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# Robust Output Agreement of Multi-Agent Systems with Flexible Topologies ⋆

**Zhanxiu Wang** * **Tengfei Liu** * **Zhong-Ping Jiang** **

* *State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, 110004, China (e-mail: zhanxiuwang@foxmail.com; tfliu@mail.neu.edu.cn).*
** *Department of Electrical and Computer Engineering, New York University, 370 Jay Street, Brooklyn, NY 11201, USA (e-mail: zjiang@nyu.edu)*

**Abstract:** This paper studies the robust output agreement problem for second-order multi-agent systems with flexible topologies subject to measurement disturbances. A new distributed control law is proposed to guarantee the robust output agreement in the sense of input-to-state stability (ISS) as long as the union of the interconnection graphs satisfies a standard connectivity condition. It is proved that, robust output agreement can be achieved in the presence of any bounded measurement disturbances if the functions of the distributed control laws are radially unbounded, and a local result can still be guaranteed if the condition of radial unboundedness is not satisfied. Numerical simulations are employed to show the effectiveness of the main result.

*Keywords:* Multi-agent systems, robust output agreement, measurement disturbance, flexible topology.

## 1. INTRODUCTION

Considerable efforts have been made on the distributed control of multi-agent systems to accomplish cooperative tasks, e.g., consensus, flocking, swarm, rendezvous and synchronization. Representative methods include algebraic graph theory in **?**, **?**, passive systems theory in **?**, Lyapunov stability theory in **?**, **?**, output regulation in **?**, **?** and small-gain approach in **?**.

The distributed control problem for agents with second-order dynamics has been mainly studied from the perspective of second-order consensus and flocking. Great efforts have been devoted to solving the problems under switching information exchange topologies. Related results include **?**, **?**, **?** and **?**. Specifically, **?** used potential functions to define Lyapunov functions and the topologies are allowed to be switching but undirected. **?** presented a consensus result for double-integrator systems based on a refined graph theoretical method. **?** studied circular formations which did not rely on any global information but required a directed cycle graph. Several recent results on distributed control can also be found in **?**, **?**, **?** and **?**. It should be pointed out that most of the papers mentioned above do not take into account disturbances, for which specific distributed nonlinear designs are expected. In **?**, the robust consensus problem of multi-agent systems with time-varying communication graphs subject to process noises is studied. However, the method used for such first-order multi-agent systems cannot be readily applied.

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This paper shows the validity of the controllers with nested loops for robust output agreement of multi-agent systems modeled by double-integrators. From a practical point of view, we assume that the double-integrators interact with each other through the interconnection between controllers, for coordination. In this paper, it is proved that robust convergence of the outputs can be achieved as long as the information exchange digraph satisfies a mild connectivity condition. Based on the control design, any bounded measurement disturbances can be handled if the functions of the distributed control laws are radially unbounded. If the condition of radial unboundedness is not satisfied, then a local robust output agreement result can be achieved.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first introduce some preliminaries about digraphs and comparison functions, and then give the problem formulation.

### 2.1 Notions and Preliminaries

*Basic Notations* The notations used in this paper are standard. We use $\mathbb{Z}_+$ to denote the set of all nonnegative integers. For any $\omega \in \mathbb{R}^N$, $\omega^T$ is its transpose and $|\omega|$ its Euclidean norm. For any function $x : \mathbb{R}_+ \to \mathbb{R}^N$, we denote $\|x\|_\infty = \sup\{|x(t)|, t \in [0, \infty)\} \leq \infty$. For two functions $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$, use $\gamma_1 \circ \gamma_2$ to represent the composition of the two functions. We use Id to denote the identity function defined on $\mathbb{R}_+$.

*Comparison Functions* For ease of presentation, two classes of functions are introduced. A function $\beta : \mathbb{R}_+ \times$

$\mathbb{R}_+ \to \mathbb{R}_+$ is called a class $\mathcal{I}^+\mathcal{L}$ function if $\beta \in \mathcal{KL}$, $\beta(s,0) = s$ for $s \in \mathbb{R}_+$, and for any specified $T > 0$, there exist continuous, positive definite and nondecreasing functions $\alpha_1, \alpha_2$ which depend on $T$ and less than the identity function, such that for all $s \in \mathbb{R}_+$, $\beta(s,t) \geq \alpha_1(s)$ for $t \in [0,T]$ and $\beta(s,t) \leq \alpha_2(s)$ for $t \in [T,\infty)$. A function $\beta : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is called a class $\mathcal{IL}$ function if there exist $\beta', \beta'' \in \mathcal{I}^+\mathcal{L}$ such that for $t \geq 0$, $\beta(s,t) = \beta'(s,t)$ for $s \geq 0$, and $\beta(s,t) = -\beta''(-s,t)$ for $s < 0$. The definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions can be found in **?**.

