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Quantized Consensus of Linear Multi-Agent Systems Under An Event-triggered Strategy ⋆

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Abstract: This paper addresses a quantized consensus problem of general linear multi-agent systems in a symmetric network under an event-triggered scheme. Firstly, a distributed event-triggered strategy is developed with a dynamic threshold to reduce the unnecessary control update. Then, based on absolute quantized state measurements, a distributed controller is proposed and then a consensus criterion is derived, which ensures bounded consensus of linear multi-agent systems. The Zeno behavior is also successfully excluded. Finally, a numerical simulation is presented to validate theoretical results.

Keywords: Multi-agent systems, quantized consensus, event-triggered strategy

1. INTRODUCTION

Over the past decade, consensus of multi-agent systems (MASs), as a fundamental problem of cooperative control, has attracted considerable attention due to its wide applications in unmanned air vehicles, robotic systems and sensor networks (Cao et al. (2013); He et al. (2017)). A crucial issue is how to design an efficient distributed control algorithm such that all agents are able to achieve a common quantity of interest.

Traditional control strategies assume that continuous measurements and continuous update of the controller are available (Li et al. (2013); He et al. (2020)), which is known as the time-scheduled strategy. However, due to the limitation of computation and storage in practice, the event-scheduled strategy has been proposed to save the limited resources. Many practical applications illustrate that event-scheduled strategy can remarkably reduce consumption of resources, such as sound control switches, temperature valves. Great efforts have been made to study the consensus of MASs under event-triggered strategies (Dimarogonas et al. (2012); Xu et al. (2017); Peng et al. (2017); Lv et al. (2018); He et al. (2019)). In 2012, Dimarogonas et al. (2012) studied the average consensus of MASs in which both centralized and distributed event-triggered schemes were proposed. Since then, different kinds of event-triggering strategies have been developed, such as model-based event-triggering strategies (Xu et al. (2017)), self-triggered event-triggering strategies (Peng et al. (2017)) and event-triggering strategies based on combinational measurements (Lv et al. (2018); Xu et al. (2019b)).

In additional, with the development of the digital communication technology, measurements needs to be quantized when communication constraints and communication load are taken into consideration (Delchamps (2018)). Studies on quantized consensus of discrete-time single-integrator systems were investigated based on uniform quantizer and logarithmic quantizer in Frasca et al. (2010) and Maestrelli et al. (2016), respectively. Continuous-time quantized consensus was solved by introducing the Krasovskii solution of discontinuous differential equation (DDE) to solve the discontinuity of the quantization error (Frasca (2010)). Bounded consensus of continuous-time and discrete-time single-integral systems with an undirected topology was addressed under logarithmic quantization, respectively (Liu et al. (2013)). It is worth noting that the results mentioned above focus on the single-integral systems. For general linear systems, an observer-based consensus protocol was proposed to guarantee quantized consensus of general linear systems in Ma et al. (2018).

Due to the communication constraints, it is essential to alleviate the resource consumption on communication and computation. The combination of the quantized control and the event-triggered strategy provides a promising way to solve the problem. Quantized consensus with event-triggering strategies of single-integral systems and general linear systems were investigated in Liu et al. (2016) and Wu et al. (2018), respectively. Quantization of relative state measurements was considered in above mentioned literature. However, in a remote control mode, the states need to be quantized before being transmitted to the neighbors in practice. Thus the controller can only obtain absolute quantized state measurements. Based on absolute logarithmic quantization, synchronization of master-slave

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system under a dynamic event-triggered strategy was addressed in He et al. (2019). Following this line, this paper further studies the quantized consensus of networked linear systems under a state-dependent dynamic event-triggered strategy based on absolute quantized state measurements.

In this paper, an effective event-triggered scheme and a consensus protocol based on absolute quantized state measurements are proposed firstly. Compared to the event-triggered condition in Wu et al. (2018), our proposed state-dependent triggering condition is more flexible. In addition, a sufficient condition of the quantized consensus for general linear systems is derived, under which bounded consensus is achieved, due to the quantization error, and the upper bound of the error is also given. Finally, the theoretical analysis for the exclusion of the Zeno behavior is presented.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section, some basic concepts about algebraic graph theory, logarithmic quantization and DDE are introduced.

