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Extremum Seeking for Nonlinear Uncertain Systems: A Small-Gain Synthesis

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Abstract: This paper studies the extremum seeking problem for static maps with the inputs of the maps generated by a nonlinear uncertain system. A new small-gain approach is developed which uses an extremum seeking strategy to generate a reference signal, and employs a control law for reference-tracking of the nonlinear uncertain systems. The notions of input-to-state stability (ISS) and input-to-output stability (IOS) are used to characterize the interconnection between the extremum seeking strategy and the reference-tracking controller, and the nonlinear small-gain theorem is employed to guarantee the stability of the closed-loop extremum seeking system. With the proposed approach, the extremum seeking problem for a complex nonlinear system is solvable as long as one can design a proper reference-tracking controller for the system. Examples are given to show the feasibility of the proposed approach, and a numerical simulation is employed to show the effectiveness of the proposed design.

Keywords: Extremum seeking; small-gain theorem; semi-global practical stabilization.

1. INTRODUCTION

The study of extremum seeking can be traced back to the work of Leblanc (1922), when the technique was popularly applied to control applications. By using input-output data, extremum seeking aims to solve the optimum searching problem for uncertain plants through online tuning of parameters or setpoints. In the past two decades, there is a renewed interest in investigating its underlying theory and applications to real-time optimization of nonlinear control systems Ariyur and Krstić (2003); Scheinker and Krstić (2017).

By taking advantage of the newly developed tools in nonlinear and adaptive control theory, a lot of efforts have been devoted to improving the capability of extremum seeking by enlarging the range of operations from local to semi-global and global, and dealing with plants with more complex dynamics. The tools including singular perturbation and averaging Krstić and Wang (2000); Kutadinata et al. (2017), Lyapunov methods Haring and Johansen (2017), small-gain theorem Tan et al. (2006), stochastic analysis Manzie and Krstić (2009); Liu and Krstić (2010), time-varying estimation Guay and Dochain (2015) and Lie algebra Dürr et al. (2017); Labar et al. (2019) have been introduced to the literature.

This paper studies the extremum seeking problem for static maps with the inputs of the maps generated by nonlinear dynamical systems. It is shown that the problem can be solved by using an extremum seeking strategy to generate a reference signal, and designing a control law for the nonlinear uncertain system to track the reference signal. Based on this idea, the closed-loop extremum seeking system can be considered to be composed of two parts, extremum seeking for a static map and reference-tracking of the nonlinear uncertain system. The two parts interact with each other through the reference signal and the reference-tracking error, which leads to the major difficulty for synthesis. In this paper, the notions of input-to-state stability (ISS) Sontag (1989) and input-to-output stability (IOS) Jiang et al. (1994) are employed to characterize the robustness of the extremum seeking algorithm with respect to reference-tracking error, and the robustness of the reference-tracking capability with respect to the changes of reference signals. Moreover, nonlinear gains are used to describe the interconnections, and the nonlinear small-gain theorem Jiang et al. (1994) is employed to guarantee the stability of the closed-loop extremum seeking system, see Sontag (2007) and Jiang and Liu (2018) for tutorials of ISS and the nonlinear small-gain theorem.

In this paper, an assumption is used to represent the reference-tracking capability of the controlled systems, which is weaker than the conditions in many of the existing results, such as Krstić and Wang (2000); Ariyur and Krstić (2003); Tan et al. (2006); Guay and Dochain (2015). This

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makes the results of this paper more applicable. We also give three examples to show how to make this assumption satisfied for general systems.

The rest of the paper is organized as follows. Section 2 gives the problem formulation. The main result is proposed in Section 3. Section 4 gives three examples on the validity of the main assumption on the reference-tracking capability. In Section 5, a numerical example is employed to show the effectiveness of the proposed design. Section 6 contains some concluding remarks.

2. PROBLEM FORMULATION

Consider the following nonlinear uncertain systems

$$\dot{x} = f(x, u), \quad z = g(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $z \in \mathbb{R}$ is the output, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions (maybe unknown) representing the system dynamics and the output map, respectively. We aim at solving the extremum seeking problem for system (1), i.e., designing controller u such that the output z converges to the extremum of some objective function $h : \mathbb{R} \to \mathbb{R}$. Specifically, if h is strongly concave, it is desired that z converges to

$$z^* = \operatorname*{argmax}_{z \in \mathbb{R}} h(z). \tag{2}$$

Extremum seeking aims to solve the optimum searching problem when the function h is analytically unknown and its gradient is not available. The problem becomes more challenging when the system (1) which generates z involves nonlinear and uncertain dynamics, and when only the output or partial state is available for feedback.