*Digraph* We begin by introducing some basic concepts from graph theory; see e.g., **?**. A digraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is called quasi-strongly connected (QSC) if there exists a $c \in \mathcal{N}$, referred as the center of $\mathcal{G}$, such that there is a directed path from $c$ to $i$ for each $i \in \mathcal{N}$. For a switching digraph $\mathcal{G}(t) = (\mathcal{N}, \mathcal{E}(t))$, we denote the union digraph over time interval $[t_1, t_2]$ as $\mathcal{G}([t_1,t_2]) = (\mathcal{N}, \bigcup_{t\in[t_1,t_2]} \mathcal{E}(t))$. A switching digraph $\mathcal{G}(t)$ is said to be uniformly quasi-strongly connected (UQSC) with time constant $T > 0$ if $\mathcal{G}([t, t+T])$ is QSC for all $t \geq 0$. A switching digraph $\mathcal{G}(t)$ has a positive edge dwell time $\tau_D > 0$ if for any $t \in [0,\infty)$ and for any directed edge $(i_1, i_2) \in \mathcal{E}(t)$, there exists a $t^* \geq 0$ depending on $t$ and $(i_1, i_2)$ such that $t \in [t^*, t^* + \tau_D]$ and $(i_1, i_2) \in \mathcal{E}(\tau)$ for $\tau \in [t^*, t^* + \tau_D]$. In this paper, the information exchange topology between the agents is modeled by $\mathcal{G}(t) = (\mathcal{N}, \mathcal{E}(t))$ with $\mathcal{N}$ being the set of the $N$ agents. Let $\mathcal{P}$ be a finite set representing all the possible information exchange topologies. For each $p \in \mathcal{P}$, if $j \in \mathcal{N}_i(p)$, then there is a directed edge $(j,i)$ belonging to $\mathcal{G}(\mathcal{P})$. By default, $(i,i)$ for $i \in \mathcal{N}$ belongs to $\mathcal{G}(p)$ for all $p \in \mathcal{P}$.

*Remark 1.* Consider a switching digraph $\mathcal{G}(\sigma(t)) = (\mathcal{N}, \mathcal{E}(\sigma(t)))$ with $\sigma : [0,\infty) \to \mathcal{P}$, which is UQSC with time constant $T > 0$ and has an edge dwell time $\tau_D > 0$. If $c \in \mathcal{N}$ is a center of $\mathcal{G}(\sigma([t, t+T]))$, then for any $\mathcal{N}_1$ such that $c \in \mathcal{N}_1$, there exist $i_1 \in \mathcal{N}_1$, $i_2 \in \mathcal{N}\backslash\mathcal{N}_1$, and $t_e \in [t - \tau_D, t + T]$ such that $(i_1, i_2) \in \mathcal{E}(\sigma(\tau))$ for $\tau \in [t_e, t_e + \tau_D]$.

### 2.2 Problem Formulation

This paper studies the robust output agreement problem for multi-agent systems with each agent $i$ $(i = 1, \ldots, N)$ described by second-order integrators:

$$\dot{\eta}_i = \zeta_i \tag{1}$$

$$\dot{\zeta}_i = \mu_i \tag{2}$$

where $[\eta_i, \zeta_i]^T \in \mathbb{R}^2$ is the state, and $\mu_i \in \mathbb{R}$ is the control input. In practice, $\eta_i$ and $\zeta_i$ usually represent the position and the velocity of agent $i$, respectively.

In the presence of position measurement disturbances, the distributed control law is in the following general form:

$$\mu_i = \overline{\varphi}_i(\zeta_i, \xi_i) \tag{3}$$

$$\xi_i = \overline{\phi}_i^{\sigma(t)}(\eta_i - \eta_1 - \omega_{i1}, \ldots, \eta_i - \eta_N - \omega_{iN}) \tag{4}$$

where $\omega_{ij} \in \mathbb{R}$ represents measurement disturbances, $\sigma : [0,\infty) \to \mathcal{P}$ is a switching signal with $\mathcal{P} \subset \mathbb{N}$ being a finite set representing all the possible information exchange topologies, and $\overline{\varphi}_i$ and $\overline{\phi}_i$ are appropriately designed functions for each $i = 1, \ldots, N$ and each $p \in \mathcal{P}$.

The objective of this paper is to design $\overline{\varphi}_i$ and $\overline{\phi}_i$ such that a robust output agreement objective is achieved, i.e., there exist $\gamma_1^\omega$, $\gamma_2^\omega \in \mathcal{K}$ such that

$$\lim_{t\to\infty} |\eta_i(t) - \eta_j(t)| \leq \gamma_1^\omega(\|\omega\|_\infty), \quad i,j = 1, \ldots, N \tag{5}$$

$$\lim_{t\to\infty} |\zeta_i(t)| \leq \gamma_2^\omega(\|\omega\|_\infty), \quad i = 1, \ldots, N \tag{6}$$

where $\omega = [\omega_{11}, \ldots, \omega_{1N}, \ldots, \omega_{N1}, \ldots, \omega_{NN}]^T$.

## 3. ROBUST OUTPUT AGREEMENT IN THE PRESENCE OF MEASUREMENT DISTURBANCES

This section focuses on the robust output agreement problem in the presence of measurement disturbances. We first propose several properties of the controlled agents in Subsection 3.1. Based on the properties, Subsection 3.2 presents the main result with proof given by Subsection 3.3.

### 3.1 Properties of the Controlled Agents

Consider a class of distributed control laws

$$\mu_i = \varphi_i(\zeta_i - \phi_i(\eta_i - \kappa_i)) \tag{7}$$

where $\phi_i, \varphi_i : \mathbb{R} \to \mathbb{R}$ are nonincreasing and globally Lipschitz functions, and $\kappa_i$ is the disturbed information available to coordination control of agent $i$, and is defined as

$$\kappa_i = \frac{\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\eta_j + \omega_{ij})}{\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}} \tag{8}$$

where $\mathcal{N}_i(p) \subseteq \{1, \ldots, N\}$ denotes the neighbor set of agent $i$ for each $i = 1, \ldots, N$ and each $p \in \mathcal{P}$, constant $a_{ij} > 0$ if $i \neq j$, and $a_{ij} \geq 0$ if $i = j$.

By substituting control law (7)–(8) into agent (1)–(2), we have

$$\dot{\eta}_i = \zeta_i, \tag{9}$$

$$\dot{\zeta}_i = \varphi_i(\zeta_i - \phi_i(\eta_i - \kappa_i)). \tag{10}$$

We first introduce a key proposition on the invariant set property of a class of controlled agents.