2.1 Preliminaries

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ denote a graph consisting of a series of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$, a series of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $\mathcal{A} = (a_{ij})_{N\times N}$. An ordered pair of vertices (v_j, v_i) can be called an edge e_{ij} if and only if $a_{ij} > 0$, where $e_{ij} \in \mathcal{E}$. Particularly, a graph is called an undirected graph if and only if $e_{ij} \in \mathcal{E} \Leftrightarrow e_{ji} \in \mathcal{E}$, in which $\mathcal{A} = [a_{ij}]_{N\times N}$, $a_{ij} = a_{ji} = 1$ if $e_{ij} \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The in-degree matrix is denoted as $\mathcal{D} = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$, where $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. $\mathcal{N}_i = \{j \in \mathcal{V} \mid e_{ij} \in \mathcal{E}\}$ denotes the neighbor index set of agent i. The Laplacian matrix $\mathcal{L} = (l_{ij})_{N\times N}$ for graph $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

A map from $\mathbb{R}$ to a finite set Γ of quantized levels $q(\cdot) : \mathbb{R} \to \Gamma$ is called a quantizer. A uniform quantizer can be defined as follows

$$q(x) = \alpha\lfloor \frac{x}{\alpha} + \frac{1}{2}\rfloor \tag{1}$$

where $\alpha > 0$ is the gain of uniform quantizer. Define the quantization error as the deviation between $q(x)$ and x:

$$q(x) - x = \Delta_u. \tag{2}$$

From (1) and (2), we have $\Delta_u \in [-\frac{\alpha}{2}, \frac{\alpha}{2})$. Thus, the quantization error satisfies $|\Delta_u| \leq \frac{\alpha}{2}$ for all $x \in \mathbb{R}$.

Next, we will introduce the concept of the Krasovskii solution of DDE. For an ordinary function:

$$\dot{x}(t) = F(x(t)), \qquad x(0) = x_0 \in \mathbb{R} \tag{3}$$

where $F(x(t))$ is right hand discontinous. There may not exist a classic solution of (3). Thus, Ceragioli (2000) introduced the Krasovskii solution. For almost $t \in [t_0, t_1] \in \mathbb{R}$, if $F(x(t))$ satisfies the following differential inclusion

$$\frac{dx}{dt} \in \mathcal{K}[F(x(t))] \triangleq \bigcap_{\zeta>0} \bar{co}F(\mathcal{B}(x, \zeta)) \tag{4}$$

where $\bar{co}$ represents the convex closure, and $\mathcal{B}(x, \zeta)$ denotes an open ball centered at x, whose radius is ζ. Then an absolutely continuous map $x : [t_0, t_1] \to \mathbb{R}^d$ is defined as a Krasovskii solution of (3). Notably, the measurability and boundedness of the function $F(\cdot)$ is the necessary and sufficient condition of the existence of a local Krasovskii solution. For more details on DDE, it can be found in Ceragioli (2000) and references therein.

2.2 Problem Statement

Consider a group of agents comprised of N identical agents, which is described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \qquad i = 1, 2, \cdots, N. \tag{5}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of agent i, and $u_i(t) \in \mathbb{R}^r$ is the control input of agent i. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are given constant matrices.

Some assumptions and definitions are presented for subsequent analysis.

Assumption 1. The pair (A, B) is stabilizable.

Based on Assumption 1, for given positive constant ξ and θ, there exists a positive symmetric matrix $P \in \mathcal{R}^{n\times n}$ satisfying the following inequality

$$PA + A^TP - \xi PBB^TP + \theta I < 0. \tag{6}$$

Assumption 2. For the MAS (5), the communication graph $\mathcal{G}$ is undirected and connected.

The object of this paper is to design an effective quantized consensus protocol under an event-triggered scheme to achieve bounded consensus. The definition of bounded consensus is given as follows.

Definition 1. The linear multi-agent system (5) is said to be achieved bounded consensus if there exists a constant $\iota > 0$ such that

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| \leq \iota, \quad i, j = 1, 2, \cdots, N. \tag{7}$$

3. MAIN RESULTS

In this section, firstly, a consensus criterion of a general linear MAS is derived based on a proposed event-triggered consensus protocol with quantized absolute state measurements. Then the exclusion of the Zeno behavior is discussed.