By taking advantage of the rich literature of both extremum seeking and nonlinear control theory, the objective of this paper is to propose a new hierarchical structure for output extremum seeking. In particular, the structure is composed of two levels, upper-level extremum seeking and lower-level reference-tracking control, and the extremum seeking problem is solvable for nonlinear uncertain systems through the coordination of the two levels.

By convention, we make the following assumption on the objective function h.

Assumption 1. The function h is strongly concave and twice continuously differentiable. Moreover, with $z^* = \operatorname{argmax}_{z \in \mathbb{R}} h(z)$, there exists a known $\Omega \subseteq \mathbb{R}$ such that

$$z^* \in \Omega, \tag{3}$$

and there exist $\overline{h}, \underline{h}', \overline{h}', \overline{h}'' \in \mathcal{K}_\infty$ and known constants $\overline{h}^*, \overline{h}''^* \geq 0$ such that

$$|h(z^* + \zeta)| \leq \overline{h}(|\zeta|) + \overline{h}^* \tag{4}$$

$$\underline{h}'(|\zeta|) \leq |h'(z^* + \zeta)| \leq \overline{h}'(|\zeta|) \tag{5}$$

$$|h''(z^* + \zeta)| \leq \overline{h}''(|\zeta|) + \overline{h}''^* \tag{6}$$

hold for all $\zeta \in \mathbb{R}$.

Remark 1. Assumption 1 is not a very strict condition. There are many functions that satisfy this assumption, such as $h(z) = -z^2 + 1$ and $h(z) = -z^{2.5} + 1$.

The proposed solution is based on the design of a reference-tracking controller for the plant. Without loss of generality, suppose that the controller is in the form of

$$u = \phi(x, \chi, z^r), \quad \dot{\chi} = \varphi(x, \chi, z^r) \tag{7}$$

where $\chi \in \mathbb{R}^m$ represents the internal state of the controller, $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ are locally Lipschitz functions, and $z^r \in \mathbb{R}$ is a reference signal.

Remark 2. Specifically, the state x in (7) does not always need to be fully known. In 4.3 of this paper, the problem of extremum seeking of nonlinear systems in the output-feedback form is solved without using the state.

For convenience of notations, define

$$\tilde{z} = z - z^r \tag{8}$$

as the reference-tracking error, and $X = (x, \chi)$ as the state of the controlled plant. The following assumption employs the notions of input-to-output practical stability (IOpS) and unboundedness observability (UO) to characterize the reference-tracking capability and stability of the controlled plant composed of (1) and (7).

Assumption 2. For the controlled plant composed of (1) and (7), there exist $\beta_{\tilde{z}} \in \mathcal{KL}$, $\alpha_{\tilde{z}}, \gamma_{\tilde{z}}^{\dot{z}^r}, \gamma_{\tilde{z}}^{z^r} \in \mathcal{K}_\infty$ and nonnegative constants c_1, c_2, c_3 such that for any z^r which is continuously differentiable on the time-line, and any initial state $X(0)$,

$$\begin{aligned} |\tilde{z}(t)| \leq & \max\{\beta_{\tilde{z}}(|X(0)| + \|z^r\|_t + c_1, t), \\ & \gamma_{\tilde{z}}^{\dot{z}^r}(\|\dot{z}^r\|_t), \gamma_{\tilde{z}}^{z^r}(\|z^r\|_t), c_2\} \end{aligned} \tag{9}$$

$$|X(t)| \leq \alpha_{\tilde{z}}(|X(0)| + \|\dot{z}^r\|_t + \|z^r\|_t + c_3) \tag{10}$$

hold for all $t \geq 0$, where $\|s\|_t$ means $\operatorname{ess\,sup}_{0 \leq \tau \leq t} |s(\tau)|$ for $s : \mathbb{R}_+ \to \mathbb{R}$.