*Proposition 1.* For $i = 1, \ldots, N$, consider each controlled agent defined by (9)–(10) with $\varphi_i$ and $\phi_i$ satisfying

$$\varphi_i(0) = \phi_i(0) = 0, \tag{11}$$

$$\varphi_i(r)r < 0, \ \phi_i(r)r < 0 \quad \text{for } r \neq 0, \tag{12}$$

$$\sup_{r\in\mathbb{R}}\{\max \partial\varphi_i(r)\} < 4 \inf_{r\in\mathbb{R}}\{\min \partial\phi_i(r)\}. \tag{13}$$

There exist globally Lipschitz and strictly decreasing functions $\underline{\psi}_i, \overline{\psi}_i : \mathbb{R} \to \mathbb{R}$ satisfying $\underline{\psi}_i(0) = \overline{\psi}_i(0) = 0$, $\underline{\psi}_i(r) \leq \overline{\psi}_i(r)$ for $r \in \mathbb{R}$, $\lim_{r\to\infty} \underline{\psi}_i(r) = -\infty$ and $\lim_{r\to-\infty} \overline{\psi}_i(r) = \infty$ such that system (9)–(10) has the following properties:

(1) If $\kappa_i \in [\underline{\kappa}_i, \overline{\kappa}_i]$ with $\underline{\kappa}_i \leq \overline{\kappa}_i$ being constants, then
$$S_i(\underline{\kappa}_i, \overline{\kappa}_i) = \{(\eta_i, \zeta_i) : \underline{\psi}_i(\eta_i - \underline{\kappa}_i) \leq \zeta_i \leq \overline{\psi}_i(\eta_i - \overline{\kappa}_i)\}$$
is an invariant set.

(2) For any specific initial state $(\eta_i(0), \zeta_i(0))$, there exist constants $\underline{\mu}_i, \overline{\mu}_i \in \mathbb{R}$ such that
$$\underline{\psi}_i(\eta_i(0) - \underline{\mu}_i) \leq \zeta_i(0) \leq \overline{\psi}_i(\eta_i(0) - \overline{\mu}_i). \tag{14}$$

(3) There exist $\underline{\beta}_{i1}$, $\overline{\beta}_{i1} \in \mathcal{IL}$ which are radially unbounded with respect to the first argument, such that for any specific $\overline{\mu}_i$, $\underline{\mu}_i \in \mathbb{R}$, if $(\eta_i(t), \zeta_i(t)) \in S_i(\underline{\mu}_i, \overline{\mu}_i)$ for $t \in [0, T]$, then

$$\underline{\mu}_i - \underline{\beta}_{i1}(\underline{\mu}_i - \eta_i(0), t) \leq \eta_i(t) \leq \overline{\beta}_{i1}(\eta_i(0) - \overline{\mu}_i, t) + \overline{\mu}_i \quad (15)$$

for $t \in [0, T]$.

(4) For any specific compact $C \subset \mathbb{R}$, there exist $\underline{\beta}_{i2}$, $\overline{\beta}_{i2} \in \mathcal{IL}$ such that if $(\eta_i(0), \zeta_i(0)) \in S_i(\underline{\mu}_i(0), \overline{\mu}_i(0))$ with $\underline{\mu}_i(0) \leq \overline{\mu}_i(0)$ belonging to $C$ and $\kappa_i \in [\underline{\kappa}_i, \overline{\kappa}_i]$ with $\underline{\kappa}_i \leq \overline{\kappa}_i$ belonging to $C$, then there exist $\underline{\mu}_i(t)$ and $\overline{\mu}_i(t)$ satisfying

$$-\underline{\beta}_{i2}(\underline{\kappa}_i - \underline{\mu}_i(0), t) + \underline{\kappa}_i \leq \underline{\mu}_i(t) \leq \overline{\mu}_i(t) \leq \overline{\beta}_{i2}(\overline{\mu}_i(0) - \overline{\kappa}_i, t) + \overline{\kappa}_i \quad (16)$$

such that

$$(\eta_i(t), \zeta_i(t)) \in S_i(\underline{\mu}_i(t), \overline{\mu}_i(t)) \quad (17)$$

for all $t \geq 0$. If moreover $\phi_i$ is radially unbounded, then $\underline{\beta}_{i2}$ and $\overline{\beta}_{i2}$ can be chosen to be radially unbounded with respect to the first argument.

Due to space limitation, the basic idea of the proof of Proposition 1 is given by Remark 2 and the proof of which is placed in the technical report **?**.

*Remark 2.* With $\kappa_i$ considered as the external input of agent $i$, the basic idea of the proof is to show that if $\kappa_i$ is bounded, then controlled agent $i$ admits an invariant set with useful properties.

For convenience of discussions, use $\upsilon_i = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}\eta_j / \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}$ and $\omega_i = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}\omega_{ij} / \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}$ to denote the interaction between the agents and weighted measurement disturbances acting on agent $i$, respectively, then $\kappa_i$ defined by (8) can be rewritten as $\kappa_i = \upsilon_i + \omega_i$.

Proposition 2 gives an estimation on the "worst-case" divergence rate of the controlled agents.

