< \frac{d^2}{3}, & (21)\\
-\frac{d^2}{24} < \mathsf{E}[\xi_{i^*j^*}(k)\xi_{m^*n^*}(k)] &< \frac{37d^2}{12}, & (22)\\
-\frac{23d^2}{12} < \mathsf{E}[\xi_{i^*j'}(k)\xi_{m^*j'}(k)] &< \frac{25d^2}{12}, & (23)\\
-\frac{23d^2}{12} < \mathsf{E}[\xi_{i'j^*}(k)\xi_{i'n^*}(k)] &< \frac{25d^2}{12}, & (24)\\
-\frac{d^2}{4} < \mathsf{E}[\xi_{i^*j^*}(k)\xi_{m^*n'}(k)] &< \frac{13d^2}{12}, & (25)\\
-\frac{d^2}{24} < \mathsf{E}[\xi_{i^*j^*}(k)\xi_{m'n^*}(k)] &< \frac{d^2}{2}, & (26)\\
-\frac{d^2}{4} < \mathsf{E}[\xi_{i'j^*}(k)\xi_{m^*i'}(k)] &< 2d^2. & (27)
\end{aligned}
$$

By (15), (16), (18), and (19)–(27), we can estimate performance of consensus with quantized communication once the topology of the system is given.

4. NUMERICAL SIMULATION

To validate the result in the previous section, numerical examples are shown in this part. For the system M, the number of agents is $N = 5$ and the topology of the

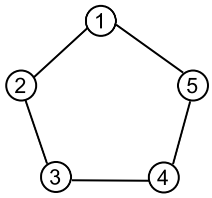


Fig. 4. Example of multi-agent system

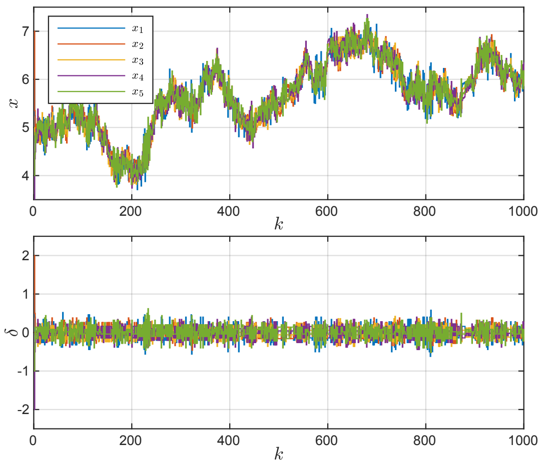


Fig. 5. Behavior of the system without irregular agents

network is given by a ring structure shown in Fig. 4. The constant value in (1) is $r = 0.5$ and the quantization interval is $d = 0.6$. The initial value of the system by $x(0) = [6\ 7\ 5\ 3\ 4]^\top$.

The behaviors of the system where $w_i = w\ \forall i$ are shown in Fig. 5. In the figure, the upper one indicates the state of agents and the lower one is the difference between each state and the average. If there is an irregular agent, *i.e.,* the case where $w_i = w, (i = 1, 2, 4, 5)$ and $w_3 \neq w$, the behavior of the system becomes like Fig. 6. We can find that the yellow line of $i = 3$ oscillates stronger than other agents.

In this case, the approximated value of J derived in the previous section is $J \approx 0.12$. Based on 10000 times trials with different dither signals, $\mathsf{E}[\|\delta(1000)\|_2^2] = 0.14$ for the system where $w_i = w,\ \forall i.$ and $\mathsf{E}[\|\delta(1000)\|_2^2] = 0.39$ for the system where $w_i = w, (i = 1, 2, 4, 5)$ and $w_3 \neq w$. Note that the system will achieve consensus at $k = 1000$ if the inputs are unquantized, that is, $\delta'(1000) = 0$. By this numerical example, we can confirm that the value of $\mathsf{E}[\|\delta(\infty)\|_2^2]$ will be close to the approximated value which is estimated in the previous section.

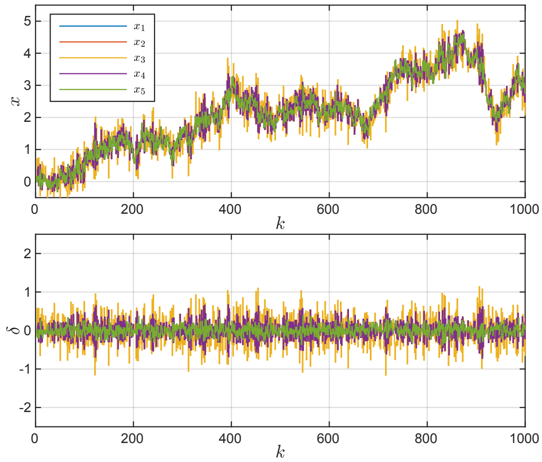


Fig. 6. Behavior of the system with an irregular agent

5. CONCLUSION

This paper has presented an analysis of quantized consensus problems with subtractive dither method. The effects of quantization noises depend on the topology of the system, and they are written by the subgraphs corresponded to the agents which produce the quantization noises. The main result (18) in this paper is a complicated formulation, so our future task is to simplify and extend to more general form.

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