Remark 3. Assumption 2 is used to represent the reference-tracking capability of the controlled systems in extremum seeking. It is weaker than many assumptions of the existing literatures. Some controlled systems with multi-equilibria or offset satisfy this assumption, while they do not satisfy the assumptions in the existing results such as Krstić and Wang (2000); Ariyur and Krstić (2003); Tan et al. (2006); Guay and Dochain (2015). What is more, the dynamic function of the system (1) does not need to be globally Lipschitz, which is required in Haring and Johansen (2017).

Three examples are given in Section 4 to show the validity of Assumption 2.

3. MAIN RESULT

This section presents the main result of the paper. Intuitively, with Assumption 2 guaranteeing the reference-tracking capability of the controlled plant, the problem of extremum seeking is reduced to the one of generating an appropriate reference signal.

As shown in Figure 1, the proposed extremum seeking control system features the standard structure of extremum seeking.

Clearly, if the reference-tracking capability of the controlled plant is perfect, say $z \equiv z^r$, then the problem is reduced to the fundamental one of static extremum seeking. In such ideal case, the block of the controlled plant (dashed bolck in Figure 1) can be removed. In the presence of the reference-tracking error, we consider the closed-loop extremum seeking control system as an interconnection of

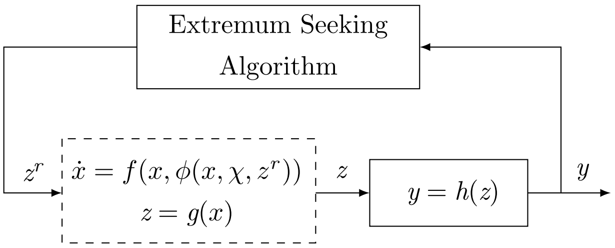

Fig. 1. Structure of the closed-loop extremum seeking system.

two parts: the upper level of static extremum seeking and the lower level of reference-tracking; see Figure 2.

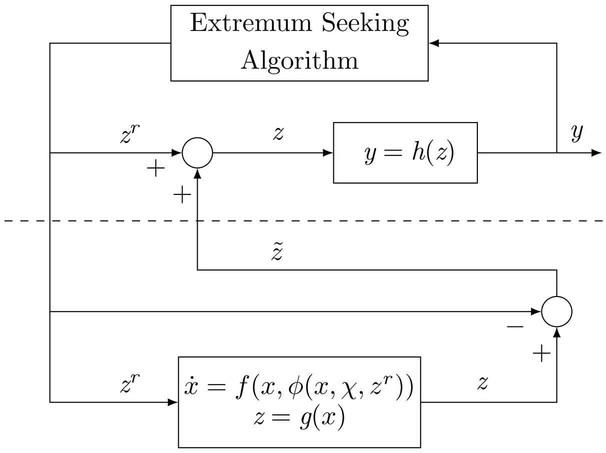

Fig. 2. Two-level structure of a closed-loop extremum seeking system.

In the remainder of this section, we first use gains to characterize the interconnection between the upper-level static extremum seeking and the lower-level reference-tracking, and then employ the nonlinear small-gain theorem to guarantee the boundedness of the signals in the closed-loop system and the convergence of z to z^*.

3.1 Gain Interconnections

Consider the extremum seeking algorithm

$$\dot{\theta} = kah(z)\sin(\omega t) \tag{11}$$

$$z^r = \theta + a\sin(\omega t) \tag{12}$$

where k, a and ω are positive constants. This subsection propose some important properties of systems (11)–(12), (1) and (7), which will be used in the next subsection to solve the extremum seeking problem (2) of system (1).

If $z \equiv z^r$, then the algorithm and its modifications have been widely used in solving extremum seeking problems, see Ariyur and Krstić (2003); Tan et al. (2006).

Recall that $\tilde{z} = z - z^r$ represents the reference-tracking error of the controlled plant. Define

$$\tilde{\theta} = \theta - z^*. \tag{13}$$

Then, the extremum seeking algorithm (11)–(12) implies

$$\dot{\tilde{\theta}} = akh(z^* + \tilde{\theta} + \tilde{z} + a\sin(\omega t))\sin(\omega t). \tag{14}$$

The robust stability of systems like the system (14) has been studied in Tan et al. (2006), with the influence of the reference-tracking error $\tilde{z}$ described by the notion of ISS. The result is given here for completeness of the paper.