*Proposition 2.* Consider the controlled multi-agent system with each agent defined by (9)–(10) with $\varphi_i$ and $\phi_i$ satisfying (11)–(13) for $i = 1, \ldots, N$. With $\underline{\psi}_i$, $\overline{\psi}_i$ and $S_i$ defined in Proposition 1, for all measurable and locally essentially bounded $\omega$, there exist $\underline{\mu}(t)$ and $\overline{\mu}(t)$ satisfying

$$\underline{\mu}(0) - M^0\|\omega\|_\infty t \leq \underline{\mu}(t) \leq \overline{\mu}(t) \leq \overline{\mu}(0) + M^0\|\omega\|_\infty t \quad (18)$$

such that

$$(\eta_i(t), \zeta_i(t)) \in S_i(\underline{\mu}(t), \overline{\mu}(t)) \quad (19)$$

$$\underline{\mu}(t) \leq \eta_i(t) \leq \overline{\mu}(t) \quad (20)$$

for all $t \geq 0$, where $M^0 = \max\left\{\frac{L^i_\phi L^i_\varphi}{L^s_{\underline{\psi}}}, \frac{L^i_\phi L^i_\varphi}{L^s_{\overline{\psi}}}\right\}$ with $\overline{\psi}(r) = \max_{i \in \mathcal{N}} \overline{\psi}_i(r)$, $\underline{\psi}(r) = \min_{i \in \mathcal{N}} \underline{\psi}_i(r)$, $L^s_{\underline{\psi}} = -\sup_{r \in \mathbb{R}_+}\{\max D^+\underline{\psi}(r)\}$, $L^s_{\overline{\psi}} = -\sup_{r \in \mathbb{R}_-}\{\max D^+\overline{\psi}(r)\}$, $L^i_\phi = -\inf_{r \in \mathbb{R}}\{\min_{i \in \mathcal{N}} \partial\phi_i(r)\}$, and $L^i_\varphi = -\inf_{r \in \mathbb{R}}\{\min_{i \in \mathcal{N}} \partial\varphi_i(r)\}$.

Here, $D^+$ denotes the Dini derivative. One may consult **?** for detailed discussions on Dini derivatives. Due to space limitation, the basic idea of the proof of Proposition 2 is given by Remark 3, and the proof of which is placed in the technical report **?**.

*Remark 3.* In the proof, the idea of the plane translational motion of the rigid body $\zeta_i = \overline{\psi}(\eta_i - \overline{\mu})$ is used; see e.g., **?**. Basic thought of the proof is to estimate an upper bound of the translational motion velocity of the rigid body.

### 3.2 Main Result

The main result of this paper is given by Theorem 1.

*Theorem 1.* Consider multi-agent system (1)–(2) with control law (7)–(8). For $i = 1, \ldots, N$, assume that $\phi_i$ and $\varphi_i$ are nonincreasing and globally Lipschitz, and satisfy (11)–(13).

- If $\mathcal{G}(\sigma(t)) = (\mathcal{N}, \mathcal{E}(\sigma(t)))$ with $\sigma : [0, \infty) \to \mathcal{P}$ is *UQSC* and has an edge dwell-time $\tau_D > 0$, then there exist $\beta_1, \beta_2 \in \mathcal{KL}$, $\gamma^\omega_1, \gamma^\omega_2 \in \mathcal{K}_\infty$ and constant $\rho > 0$ such that for all $\omega$ satisfying $\|\omega\|_\infty \leq \rho$, and for all $t \geq 0$,
$$|\eta_i(t) - \eta_j(t)| \leq \max\{\beta_1(|\eta_i(0) - \eta_j(0)|, t), \gamma^\omega_1(\|\omega\|_\infty)\}, \quad (21)$$
$$|\zeta_i(t)| \leq \max\{\beta_2(|\zeta_i(0)|, t), \gamma^\omega_2(\|\omega\|_\infty)\} \quad (22)$$
with $i, j = 1, \ldots, N$.
- If moreover $\phi_i$ is radially unbounded, then (21) and (22) hold for all measurable and locally essentially bounded $\omega$.

### 3.3 Proof of Theorem 1

Based on Proposition 1, the basic idea of the proof is to find appropriate $\underline{\mu}(t) \leq \overline{\mu}(t)$ such that the controlled multi-agent system admits properties in the form of (19) and (20), and the difference between $\overline{\mu}(t)$ and $\underline{\mu}(t)$, admits an ISS-like property with the measurement disturbance $\omega$ as the input.

Since $\underline{\psi}_i$ and $\overline{\psi}_i$ are radially unbounded, one can find $\underline{\mu}(0)$ and $\overline{\mu}(0)$ such that (19) and (20) hold for $i \in \mathcal{N}$ with $t = 0$.

We define two sets $\mathcal{O}_1$ and $\mathcal{O}_2$ as a partition of set $\mathcal{N}$ such that

$$\eta_i(0) \geq (\overline{\mu}(0) + \underline{\mu}(0))/2, \text{ for } i \in \mathcal{O}_1, \quad (23)$$

$$\eta_i(0) \leq (\overline{\mu}(0) + \underline{\mu}(0))/2, \text{ for } i \in \mathcal{O}_2. \quad (24)$$

It should be mentioned that either $\mathcal{O}_1$ or $\mathcal{O}_2$ can be an empty set, and the existence of the pair $(\mathcal{O}_1, \mathcal{O}_2)$ may not be unique. This does not influence the validity of the proof.

Define $T' = N(T + 2\tau_D + \Delta_T) + \Delta_T$ with any constant $\Delta_T > 0$. According to Proposition 2, there exist $\underline{\mu}(t)$ and $\overline{\mu}(t)$ satisfying (18) such that (19) and (20) hold for all $t \geq 0$. Define $M = M^0\|\omega\|_\infty$. And thus, for each $i \in \mathcal{O}_1$, $(\eta_i(t), \zeta_i(t)) \in S_i(\underline{\mu}(0) - MT', \overline{\mu}(0) + MT')$ holds for all $t \in [0, T']$. With property 3 in Proposition 1, there exists $\underline{\beta}_{i1} \in \mathcal{IL}$ which is radially unbounded with respect to the first argument such that

$$\begin{aligned}\eta_i(t) &\geq -\underline{\beta}_{i1}(\underline{\mu}(0) - MT' - \eta_i(0), t) + \underline{\mu}(0) - MT' \\ &\geq -\underline{\beta}_{i1}\left(\frac{\underline{\mu}(0) - \overline{\mu}(0)}{2} - MT', T'\right) + \underline{\mu}(0) - MT' \\ &=: \alpha^1_i(\overline{\mu}(0) - \underline{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' \quad (25)\end{aligned}$$

for $t \in [0, T']$. Clearly, $\alpha^1_i$ is of class $\mathcal{K}_\infty$ and less than Id.