3.1 A dynamic event-triggered scheme

A dynamic event-triggered strategy with the consideration of quantization is introduced in this subsection. Denote the event-triggering time sequence of the ith agent as t_k^i, $k = 0, 1, \ldots$. Consider the quantized combining measurement of agent i

$$\begin{aligned}\varphi_i(t) &= \sum_{j\in\mathcal{N}_i} [q(x_i(t)) - q(x_j(t))] \\ &= \sum_{j\in\mathcal{N}_i} [x_i(t) + \Delta_i(t) - x_j(t) - \Delta_j(t)] \end{aligned} \tag{8}$$

where $q(x_i(t))$ and $q(x_j(t))$ are the quantization of $x_i(t)$ and $x_j(t)$, $\Delta_i(t), \Delta_j(t)$ denote the quantized errors of agents i and j, respectively. The measurement error of agent i is defined as follows

$$e_i^q(t) = \varphi_i(t_k^i) - \varphi_i(t). \tag{9}$$

The event-triggering time sequence for agent i is given as

$$\begin{cases} t_0^i = 0, \\ t_{k+1}^i = \inf_{r>t_k} \{r | f(e_i^q(t), \varphi_i(t_k^i)) > 0, t \in [t_k, r)\} \end{cases} \tag{10}$$

where $f(e_i^q(t), \varphi_i(t_k^i))$ is the triggering function with the form

$$\begin{aligned} f(e_i^q(t), \varphi_i(t_k^i)) =& \|\Gamma\| \|e_i^q(t)\|^2 - \mu^2 \|\Gamma\| \|\varphi_i(t_k^i)\|^2 \\ & - \gamma e^{-\rho t} - \tau \end{aligned} \tag{11}$$

where $e_i^q(t)$ is the measurement error defined in (9), $\Gamma = PBB^TP$, $0 < \mu < 1, \rho > 0, \gamma > 0$ and $\tau > 0$, in which μ will be determined later.

Remark 1. The event-triggering function (11) depends on the quantized combining measurements $\varphi_i(t_k^i)$ of agent i, the exponential decay term $\gamma e^{-\rho t}$ and the constant τ. Compared with Wu et al. (2018), the triggering condition is relevant to the combining measurements, which is more flexible for MAS (5). It is worth noting that the threshold in the triggering condition is dynamic due to the exponential decay function, which could potentially reduce more communication resources compared with static one in Zhang et al. (2016).

The quantized event-triggered consensus protocol of agent i during two adjacent triggering instants $[t_k^i, t_{k+1}^i)$ is defined as

$$u_i(t) = K\varphi_i(t_k^i) \tag{12}$$

where K is the feedback gain matrix, which takes the form of $K = -B^TP$.

3.2 Consensus criteria

Taking (12) into (5), the following closed loop system can be obtained

$$\begin{aligned} \dot{x}_i(t) =& Ax_i(t) + BK\varphi_i(t_k^i) \\ =& Ax_i(t) + BK(e_i^q(t) + \varphi_i(t)), \quad t \in [t_k^i, t_{k+1}^i). \end{aligned} \tag{13}$$

Denote the average states of the MAS at t as $\bar{x}(t) = 1/N \sum_{i=1}^N x_i(t)$. Since $\mathbf{1}_N^T \mathcal{L} = 0$, one has

$$\begin{aligned} \dot{\bar{x}}(t) =& 1/N \sum_{i=1}^N (Ax_i(t) + BK\varphi(t_k^i)) \\ =& 1/N \sum_{i=1}^N (Ax_i(t) + BK(e_i^q(t) + \varphi_i(t))) \\ =& 1/N[(\mathbf{1}_N^T \otimes A)x(t) + (\mathbf{1}_N^T \otimes BK)e^q(t) \\ & + (\mathbf{1}_N^T \mathcal{L} \otimes BK)x(t) + (\mathbf{1}_N^T \mathcal{L} \otimes BK)\Delta(t)] \\ =& A\bar{x}(t) + 1/N(\mathbf{1}_N^T \otimes BK)e^q(t). \end{aligned} \tag{14}$$

Define $\delta_i(t) = x_i(t) - \bar{x}(t)$. Then (8) can be rewritten as

$$\varphi_i(t) = \sum_{j \in \mathcal{N}_i} [\delta_i(t) - \delta_j(t) + \Delta_i(t) - \Delta_j(t)]. \tag{15}$$

Taking the derivative of $\delta_i(t)$ for $t \in [t_k^i, t_{k+1}^i)$ yields

$$\dot{\delta}_i(t) = A\delta_i(t) + BK(e_i^q(t) + \varphi_i(t)) - 1/N(\mathbf{1}_N^T \otimes BK)e^q(t). \tag{16}$$