Proposition 1. (Tan et al. (2006)). Consider system (14). Design $\delta = k/\omega$. Suppose that Assumption 1 is satisfied. For any positive constants λ, $\epsilon_{\tilde{\theta}}$ and ω, one can find positive constants a and δ such that for any $\theta(0)$ and $\tilde{z}$ satisfying $\max\{|\tilde{\theta}(0)|, \|\tilde{z}\|_\infty\} \leq \lambda$,

$$|\tilde{\theta}(t)| \leq \max\{\beta_{\tilde{\theta}}(|\tilde{\theta}(0)|, t), \gamma_{\tilde{\theta}}^{\tilde{z}}(\|\tilde{z}\|_t), \epsilon_{\tilde{\theta}}\} \tag{15}$$

holds for all $t \geq 0$, where $\beta_{\tilde{\theta}}$ is a class $\mathcal{KL}$ function, and $\gamma_{\tilde{\theta}}^{\tilde{z}}$ is a class $\mathcal{K}_\infty$ function.

Next, two propositions are to describe the other interconnections in the extremum seeking system by gains.

Proposition 2. Suppose that Assumption 1 holds for system (14). Then, there exist $\gamma_{\dot{z}^r}^{\tilde{\theta}}, \gamma_{\dot{z}^r}^{\tilde{z}} \in \mathcal{K}_\infty$ and a constant $\epsilon_{\dot{z}^r}$ such that

$$|\dot{z}^r| \leq \max\{\gamma_{\dot{z}^r}^{\tilde{\theta}}(|\tilde{\theta}|), \gamma_{\dot{z}^r}^{\tilde{z}}(|\tilde{z}|), \epsilon_{\dot{z}^r}\}. \tag{16}$$

Proof Define $\sigma = \omega t$, and recall $k = \omega\delta$. Then, the system (14) can be rewritten as

$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\sigma} = a\delta h(\theta^* + \tilde{\theta} + \tilde{z} + a\sin(\sigma))\sin(\sigma). \tag{17}$$

With Assumption 1 satisfied, it can be directly proved that

$$\left|\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\sigma}\right| \leq \max\{\gamma_{\tilde{\theta}_\sigma}^{\tilde{\theta}}(|\tilde{\theta}|), \gamma_{\tilde{\theta}_\sigma}^{\tilde{z}}(|\tilde{z}|), d_{\tilde{\theta}_\sigma}\} \tag{18}$$

where $\gamma_{\tilde{\theta}_\sigma}^{\tilde{\theta}}(s) = a\delta\overline{h}(3s)$, $\gamma_{\tilde{\theta}_\sigma}^{\tilde{z}}(s) = a\delta\overline{h}(3s) \in \mathcal{K}_\infty$, and $d_{\tilde{\theta}_\sigma} = a\delta(\overline{h}(3a) + \overline{h}^*)$. Using (12), we obtain

$$\frac{\mathrm{d}z^r}{\mathrm{d}\sigma} = \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\sigma} + a\cos(\sigma), \tag{19}$$

which implies

$$\left|\frac{\mathrm{d}z^r}{\mathrm{d}\sigma}\right| \leq \max\left\{\gamma_{z_\sigma^r}^{\tilde{\theta}_\sigma}\left(\left|\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\sigma}\right|\right), d_{z_\sigma^r}\right\}, \tag{20}$$

where $\gamma_{z_\sigma^r}^{\tilde{\theta}_\sigma}(s) = 2s$ and $d_{z_\sigma^r} = 2a$. Properties (18) and (20) together guarantee (16) with $\gamma_{\dot{z}^r}^{\tilde{\theta}}(s) = \omega\gamma_{z_\sigma^r}^{\tilde{\theta}_\sigma} \circ \gamma_{\tilde{\theta}_\sigma}^{\tilde{\theta}}(s)$, $\gamma_{\dot{z}^r}^{\tilde{z}}(s) = \omega\gamma_{z_\sigma^r}^{\tilde{\theta}_\sigma} \circ \gamma_{\tilde{\theta}_\sigma}^{\tilde{z}}(s)$, and $\epsilon_{\dot{z}^r} = \max\{\omega d_{z_\sigma^r}, \omega\gamma_{z_\sigma^r}^{\tilde{\theta}_\sigma}(d_{\tilde{\theta}_\sigma})\}$. This ends the proof of Proposition 2. □