Denote $N_2$ and $i^*$ as the number of elements of set $\mathcal{O}_2$ and the center of the union digraph $\mathcal{G}([0,T'])$, respectively. Due to symmetry, we only consider the case of $i^* \in \mathcal{O}_1$. Recursively define $\mathcal{I}_k = \{i_1, \ldots, i_k\}$ for $k = 1, \ldots, N_2$ such that

- there exist $t'_{i_1}$ satisfying $[t'_{i_1}, t'_{i_1} + \tau_D] \subseteq [0, T + 2\tau_D]$ and $l_1 \in \mathcal{O}_1$ such that $(l_1, i_1) \in \mathcal{E}(\sigma([t'_{i_1}, t'_{i_1} + \tau_D]))$;
- for $k = 2, \ldots, N_2$, there exist $t'_{i_k}$ satisfying $[t'_{i_k}, t'_{i_k} + \tau_D] \subseteq [(k-1)(T + 2\tau_D + \Delta_T), k(T + 2\tau_D + \Delta_T) - \Delta_T]$ and $l_k \in \mathcal{O}_1 \cup \mathcal{I}_{k-1}$ such that $(l_k, i_k) \in \mathcal{E}(\sigma([t'_{i_k}, t'_{i_k} + \tau_D]))$.

The existence of such $i_k$ is guaranteed by the UQSC property of $\mathcal{G}$, which is discussed in Remark 1. For convenience of notations, denote $\mathcal{I}_0 = \emptyset$.

Denote $\tilde{\mu}(t) = \overline{\mu}(t) - \underline{\mu}(t)$. Note that (25) holds for $i \in \mathcal{O}_1$ for all $t \in [0, T']$. It then follows that $\eta_i(t) \geq \alpha^1_{l_1}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT'$ holds for $i \in \mathcal{O}_1 \cup \mathcal{I}_0$ for all $t \in [0, T']$. We assume that there exists a function $\alpha^1_{l_k} < \mathrm{Id}$ which is continuous and positive definite such that $\eta_i(t) \geq \alpha^1_{l_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - (k-1)\|\omega\|_\infty$ holds for $i \in \mathcal{O}_1 \cup \mathcal{I}_{k-1}$ for all $t \in [(k-1)(T + 2\tau_D + \Delta_T), T']$. In what follows, we prove the existence of $\alpha^1_{l_{k+1}} < \mathrm{Id}$ which is continuous and positive definite such that $\eta_i(t) \geq \alpha^1_{l_{k+1}}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty$ holds for $i \in \mathcal{O}_1 \cup \mathcal{I}_k$ for all $t \in [k(T + 2\tau_D + \Delta_T), T']$.
1) Consider the motion of $(\eta_{i_k}, \zeta_{i_k})$ during the time interval $t \in [t'_{i_k}, t'_{i_k} + \tau_D] \subseteq [(k-1)(T + 2\tau_D + \Delta_T), k(T + 2\tau_D + \Delta_T) - \Delta_T]$. One can show that

$$\begin{aligned}
\upsilon_{i_k}(t) &= \frac{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}\eta_j(t)}{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}} \\
&= \frac{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))\setminus\{l_k\}} a_{i_kj}\eta_j(t) + a_{i_kl_k}\eta_{l_k}(t)}{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}} \\
&\geq \frac{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))\setminus\{l_k\}} a_{i_kj}(\underline{\mu}(0) - MT' - (k-1)\|\omega\|_\infty)}{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}} + \\
&\quad \frac{a_{i_kl_k}(\alpha^1_{l_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - (k-1)\|\omega\|_\infty)}{\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}} \\
&= \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - (k-1)\|\omega\|_\infty
\end{aligned}$$

for $t \in [t'_{i_k}, t'_{i_k} + \tau_D]$, with $\alpha^2_{i_k}(s) = a_{i_kl_k}\alpha^1_{l_k}(s)/\sum_{j\in\mathcal{N}_{i_k}(\sigma(t))} a_{i_kj}$ for $s \in \mathbb{R}_+$, where $\sigma(t)$ represents the fixed topology during the time interval. It can be directly checked that $\alpha^2_{i_k}$ is continuous, positive definite and less than Id.

Then, with property 4 of Proposition 1, there exists $\underline{\beta}_{i_k2} \in \mathcal{IL}$ such that

$$(\eta_{i_k}(t), \zeta_{i_k}(t)) \in S_{i_k}(\underline{\mu}'_{i_k}(t), \overline{\mu}(0) + MT') \tag{26}$$

holds with $\underline{\mu}'_{i_k}(t) = -\underline{\beta}_{i_k2}(\underline{\kappa}_{i_k} - \underline{\mu}_{i_k0}, t) + \underline{\kappa}_{i_k}$, $\underline{\kappa}_{i_k} = \min_{t'_{i_k} \leq t \leq t'_{i_k} + \tau_D}\{\upsilon_{i_k}(t) + \omega_{i_k}(t)\} \geq \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty$ and $\underline{\mu}_{i_k0} = \underline{\mu}(0) - Mt'_{i_k}$. Then,

$$\begin{aligned}
\underline{\mu}'_{i_k}(t) \geq &- \underline{\beta}_{i_k2}(\alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') - k\|\omega\|_\infty - M(T' \\
&- t'_{i_k}), t - t'_{i_k}) + \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) \\
&- MT' - k\|\omega\|_\infty \\
\geq &- \underline{\beta}_{i_k2}(\alpha^2_{i_k}(\tilde{\mu}(0) + 2MT'), t - t'_{i_k}) + \alpha^2_{i_k}(\tilde{\mu}(0) \\
&+ 2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty.
\end{aligned} \tag{27}$$