According to (15), one has

$$\begin{aligned} \dot{\delta}(t) =& (I_N \otimes A + \mathcal{L} \otimes BK)\delta(t) + (\mathcal{L} \otimes BK)\Delta(t) \\ & + ((I_N - 1/N\mathbf{1}_N\mathbf{1}_N^T) \otimes BK)e^q(t) \end{aligned} \tag{17}$$

where $\delta(t) = \mathrm{col}(\delta_1(t), \delta_2(t), \ldots, \delta_N)$, $\Delta(t) = \mathrm{col}(\Delta_1(t), \Delta_2(t), \ldots, \Delta_N(t))$, $e^q(t) = \mathrm{col}(e_1^q(t), e_2^q(t), \ldots, e_N^q(t))$, and $\varphi(t)$ has the compact form

$$\varphi(t) = (\mathcal{L} \otimes I_N)\delta(t) + (\mathcal{L} \otimes I_N)\Delta(t). \tag{18}$$

Note that (17) is a DDE. Hence according to (4), a complete Krasovskii solution to the following differential inclusion exists

$$\begin{aligned} \dot{\delta}(t) \in& (I_N \otimes A + \mathcal{L} \otimes BK)\delta(t) + (\mathcal{L} \otimes BK)\mathcal{K}(\Delta(t)) \\ & + (M \otimes BK)e^q(t) \end{aligned} \tag{19}$$

where $M = I_N - 1/N\mathbf{1}_N\mathbf{1}_N^T$.

Choose $\nu(t)$ satisfying $\nu(t) \in \mathcal{K}(\Delta(t))$. It is easy to conduct that $\|\nu(t)\| \leq \frac{Nn\alpha}{2}$. Thus (19) can be rewritten as

$$\begin{aligned} \dot{\delta}(t) =& (I_N \otimes A + \mathcal{L} \otimes BK)\delta(t) + (\mathcal{L} \otimes BK)\nu(t) \\ & + (M \otimes BK)e^q(t). \end{aligned} \tag{20}$$

Theorem 1. Under Assumptions 1 and 2, for given A, B, if there exist positive constants κ_1, κ_2 such that

$$1 - p - \frac{\kappa_2}{2}|1 - 2q| > 0 \tag{21}$$

where $p = \frac{\kappa_1}{2} + \frac{2\mu^2}{\kappa_1(1-\mu^2)}$ and $q = \frac{2\mu^2}{\kappa_1(1-\mu^2)}$. Then the MAS (5) achieves bounded consensus, and the error system (20) converges exponentially to the set $\mathcal{M}$

$$\mathcal{M} = \{\delta(t) \in \mathbb{R}^n | \|\delta(t)\| \leq \sqrt{\frac{2\lambda_{\max}(\mathcal{L} \otimes P)}{\theta \lambda_2^2 \lambda_{\min}(P)} \Omega}\} \tag{22}$$

where $\Omega = \tilde{q}\lambda_N^2 \|\Gamma\| (\frac{(N-1)n\alpha}{2})^2 + \frac{2N}{\kappa_1(1-\mu^2)}(\gamma + \tau)$, $\tilde{q} = q + \frac{2}{\kappa_2}$, λ_2 and λ_N denote the smallest nonzero eigenvalue and maximum eigenvalue of $\mathcal{L}$, respectively.

Proof. Consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}\delta^T(t)(\mathcal{L} \otimes P)\delta(t). \tag{23}$$

Since $K = -B^TP$, taking the derivative of $V(t)$ with respect to t yields

$$\begin{aligned} \dot{V}(t) =& \delta^T(t)(\mathcal{L} \otimes P)[(I_N \otimes A - \mathcal{L} \otimes BB^TP)\delta(t) \\ & - (\mathcal{L} \otimes BB^TP)\nu(t) - (M \otimes BB^TP)e^q(t)] \\ =& \delta^T(t)(\mathcal{L} \otimes PA - \mathcal{L}^2 \otimes PBB^TP)\delta(t) \\ & - \delta^T(t)(\mathcal{L}^2 \otimes PBB^TP)\nu(t) \\ & - \delta^T(t)(\mathcal{L}M \otimes PBB^TP)e^q(t). \end{aligned} \tag{24}$$

Note that $\mathcal{L}M = \mathcal{L}$. For a constant $\kappa_1 > 0$, the following inequality can be conducted

$$\begin{aligned} & - \delta^T(t)(\mathcal{L}M \otimes PBB^TP)e^q(t) \\ =& - \delta^T(t)(\sqrt{\frac{\kappa_1}{2}}\mathcal{L} \otimes PB)(\sqrt{\frac{2}{\kappa_1}}I_N \otimes B^TP)e^q(t) \\ \leq& \frac{\kappa_1}{2}\delta^T(t)(\mathcal{L}^2 \otimes PBB^TP)\delta(t) \\ & + \frac{2}{\kappa_1}(e^q(t))^T(I_N \otimes PBB^TP)e^q(t). \end{aligned} \tag{25}$$