Proposition 3. Suppose that Assumptions 1 and 2 are satisfied. Consider the extremum seeking algorithm defined by (11) and (12), and the controlled plant composed of (1) and (7). There exist a class $\mathcal{K}_\infty$ function $\gamma_{\tilde{z}}^{\tilde{\theta}}$ and a constant $\epsilon_{\tilde{z}}$ such that

$$\begin{aligned}|\tilde{z}(t)| \leq &\max\{\beta_{\tilde{z}}(|X(0)| + \|z^r\|_t + c_1, t),\\ &\gamma_{\tilde{z}}^{\dot{z}^r}(\|\dot{z}^r\|_t), \gamma_{\tilde{z}}^{\tilde{\theta}}(\|\tilde{\theta}\|_t), \epsilon_{\tilde{z}}\}\end{aligned} \tag{21}$$

where $\beta_{\tilde{z}}$, $\gamma_{\tilde{z}}^{\dot{z}^r}$ and c_1 are given by Assumption 2.

Proof Under Assumption 1, there exist s a known positive constant $\overline{z}^*$ such that $|z^*| \leq \overline{z}^*$. With (12) and (13), it can be directly verified that

$$|z^r| \leq \max\{\gamma_{z^r}^{\tilde{\theta}}(\tilde{\theta}), d_{z^r}\} \tag{22}$$

where $\gamma_{z^r}^{\tilde{\theta}}(s) = 2s$ and $d_{z^r} = 2\overline{z}^* + 2a$. Then, Assumption 2 directly implies (21) with $\gamma_{\tilde{z}}^{\tilde{\theta}}(s) = \gamma_{\tilde{z}}^{z^r} \circ \gamma_{z^r}^{\tilde{\theta}}(s)$ and $\epsilon_{\tilde{z}} = \max\{c_2, \gamma_{\tilde{z}}^{z^r}(d_{z^r})\}$. This ends the proof of Proposition 3. □

3.2 Small-Gain-Based Synthesis

With Propositions 1, 2 and 3, the closed-loop extremum seeking system is transformed into an interconnected system with the interconnections described by gains; as shown in Figure 3.

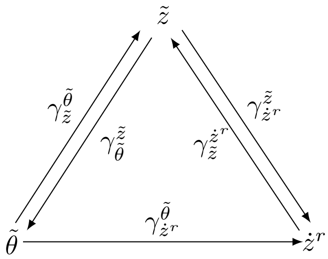

Fig. 3. Gain interconnection of the closed-loop extremum seeking system.

The following theorem gives the main result of the paper.

Theorem 4. Suppose that Assumptions 1 and 2 are satisfied. Consider the closed-loop extremum seeking system composed of (1), (7), (11) and (12). For any constants $\rho, \omega > 0$, by choosing suitably $a, \delta > 0$ such that

$$\gamma_{\tilde{z}}^{\dot{z}^r} \circ \gamma_{\dot{z}^r}^{\tilde{z}} < \mathrm{Id}, \quad \gamma_{\tilde{z}}^{\tilde{\theta}} \circ \gamma_{\tilde{\theta}}^{\tilde{z}} < \mathrm{Id}, \quad \gamma_{\dot{z}^r}^{\tilde{\theta}} \circ \gamma_{\tilde{\theta}}^{\tilde{z}} \circ \gamma_{\tilde{z}}^{\dot{z}^r} < \mathrm{Id}, \tag{23}$$

there exists a class $\mathcal{KL}$ function β_{final} such that for any $\theta(0)$ and $X(0)$ satisfying $\max\{|\tilde{\theta}(0)|, |X(0)|\} \leq \rho$,

$$|\tilde{\theta}(t)| \leq \max\{\beta_{\mathrm{final}}(|[\tilde{\theta}(0), X(0)]|, t), \epsilon\} \tag{24}$$

holds for all $t \geq 0$, with $\epsilon = \max\{\epsilon_{\tilde{\theta}}, \gamma_{\tilde{\theta}}^{\tilde{z}}(\epsilon_{\tilde{z}}), \gamma_{\tilde{\theta}}^{\tilde{z}} \circ \gamma_{\tilde{z}}^{\dot{z}^r}(\epsilon_{\dot{z}^r})\}$.