Thus, we have $\underline{\mu}'_{i_k}(t'_{i_k} + \tau_D) \geq (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty$ with $\alpha^3_{i_k}(s) = \underline{\beta}_{i_k2}(s, \tau_D)$ for $s \in \mathbb{R}_+$, where it can be verified that $\alpha^3_{i_k}$ is of class $\mathcal{K}$ and less than Id. Specially, when $\phi_i$ is radially unbounded, $\alpha^3_{i_k} \in \mathcal{K}_\infty$. By Lemma 1, it also holds that $(\mathrm{Id} - \alpha^3_{i_k}) \in \mathcal{K}_\infty$.
2) $t \in [t'_{i_k} + \tau_D, T']$. It holds that

$$\upsilon_{i_k}(t) \geq \underline{\mu}(0) - MT' - (k-1)\|\omega\|_\infty$$

for $t \in [t'_{i_k} + \tau_D, T']$. By using property 4 of Proposition 1, we can prove that (26) holds with

$$\underline{\mu}'_{i_k}(t) = -\underline{\beta}_{i_k2}(\underline{\kappa}'_{i_k} - \underline{\mu}'_{i_k0}, t) + \underline{\kappa}'_{i_k}$$

where $\underline{\kappa}'_{i_k} = \min_{t'_{i_k} + \tau_D \leq t \leq T'}\{\upsilon_{i_k}(t) + \omega_{i_k}(t)\} \geq \underline{\mu}(0) - MT' - k\|\omega\|_\infty$ and $\underline{\mu}'_{i_k0} = \underline{\mu}'_{i_k}(t'_{i_k} + \tau_D)$. Then, it follows that

$$\begin{aligned}
\underline{\mu}'_{i_k}(t) \geq &- \underline{\beta}_{i_k2}(-(\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT'), t - \\
&t'_{i_k} - \tau_D) + \underline{\mu}(0) - MT' - k\|\omega\|_\infty \\
\geq &\alpha^4_{i_k} \circ (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) \\
&- MT' - k\|\omega\|_\infty
\end{aligned} \tag{28}$$

where $\alpha^4_{i_k}(s) = \underline{\beta}_{i_k2}(s, T')$ for $s \in \mathbb{R}_+$, which is of class $\mathcal{K}$ and less than Id.
3) $t \in [k(T + 2\tau_D + \Delta_T), T']$. Property (20) implies that $\eta_{i_k}(k(T + 2\tau_D + \Delta_T)) \geq \underline{\mu}(0) - Mk(T + 2\tau_D + \Delta_T)$. By using property 3 of Proposition 1, we have

$$\begin{aligned}
\eta_{i_k}(t) \geq &- \underline{\beta}_{i_k1}(\alpha^4_{i_k} \circ (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + 2MT') - \\
&M(T' - k(T + 2\tau_D + \Delta_T)) - k\|\omega\|_\infty, t - k(T \\
&+ 2\tau_D + \Delta_T)) + \alpha^4_{i_k} \circ (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + \\
&2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty \\
\geq &(\mathrm{Id} - \alpha^5_{i_k}) \circ \alpha^4_{i_k} \circ (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}(\tilde{\mu}(0) + 2M \\
&T') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty
\end{aligned}$$

where $\alpha^5_{i_k}(s) = \underline{\beta}_{i_k1}(s, \Delta_T)$ for $s \in \mathbb{R}_+$. Since $\underline{\beta}_{i_k1} \in \mathcal{IL}$ is radially unbounded with respect to the first argument, one sees that $\alpha^5_{i_k} \in \mathcal{K}_\infty$ and less than Id. By also using Lemma 1, we have $(\mathrm{Id} - \alpha^5_{i_k}) \in \mathcal{K}_\infty$.

By defining $\alpha^1_{l_{k+1}} = (\mathrm{Id} - \alpha^5_{i_k}) \circ \alpha^4_{i_k} \circ (\mathrm{Id} - \alpha^3_{i_k}) \circ \alpha^2_{i_k}$, one sees that $\eta_i(t) \geq \alpha^1_{l_{k+1}}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - k\|\omega\|_\infty$ holds for $i \in \mathcal{O}_1 \cup \mathcal{I}_k$ for all $t \in [k(T + 2\tau_D + \Delta_T), T']$. Applying this reasoning repeatedly with $k = 1, \ldots, N_2$, and using property (25), one can show that

$$(\eta_i(T'), \zeta_i(T')) \in S_i(\underline{\mu}'_i(T'), \overline{\mu}(0) + MT')$$

holds for all $i \in \mathcal{N}$, with

$$\begin{aligned}
\underline{\mu}'_i(T') \geq &\alpha^4_i \circ (\mathrm{Id} - \alpha^3_i) \circ \alpha^2_i(\tilde{\mu}(0) + 2MT') \\
&+ \underline{\mu}(0) - MT' - N\|\omega\|_\infty
\end{aligned} \tag{29}$$

where $\alpha^2_i$ is continuous, positive definite and less than Id and $\alpha^3_i, \alpha^4_i \in \mathcal{K}$.