Since $|\|\varphi_i(t_k^i)\|^2 - \|\varphi_i(t)\|^2| \leq \|e_i^q(t)\|^2$, according to (9), (10) and (11), we have

$$\|\Gamma\| \|\varphi_i(t_k^i)\|^2 \leq \frac{\|\Gamma\| \|\varphi_i(t)\|^2 + \gamma e^{-\rho t} + \tau}{1 - \mu^2} \tag{26}$$

According to (10) and (11), the inequality (25) can be rewritten as

$$
\begin{aligned}
&\frac{\kappa_1}{2}\delta^T(t)(\mathcal{L}^2\otimes PBB^TP)\delta(t)\\
&+\frac{2}{\kappa_1}(e^q(t))^T(I_N\otimes PBB^TP)e^q(t)\\
\leq&\frac{\kappa_1}{2}\delta^T(t)(\mathcal{L}^2\otimes PBB^TP)\delta(t)\\
&+\frac{2\mu^2}{\kappa_1(1-\mu^2)}\varphi^T(t)(I_N\otimes PBB^TP)\varphi(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau)\\
\leq&(\frac{\kappa_1}{2}+\frac{2\mu^2}{\kappa_1(1-\mu^2)})\delta^T(t)(\mathcal{L}^2\otimes PBB^TP)\delta(t)\\
&+\frac{2\mu^2}{\kappa_1(1-\mu^2)}[2\delta^T(t)(\mathcal{L}^2\otimes PBB^TP)\nu(t)\\
&+\nu^T(t)(\mathcal{L}^2\otimes PBB^TP)\nu(t)]\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau). \qquad (27)
\end{aligned}
$$

Substituting (27) into (24) yields

$$
\begin{aligned}
\dot V(t)\leq&\delta^T(t)(\mathcal{L}\otimes\frac{PA+A^TP}{2}-(1-p)(\mathcal{L}^2\otimes\Gamma))\delta(t)\\
&-(1-2q)\delta^T(t)(\mathcal{L}^2\otimes\Gamma)\nu(t)\\
&+q\nu^T(t)(\mathcal{L}^2\otimes\Gamma)\nu(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau) \qquad (28)
\end{aligned}
$$

where p, q is defined in Theorem 1.

Similar to (25), there exist $\kappa_2>0$ such that

$$
\begin{aligned}
&-(1-2q)\delta^T(t)(\mathcal{L}^2\otimes\Gamma)\nu(t)\\
\leq&|1-2q|[\frac{\kappa_2}{2}\delta^T(t)(\mathcal{L}^2\otimes\Gamma)\delta(t)\\
&+\frac{2}{\kappa_2}\nu^T(t)(\mathcal{L}^2\otimes\Gamma)\nu(t)]. \qquad (29)
\end{aligned}
$$

Thus, the inequality (28) can be rewritten as

$$
\begin{aligned}
\dot V(t)\leq&\delta^T(t)(\mathcal{L}\otimes\frac{PA+A^TP}{2}-\tilde p(\mathcal{L}^2\otimes\Gamma))\delta(t)\\
&+\tilde q\nu^T(t)(\mathcal{L}^2\otimes\Gamma)\nu(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau) \qquad (30)
\end{aligned}
$$

where $\tilde p=1-p-\frac{\kappa_2}{2}|1-2q|$, $\tilde q=q+\frac{2}{\kappa_2}$.

Under Assumption 1, denote $\lambda_1,\lambda_2,...,\lambda_N$ as the eigenvalues of matrix $\mathcal{L}$, satisfying $0=\lambda_1<\lambda_2\leq\lambda_3...\leq\lambda_N$. Since $\mathcal{L}$ is symmetric, there exists an orthogonal matrix U such that

$$
U^T\mathcal{L}U=J=diag\{\lambda_1,\lambda_2,...,\lambda_N\} \qquad (31)
$$

where U satisfies $U^TU=I$. We can conduct that $\mathcal{L}=UJU^T$. Define $\tilde\delta(t)=(U^T\otimes I_n)\delta(t)$ and $\tilde\nu(t)=(U^T\otimes I_n)\nu(t)$. Then inequality (30) is equivalent to