Since the closed-loop extremum seeking system has been transformed into a network with the interconnections described by gains, the main result is proved by directly using the recently developed nonlinear cyclic-small-gain theorem for dynamic networks Sontag (1989); Bao et al. (2019). Specifically, the three inequalities in (23) correspond to the three simple cycles in the graph shown in Figure 3. The details of the proof are omitted due to space limitation.

4. EXAMPLES

This section employs examples to show the validity of Assumption 2.

4.1 A Nonlinear System with Multi-Equilibria

For the system in Sontag and Wang (1999):

$$\dot{x}_1 = 0, \quad \dot{x}_2 = -\frac{x_2 - u}{1 + x_1^2}, \quad z = x_2. \tag{25}$$

Consider the control law

$$u = z^r - b(1 + x_1^2)(x_2 - z^r) \tag{26}$$

where z^r represents the reference signal, and b is a positive constant. Define $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2 - z^r$ and $\tilde{z} = z - z^r$. Then, the closed-loop system is

$$\dot{\tilde{x}}_1 = 0, \quad \dot{\tilde{x}}_2 = -\frac{\tilde{x}_2}{1 + \tilde{x}_1^2} - b\tilde{x}_2 - \dot{z}^r, \quad \tilde{z} = \tilde{x}_2, \tag{27}$$

which satisfies Assumption 2 with $X = [\tilde{x}_1, \tilde{x}_2]^T$, $\beta_{\tilde{z}}(s,t) = e^{-0.5bt}s$, $\gamma_{\tilde{z}}^{\dot{z}^r}(s) = 2s/b$, $\gamma_{\tilde{z}}^{z^r}(s) = 0.0001s$, $\alpha_{\tilde{z}}(s) = (1.0001 + 2/b)(s)$ and $c_1 = c_2 = c_3 = 0$.

4.2 A General Class of Linear Systems

Consider a linear system represented by the transfer function

$$G(s) = K\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}. \tag{28}$$

Suppose that the numerator and denominator polynomials are relatively prime. Then, one of its minimal realizations is

$$\dot{x} = Ax + Bu, \quad z = Cx \tag{29}$$

where

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K \end{bmatrix} \tag{30}$$

$$C = [b_0, \cdots, b_{n-r-1}, 1, 0, \cdots, 0]. \tag{31}$$

with $K \neq 0$ and $b_0 \neq 0$.

Define matrices $K_1 = \frac{1}{K}[a_0, \cdots, a_{n-1}]$ and K_2 satisfying $A_2 = A + BK_1 - BK_2$ is Hurwitz. This is achievable due to the controllability of the minimal realization. Then, it can be verified that $CA_2^{-1}B \neq 0$. Define $\tilde{x} = \mathrm{diag}\{1, \mu, \cdots, \mu^{n-1}\}x - \frac{A_2^{-1}B}{CA_2^{-1}B}z^r$ and choose $u = K_1 x - K_2 \mathrm{diag}\{\mu^{-n}, \cdots, \mu^{-1}\}x - \mu^{-n}\frac{1}{CA_2^{-1}B}z^r$ with z^r representing the reference signal, and μ being a positive parameter to be determined later. Then, direct calculation yields

$$\dot{\tilde{x}} = \mu^{-1}A_2\tilde{x} - \frac{A_2^{-1}B}{CA_2^{-1}B}\dot{z}^r. \tag{32}$$

This guarantees the satisfaction of Assumption 2 by taking proper μ.

4.3 Nonlinear Systems in the Output-Feedback Form

Consider the class of nonlinear uncertain systems in the output-feedback form, which has been widely studied in the literature of nonlinear control Krstić et al. (1995):

$$\begin{aligned} \dot{x}_j &= x_{j+1} + \Delta_j(z), \ 1 \leq j \leq n-1, \\ \dot{x}_n &= u + \Delta_n(z), \\ z &= x_1 \end{aligned} \tag{33}$$

where $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $z \in \mathbb{R}$ is the output, and $\Delta_i : \mathbb{R} \to \mathbb{R}$ with $i = 1, \ldots, n$ are nonlinear uncertain functions.