Define $\overline{\mu}(T') = \overline{\mu}(0) + MT'$ and $\underline{\mu}(T') = \tilde{\alpha}(\tilde{\mu}(0) + 2MT') + \underline{\mu}(0) - MT' - N\|\omega\|_\infty$ with $\tilde{\alpha} = \min_{i\in\mathcal{N}}\{\alpha^4_i \circ (\mathrm{Id} - \alpha^3_i) \circ \alpha^2_i, (\mathrm{Id} -$

$\alpha_i^5)\circ\alpha_i^4\circ(\mathrm{Id}-\alpha_i^3)\circ\alpha_i^2\}$. From the definitions of $\alpha_{i_k}^2$, $\alpha_{i_k}^3$, $\alpha_{i_k}^4$ and $\alpha_{i_k}^5$, we obtain that $\alpha_i^2$ and $(\mathrm{Id}-\alpha_i^3)$ are continuous, positive definite and less than Id, and $\alpha_i^4$ is of class $\mathcal{K}$ and $\mathrm{Id}-\alpha_i^5$ is of class $\mathcal{K}_\infty$. Then, $\tilde{\alpha}$ is continuous, positive definite and less than Id. Thus, we have

$$\tilde{\mu}(T') \le \tilde{\mu}(0) - \tilde{\alpha}(\tilde{\mu}(0) + 2MT') + 2MT' + N\|\omega\|_\infty.$$

Recall the definition of $M$. By recursively applying the reasoning above, one can show that

$$\begin{aligned}\tilde{\mu}((k+1)T') \le& \tilde{\mu}(kT') - \tilde{\alpha}(\tilde{\mu}(kT') + 2M^0T'\|\omega\|_\infty) \\ &+ (2M^0T' + N)\|\omega\|_\infty, k \in \mathbb{Z}_+. \quad (30)\end{aligned}$$

Define $\breve{\mu}(kT') = \tilde{\mu}(kT') + 2M^0\|\omega\|_\infty T'$ for $k \in \mathbb{Z}_+$. Then it follows from (30) that

$$\begin{aligned}\breve{\mu}((k+1)T') \le& \breve{\mu}(kT') - \tilde{\alpha}(\breve{\mu}(kT')) \\ &+ (2M^0T' + N)\|\omega\|_\infty, k \in \mathbb{Z}_+. \quad (31)\end{aligned}$$

Property (31) is in the form of standard ISS definition. If $\tilde{\alpha} \in \mathcal{K}_\infty$, then ISS property can be proved. The rest of the proof studies the ISS property by considering the following two cases regarding the radial unboundedness of $\phi_i$.

For the case that $\phi_i$ is bounded for some $i \in \mathcal{N}$, one can show that there exists a constant $\lambda$ such that $\lim_{r\to\infty}\sup\tilde{\alpha}(r) \ge \lambda$. Given a positive definite $\epsilon$ satisfying $(\mathrm{Id}-\epsilon) \in \mathcal{K}_\infty$, if

$$\|\omega\|_\infty \le \frac{(\mathrm{Id}-\epsilon)\circ\lambda}{2\max\left\{\frac{L_\phi^i L_\varphi^i}{L_{\overline{\psi}}^s}, \frac{L_\phi^i L_\varphi^i}{L_{\underline{\psi}}^s}\right\}T' + N} =: \rho, \quad (32)$$

we obtain

$$\begin{aligned}\breve{\mu}(kT') \ge& \gamma(\|\omega\|_\infty) \Rightarrow \\ &\breve{\mu}((k+1)T') - \breve{\mu}(kT') \le -\epsilon\circ\tilde{\alpha}(\breve{\mu}(kT'))\end{aligned}$$

where $\gamma(s) = \tilde{\alpha}^{-1}\circ(\mathrm{Id}-\epsilon)^{-1}((2M^0T'+N)s)$ for $s \in \mathbb{R}_+$. By a standard comparison lemma in **?**, there exists a $\beta \in \mathcal{KL}$ such that

$$\breve{\mu}(kT') \le \max\{\beta(\breve{\mu}(0), kT'), \gamma(\|\omega\|_\infty)\} \quad (33)$$

holds for all $k \in \mathbb{Z}_+$.

By (18), one sees that $\tilde{\mu}(t) \le \tilde{\mu}(kT') + 2M^0T'\|\omega\|_\infty$ for all $t \in [kT', (k+1)T')$. This together with the definition of $\breve{\mu}(kT')$ yields

$$\tilde{\mu}(t) \le \max\{\beta(\tilde{\mu}(0) + 2M^0T'\|\omega\|_\infty, t), \gamma(\|\omega\|_\infty)\} \quad (34)$$

for all $t \ge 0$.

Consider the case that $\phi_i$ is radially unbounded for all $i \in \mathcal{N}$. By using properties 3 and 4 in Proposition 1, one can prove that $\alpha_i^2$, $(\mathrm{Id}-\alpha_i^3)$, $\alpha_i^4$ and $(\mathrm{Id}-\alpha_i^5)$ are of class $\mathcal{K}_\infty$ and less than Id. Thus, there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that (31) holds. Clearly, (34) holds for all measurable and locally essentially bounded $\omega$.

From the discussions above, it always holds that (19)–(20). Properties (21) and (22) can be proved as $\tilde{\mu}$ satisfies (34). This ends the proof of Theorem 1.

## 4. NUMERIAL SIMULATION

In this section, numerical simulation examples are employed to verify the main results. We consider a group of six agents with indices $1,\dots,6$.

**Case 1.** $\phi_i$ is bounded for some $i \in \mathcal{N}$. We choose $\varphi_i(r) = -6r$ and $\phi_i(r) = -\operatorname{sgn}(r)\min\{0.3, 0.3r\}$ for $i = 1,\dots,6$. Here, sgn represents the sign function. With direct calculation, it can be verified that the selected $\varphi_i$ and $\phi_i$ satisfy (13).