$$
\begin{aligned}
\dot V(t)\leq&\tilde\delta^T(t)(J\otimes\frac{PA+A^TP}{2}-\tilde p(J^2\otimes\Gamma))\tilde\delta(t)\\
&+\tilde q\tilde\nu^T(t)(J^2\otimes\Gamma)\tilde\nu(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau)\\
=&\sum_{i=2}^N\tilde\delta_i^T(t)(\lambda_i\frac{PA+A^TP}{2}-\tilde p\lambda_i^2\Gamma)\tilde\delta_i(t)\\
&+\tilde q\sum_{i=2}^N\tilde\nu_i^T(t)\lambda_i^2\Gamma\tilde\nu_i(t)+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau). \qquad (32)
\end{aligned}
$$

According to (6), one has

$$
\begin{aligned}
\dot V(t)\leq&-\sum_{i=2}^N\frac{\lambda_i\theta}{2}\tilde\delta_i^T(t)\tilde\delta_i(t)+\tilde q\sum_{i=2}^N\tilde\nu_i^T(t)\lambda_i^2\Gamma\tilde\nu_i(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau)\\
\leq&-\frac{\lambda_2\theta}{2}\sum_{i=2}^N\tilde\delta_i^T(t)\tilde\delta_i(t)+\tilde q\lambda_N^2\sum_{i=2}^N\tilde\nu_i^T(t)\Gamma\tilde\nu_i(t)\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma e^{-\rho t}+\tau)\\
\leq&-\frac{\theta}{2}\lambda_2\delta^T(t)\delta(t)+\tilde q\lambda_N^2\|\Gamma\|(\frac{(N-1)n\alpha}{2})^2\\
&+\frac{2N}{\kappa_1(1-\mu^2)}(\gamma+\tau)\\
=&-\frac{\theta}{2}\lambda_2\delta^T(t)\delta(t)+\Omega\\
\leq&-\frac{\theta\lambda_2}{\lambda_{max}(\mathcal{L}\otimes P)}V(t)+\Omega \qquad (33)
\end{aligned}
$$

where Ω is given in Theorem 1.

By the comparison principle and induction, it is conducted that

$$
\begin{aligned}
V(t)\leq&e^{-\frac{\theta\lambda_2}{\lambda_{max}(\mathcal{L}\otimes P)}t}V(\delta(0))\\
&+\frac{\lambda_{max}(\mathcal{L}\otimes P)}{\theta\lambda_2}\Omega(1-e^{-\frac{\theta\lambda_2}{\lambda_{max}(\mathcal{L}\otimes P)}t}). \qquad (34)
\end{aligned}
$$

When $\delta_1(t)=\delta_2(t)=...=\delta_N(t)$ is not satisfied, we have $\frac{1}{2}\lambda_2\lambda_{min}(P)\delta^T(t)\delta(t)\leq V(t)$. Since $\theta>0, P>0$, one obtains

$$
\begin{aligned}
\|\delta(t)\|\leq&e^{-\frac{\theta\lambda_2}{\lambda_{max}(\mathcal{L}\otimes P)}t}\sqrt{\frac{2V(\delta(0))}{\lambda_2\lambda_{min}(P)}}\\
&+\sqrt{\frac{2\lambda_{max}(\mathcal{L}\otimes P)}{\theta\lambda_2^2\lambda_{min}(P))}\Omega(1-e^{-\frac{\theta\lambda_2}{\lambda_{max}(\mathcal{L}\otimes P)}t})}. \qquad (35)
\end{aligned}
$$

Thus, the error $\delta(t)$ converges to the set M exponentially as $t\to\infty$. This completes the proof.

Theorem 2. Under the event-triggering condition (10), the Zeno behavior of the error system (17) is avoided if Theorem 1 is satisfied and $0<\tau<\|\Gamma\|(2d_{\max}n\alpha)^2$, where $d_{\max}$ represents the largest diagonal element of the in-degree matrix $\mathcal{D}$.