The following assumption is made on the system dynamics.

Assumption 3. There exist $\psi_{1\Delta_j}, \psi_{2\Delta_j}, \psi_{\Delta_j}$ which are known, locally Lipschitz, class $\mathcal{K}_\infty$ functions, and c_{Δ_j} which is a known nonnegative constant for $1 \leq j \leq n$, such that

$$|\Delta_j(z) - \Delta_j(z^r)| \leq \psi_{1\Delta_j}(|z - z^r|) + \psi_{2\Delta_j}(|z^r|), \quad 1 \leq j \leq n \tag{34}$$

$$\left|\frac{\partial \Delta_j(z)}{\partial z}\right| \leq \psi_{\Delta_j}(|z|) + c_{\Delta_j}, \quad 1 \leq j \leq n-1 \tag{35}$$

$$|\Delta_n(z^r)| \leq \psi_{\Delta_n}(|z^r|) + c_{\Delta_n} \tag{36}$$

hold for all $z, z^r \in \mathbb{R}$.

The following design is motivated by the observer-based control law developed by Liu and Jiang (2013).

Define $\tilde{x}_1 = x_1 - z^r$ and $\tilde{x}_j = x_j + \Delta_{j-1}(z^r)$ for $2 \le j \le n$. Then, the system (33) can be rewritten as

$$\begin{aligned}\dot{\tilde{x}}_j =& \tilde{x}_{j+1} + \tilde{\Delta}_j(z, z^r, \dot{z}^r), \quad 1 \le j \le n-1 \\ \dot{\tilde{x}}_n =& u + \tilde{\Delta}_n(z, z^r, \dot{z}^r) \\ \tilde{z} =& \tilde{x}_1 \end{aligned} \tag{37}$$

where $\tilde{\Delta}_1(z, z^r, \dot{z}^r) = \Delta_1(z) - \Delta_1(z^r) - \dot{z}^r$, $\tilde{\Delta}_j(z, z^r, \dot{z}^r) = \Delta_j(z) - \Delta_j(z^r) + \frac{\partial \Delta_{j-1}(z^r)}{\partial z^r}\dot{z}^r$ for $j = 2, \cdots, n-1$ and $\tilde{\Delta}_n(z, z^r, \dot{z}^r) = \Delta_n(z) - \Delta_n(z^r) + \frac{\partial \Delta_{n-1}(z^r)}{\partial z^r}\dot{z}^r + \Delta_n(z^r)$. From Assumption 3, we have

$$\begin{aligned}|\tilde{\Delta}_1(z, z^r, \dot{z}^r)| \le& \psi_{1\Delta_1}(|\tilde{x}_1|) + \psi_{2\Delta_1}(|z^r|) + |\dot{z}^r| \\ :=& \tilde{\Delta}_j^*(|\tilde{x}_1|, |z^r|, |\dot{z}^r|), \\ |\tilde{\Delta}_j(z, z^r, \dot{z}^r)| \le& \psi_{\Delta_j}(|\tilde{x}_1|) + \psi_{2\Delta_j}(|z^r|) \\ &+ 0.5\psi^2_{\Delta_{j-1}}(|z^r|) + 0.5|\dot{z}^r|^2 + c_{\Delta_j}|\dot{z}^r| \\ :=& \tilde{\Delta}_j^*(|\tilde{x}_1|, |z^r|, |\dot{z}^r|), j = 2, \cdots, n-1, \\ |\tilde{\Delta}_n(z, z^r, \dot{z}^r)| \le& \psi_{\Delta_n}(|\tilde{x}_1|) + \psi_{2\Delta_n}(|z^r|) + 0.5\psi^2_{\Delta_{n-1}}(|z^r|) \\ &+ 0.5|\dot{z}^r|^2 + c_{\Delta_n}|\dot{z}^r| + \psi_{\Delta_n}(|z^r|) + c_{\Delta_n} \\ :=& \tilde{\Delta}_n^*(|\tilde{x}_1|, |z^r|, |\dot{z}^r|, c_{\Delta_n}).\end{aligned}$$