In the simulation, the initial states of the agents are chosen as: $\eta_1(0) = 0$, $\eta_2(0) = -24$, $\eta_3(0) = 36$, $\eta_4(0) = -40$, $\eta_5(0) = 35$, $\eta_6(0) = 20$ and $\zeta_i(0) = 0$ for all $i = 1,\dots,6$.

The information exchange topology switches between six digraphs $\mathcal{G}_i$ $(i = 1,\dots,6)$ with their links defined as: $\mathcal{G}_1 = \{(1,2),(3,4),(4,5),(6,1)\}$; $\mathcal{G}_2 = \{(2,3),(6,5),(6,1)\}$; $\mathcal{G}_3 = \{(3,4),(5,6),(6,1)\}$; $\mathcal{G}_4 = \{(2,3),(6,4),(6,1)\}$; $\mathcal{G}_5 = \{(1,2),(6,4),(6,1)\}$; $\mathcal{G}_6 = \{(3,2),(4,5),(1,6),(6,1)\}$. The switching sequence is shown in Figure 1. Figures 2 and 3 show the state trajectories of the agents in the presence of measurement disturbances $\omega_i(t) = 0.1\sin(0.01t)$ and $\omega_i(t) \equiv 2$, respectively, which are in accordance with the local robust output agreement results given by Theorem 1.

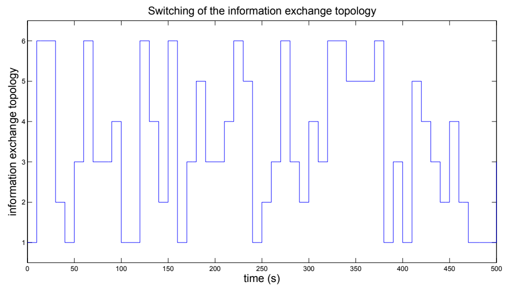


Fig. 1. The switching sequence of the information exchange topology.

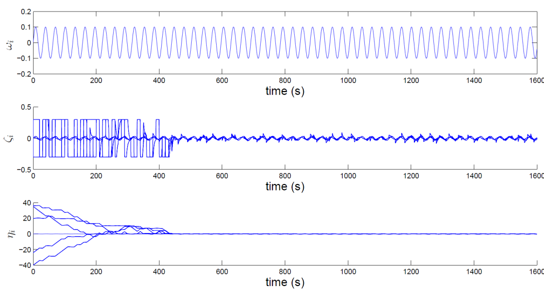


Fig. 2. The state trajectories of the agents with $\omega_i(t) = 0.1\sin(0.01t)$.

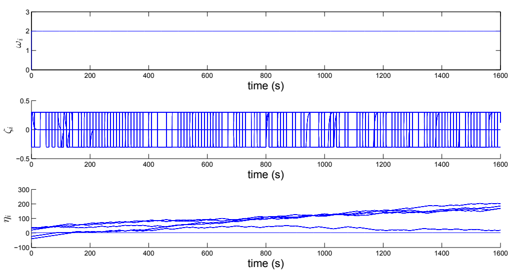


Fig. 3. The state trajectories of the agents with $\omega_i(t) \equiv 2$.

**Case 2.** $\phi_i$ is radially unbounded for all $i \in \mathcal{N}$. We choose functions $\varphi_i(r) = -6r$ and $\phi_i(r) = -0.3r$ for $i = 1,\dots,6$.

The initial states of the agents are the same with the first case.

Figure 4 shows the state trajectories of the agents subject to measurement disturbances $\omega_i(t) = 6\sin(0.01t)$, which is in accordance with the robust output agreement result given by Theorem 1.

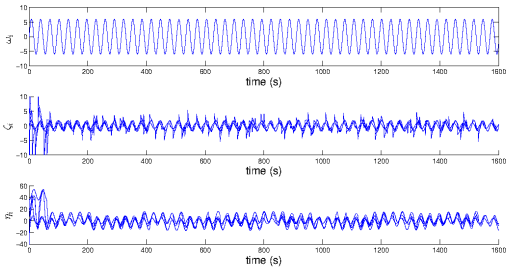

Fig. 4. The state trajectories of the agents with $\omega_i(t) = 6\sin(0.01t)$.

## 5. CONCLUSIONS

This paper has studied the robust output agreement problem for multi-agent systems with flexible topologies subject to measurement disturbances. A class of nonlinear distributed control laws has been proposed for robust output agreement in the sense of ISS, as long as the switching interconnection digraph satisfies a mild connectivity condition. It is proved that, robust output agreement can be guaranteed in the presence of any bounded measurement disturbances if the functions of the distributed control laws are radially unbounded, while a local result can still be achieved if the condition of radial unboundedness is not satisfied.

## Appendix A. A TECHNICAL LEMMA

*Lemma 1.* Consider the initial value problem

$$\dot{\varsigma} = \phi(\varsigma), \quad \varsigma(0) = \varsigma_0 \tag{A.1}$$

where $\varsigma \in \mathbb{R}$ is the state, and $\phi$ is nonincreasing and locally Lipschitz and satisfies $\phi(0) = 0$ and $r\phi(r) < 0$ for all $r \neq 0$. For $\varsigma_0 \in \mathbb{R}$ and $t \geq 0$, denote $\bar{\varsigma}(\varsigma_0, t)$ as the solution. Then, $\bar{\varsigma} \in \mathcal{IL}$. If moreover there exists a $k_\phi > 0$ such that $|\phi(r)| \leq k_\phi |r|$ for all $r \in \mathbb{R}$ and $\phi$ is radially unbounded, then $\bar{\varsigma}$ and $\varsigma_0 - \bar{\varsigma}$ are radially unbounded with respect to $\varsigma_0$.

Due to space limitation, the proof of Lemma 1 is placed in the technical report **?**.