Proof. Since the quantized state measurement error e_i^q is not continuous due to quantization. Thus, define a continuous measurement error: $e_i(t)=\psi_i(t_k^i)-\psi_i(t), t\in[t_k^i,t_{k+1}^i)$, where $\psi_i(t)=\sum_{i\in\mathcal{N}_i}(x_i(t)-x_j(t))=\sum_{i\in\mathcal{N}_i}(\delta_i(t)-\delta_j(t))$. The relationship between $e_i^q(t)$ and $e_i(t)$ is given as

$$
\begin{aligned}
\|e_i(t)\|=&\|\psi_i(t_k^i)-\psi_i(t)\|\\
=&\|\sum_{j\in\mathcal{N}_i}[q(x_i(t_k^i))-q(x_j(t_k^i))-\Delta_i(t_k^i)+\Delta_j(t_k^i)]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j\in\mathcal{N}_i}[q(x_i(t)) - q(x_j(t)) - \Delta_i(t) + \Delta_j(t)]\| \\
=&\|e_i^q(t) - \sum_{j\in\mathcal{N}_i}[\Delta_i(t_k^i) - \Delta_j(t_k^i)] \\
&+ \sum_{j\in\mathcal{N}_i}[\Delta_i(t) - \Delta_j(t)]\|. \qquad (36)
\end{aligned}
$$

When the event-triggering condition (10) is fulfilled, $e_i(t)$ is reset to zero.

Taking the derivative of (36) for $t \in (t_k^i, t_{k+1}^i)$ yields

$$
\begin{aligned}
\frac{d\|e_i\|}{dt} =& \frac{e_i^T(t)\dot{e}_i(t)}{\|e_i(t)\|} \le \|\dot{e}_i(t)\| \\
=&\| \sum_{j\in\mathcal{N}_i}[\dot{\delta}_i(t) - \dot{\delta}_j(t)]\| \\
=&\| \sum_{j\in\mathcal{N}_i}[A(\delta_i(t) - \delta_j(t)) - BB^TP(\varphi_i(t_{k_i}^i) - \varphi_j(t_{k_j}^j))]\| \\
\le& \sum_{j\in\mathcal{N}_i}[\|A\|\|\delta_i(t) - \delta_j(t)\| \\
&+ \|BB^TP\|\|(\varphi_i(t_{k_i}^i) - \varphi_j(t_{k_j}^j))\|] \\
\le& d_{\max}[\|A\|(\|\delta_i(t)\| + \|\delta_j(t)\| \\
&+ \|BB^TP\|(\|\varphi_i(t_{k_i}^i)\| + \|\varphi_j(t_{k_j}^j)\|)] \qquad (37)
\end{aligned}
$$

where $t_{k_i}^i$ is equal to t_k^i, $t_{k_j}^j$ denotes the last triggering-instant of agent j before t, $d_{\max}$ is defined in Theorem 2.

According to (15), we have

$$
\begin{aligned}
\|\varphi_i(t_k^i)\| =&\| \sum_{j\in\mathcal{N}_i}[\delta_i(t) - \delta_j(t) + \Delta_i(t) - \Delta_j(t)]\| \\
\le& \sum_{j\in\mathcal{N}_i}[\|\delta_i(t) - \delta_j(t)\| + \|\Delta_i(t) - \Delta_j(t)\|] \\
\le& 2d_{\max}\|\delta(t)\| + d_{\max}Nn\alpha. \qquad (38)
\end{aligned}
$$

According to (33), it is obtained that $\|\delta(t)\|$ has the upper bound with $\Xi_2 = \sqrt{\frac{\lambda_{\max}(\mathcal{L}\otimes P)}{\lambda_2\lambda_{\min}P}}\|\delta(0)\| + \Xi_1$, where $\Xi_1 = \sqrt{\frac{2\lambda_{\max}(\mathcal{L}\otimes P)}{\theta\lambda_2^2\lambda_{\min}(P)}}\Omega$. Thus

$$\|\varphi_i(t_k^i)\| \le 2d_{\max}\Xi_2 + d_{\max}Nn\alpha = \Xi_3. \qquad (39)$$

Substituting (39) into (37) yields

$$\frac{d\|e_i\|}{dt} \le 2d_{\max}(\|A\|\Xi_2 + \|BB^TP\|\Xi_3) = \omega. \qquad (40)$$

For $t \in [t_k^i, t_{k+1}^i)$, one has

$$\int_{t_k^i}^{t}\frac{d\|e_i(t)\|}{dt} \le \omega(t - t_k^i). \qquad (41)$$

Then the following inequality holds

$$t - t_k^i \ge \frac{\|e_i(t)\| - \|e_i(t_k^i)\|}{\omega}. \qquad (42)$$

According to (10), (11) and (36), when the triggered condition (10) is satisfied, one has

$$
\begin{aligned}
\|e_i(t)\| - \|e_i(t_k^i)\| \ge& \|e_i^q(t)\| - 2d_{\max}n\alpha \\
\ge& \mu\|\varphi_i(t_k^i)\| + \tilde{\tau} - 2d_{\max}n\alpha \\
\ge& \tilde{\tau} - 2d_{\max}n\alpha \qquad (43)
\end{aligned}
$$