Owing to the output-feedback structure, we design an observer

$$\begin{aligned}\dot{\xi}_1 =& \xi_2 + L_2\xi_1 + \rho_1(\xi_1 - \tilde{x}_1) \\ \dot{\xi}_j =& \xi_{j+1} + L_{j+1}\xi_1 - L_j(\xi_2 + L_2\xi_1), \ 2 \le j \le n-1 \\ \dot{\xi}_n =& u - L_n(\xi_2 + L_2\xi_1)\end{aligned} \tag{38}$$

where $\rho_1 : \mathbb{R} \to \mathbb{R}$ is an odd and strictly decreasing function, and $L_2, \cdots, L_n$ are positive constants. In the observer, ξ_1 is an estimate of $\tilde{z}$, and ξ_j is an estimate of $\tilde{x}_j - L_j\tilde{z}$ for $2 \le j \le n$. Then, we consider the observer-based nonlinear control law

$$\begin{aligned}&e_1 = \xi_1, \\ &e_j = \xi_j - \kappa_{j-1}(e_{j-1}), \quad 2 \le j \le n, u = \kappa_n(e_n)\end{aligned} \tag{39}$$

where $\kappa_1, \cdots, \kappa_n$ are continuously differentiable, odd, strictly decreasing and radially unbounded functions. Define $X = [\tilde{x}_1, \cdots, \tilde{x}_n, \xi_1, \cdots, \xi_n]^T$ as the internal state of the controlled system composed of the transformed system (37) and the observer-based controller (38)-(39). Then, Assumption 2 can be satisfied following an analysis similar to that in (Liu and Jiang, 2013, Section V).

5. NUMERICAL SIMULATION

Consider the system (25) in Section 4.1, with the objective function

$$y = h(z) = -0.4(z+1)^2 + 1. \tag{40}$$

Define $\Omega = [-3, 1]$. Then, it is clear that $z^* \in \Omega$ and the maximum value of y is 1. Also,

$$|h(z^* + \zeta)| \le 2|\zeta| + 1.2 \tag{41}$$
$$0.5|\zeta| \le |h'(z^* + \zeta)| \le |\zeta| \tag{42}$$
$$|h''(z^* + \zeta)| \le |\zeta| + 0.2 \tag{43}$$

hold for all $z^* \in \Omega$ and $|\zeta| \le 5$.

By choosing $a = 0.2$, $\delta = 0.5$ and $\omega = 1$ and using control law (26) with $b = 100$, Assaumption 2 and the cyclic-small-gain condition (23) are satisfied. Figure 4 shows the simulation result with initial states $x_1(0) = -0.6$, $x_2(0) = 3$ and $\theta(0) = 2.7$.

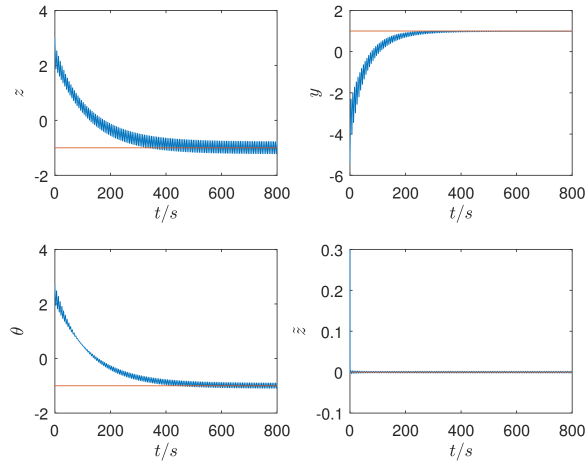

Fig. 4. Numerical simulation result.

It can be observed that z and θ converge to a small neighborhood of the extremum point $z^* = -1$, and y converges to a small neighborhood of the maximum $y = 1$.

6. CONCLUSIONS

This paper has introduced a small-gain approach to extremum seeking for static maps with the inputs of the maps generated by nonlinear uncertain systems. It has been shown that the problem can be solved by using an extremum seeking strategy to generate a reference signal, and designing a control law for the nonlinear uncertain system to track the reference signal. The closed-loop system has been transformed into an interconnected system, and the nonlinear small-gain theorem has been used for stability and convergence analysis. With the proposed approach, the extremum seeking problem for static maps with the inputs of the maps generated by nonlinear uncertain system is solvable as long as one can design an appropriate reference-tracking controller for the system.

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