where$\tilde{\tau} = \sqrt{\frac{\tau}{\|\Gamma\|}}$. Thus

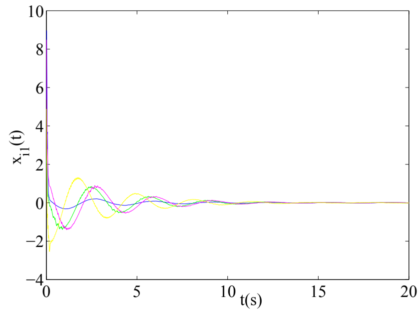

Fig. 1. Time evolutions of $x_{i1}(t), i = 1, 2, 3, 4$.

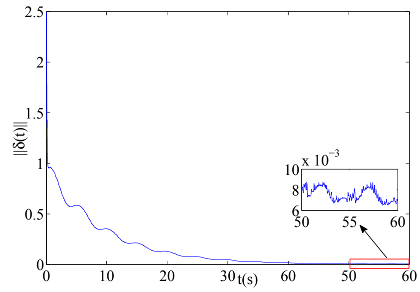

Fig. 2. The time evolution of $\|\delta(t)\|$.

$$t_{k+1}^i - t_k^i \ge \frac{\tilde{\tau} - 2d_{\max}n\alpha}{\omega}. \qquad (44)$$

If $\tau > \|\Gamma\|(2d_{\max}n\alpha)^2$, then $t_{k+1}^i - t_k^i > 0$. Thus, the Zeno behavior can be avoided. This completes the proof.

4. SIMULATIONS

In this section, a numerical example is provided to show the effectiveness of the proposed event-triggered strategy. Take the multi-agent system (5) as an example with

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The communication topology of four agents is described by Laplacian matrix

$$\mathcal{L} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

It is easy to compute that eigenvalues of $\mathcal{L}$ are $0, 1, 1, 4$.

The initial states are $x_1(0) = (2.5469, -2.2397, 1.797)^T$, $x_2(0) = (4.551, -0.3739, -0.81)^T$, $x_3(0) = (7.9836, 12.5974, 6.4039)^T$, and $x_4(0) = (11.8527, 8.2381, 13.5127)^T$. Select $\mu = 0.01$, $\gamma = 20$, $\rho = 0.1$ and $\tau = 0.04$ as parameters of (11). The parameter of the logarithmic quantizer (1) is chosen as $\alpha = 0.025$. Based on Theorem 1, one has

$$P = \begin{pmatrix} 19.0627 & 1.0594 & -2.0880 \\ 1.0594 & 15.5212 & 1.9698 \\ -2.0880 & 1.9698 & 10.4304 \end{pmatrix},$$

$$K = \begin{pmatrix} -16.9747 & -3.0292 & -8.3424 \end{pmatrix}.$$

Fig. 1 demonstrates the time evolutions of first variables of agents 1, 2, 3, 4. Fig. 2 shows the time evolution of the consensus error $\|\delta(t)\|$. It can be seen from Fig. 2

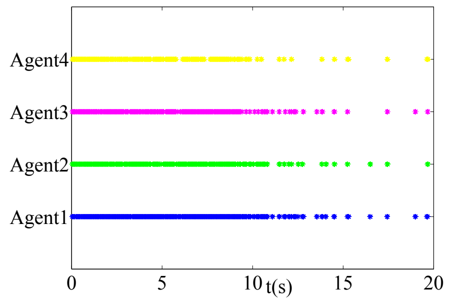

Fig. 3. Triggering time sequences of the ith agent($i = 1, 2, 3, 4$).

that the MAS (5) can achieve bounded consensus under the event-triggered consensus protocol (12). The triggering sequences of each agent is depicted in Fig. 3.

5. CONCLUSION

This paper has studied bounded consensus of multi-agent systems under an event-triggered strategy with absolute quantization of state measurements. First, a distributed dynamic event-triggered scheme has been constructed with the local quantized states of the neighboring agents. Then a sufficient consensus condition has been derived to guarantee bounded consensus with any given error, and meanwhile the Zeno behavior has been successfully excluded. Finally, a numerical example has been presented to verify the effectiveness of proposed consensus protocol. While, some issues such as quantized consensus of general linear systems under directed topologies, improvement of event-triggering conditions will be discussed in the near future.

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