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Robust Sliding Window Observer-Based Controller Design for Lipschitz Discrete-Time Systems ⋆

Noussaiba GASMI * Mohamed BOUTAYEB * Assem THABET ** Mohamed AOUN ** Ghazi BEL HAJ FREJ ***

* *Centre de Recherche en Automatique de Nancy, CRAN UMR 7039 CNRS, University of Lorraine, 54500 Vandoeuvre-lès-Nancy, France.*
** *MACS Laboratory, National Engineering School of Gabes (ENIG), University of Gabes, Tunisia.*
*** *IMS Laboratory, UMR 5218, University of Bordeaux, 33405 Talence, France.*

Abstract: The aim of this paper is to develop a new observer-based stabilization strategy for a class of Lipschitz uncertain systems. This new strategy improves the performances of existing methods and ensures better convergence conditions. The observer and the controller are enriched with sliding windows of measurements and estimated states, respectively. This technique allows to increase the number of decision variables and thus get less restrictive and more general LMI conditions. The established sufficient stability conditions are in the form of Bilinear Matrix Inequality (BMI). The obtained constraint is transformed, through a useful approach, to a more suitable one easily tractable by standard software algorithms. Numerical example is given to illustrate the performances of the proposed approach.

Keywords: Discrete-time systems, uncertain systems, Lipschitz non-linearities, observer-based stabilization design, $\mathcal{H}_\infty$ criterion, sliding window approach.

1. INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Robust stabilization of nonlinear uncertain systems have been extensively investigated in recent years and numerous design methodologies have been established to filter disturbances and uncertainties and ensure good degree of noise sensitivity, good performance and robustness (Ho and Lu (2003); Kheloufi et al. (2014); Song and S.He (2015); Kheloufi et al. (2015); Ito and Dinh (2018)).

Many researchers focus on the observer-based controller design for nonlinear continuous or discrete-time systems. Interesting results are presented in Ibrir (2008); Grandvallet et al. (2013); Kheloufi et al. (2016). For example, an LMI condition for observer-based controller design of Lipschitz systems is given in Ibrir (2008). In this paper, the author proposes to compute the controller and the observer gains in two steps. Then, an other design methodology is proposed in Grandvallet et al. (2013). This method uses a diagonal Lyapunov matrix and allows to compute the observer and the controller gains simultaneously via the same optimization problem. Recently, an interesting publication of Kheloufi et al. (2016) presents a new and useful design procedure to synthesize a robust observer-based stabilization for nonlinear systems using a symmetric Lyapunov function. In order to handle the difficulty of Non-deterministic Polynomial-time hard (NP-hard) nature of the problem, the authors propose to use a slack variable technique inspired from Heemels et al. (2010) with the standard Young's inequality. All these interesting results use only the last available state and measurement to synthesis the observer and the controller gains.

In this paper, a new observer-based control design methodology for a class of Lipschitz discrete-time systems in the presence of bounded disturbances and parametric uncertainties is proposed. This work is motivated by the recent results on observer design of Grandvallet et al. (2014); Gasmi et al. (2018). The main contribution lies in the use of a sliding window of measurements in a Luenberger observer and a sliding window of delayed states in the control law which allows to introduce additional decision variables to enhance the constraint to be resolved. The use of a sliding window approach, in the synthesis of the robust observer-based controller, improves the disturbance rejection by involving a fixed number of previous states and measurements and allows to improve the robustness of control. This new formulation is a significant contribution, contrary to conventional approaches that consider only the last available measurement for the observer and the last available state estimate for the controller (Kheloufi et al. (2016)). A judicious use of a slack variable technique (Heemels et al. (2010)) with a useful reformulation of the Young's inequality (Zemouche et al. (2017)) allows to add more degree of freedom.

⋆ Corresponding author : N. GASMI (noussaiba.gasmi@univ-lorraine.fr).

This contribution is organized as follows: In the next part of this section, useful preliminaries are presented. The problem statement is introduced in the second section. The third section details the synthesis procedure of the robust sliding window observer-based controller. Numerical example is provided, in the last section, to demonstrate the validity of the proposed results for the considered class of nonlinear systems.

1.2 Notation

The notation used in this paper are as following:

- $e_p(i) = \Big(\underbrace{0,...,0,\overbrace{1}^{\mathrm{i^{th}}},0,...,0}_{\mathrm{p-components}}\Big)^T \in \mathbb{R}^p$, $p \geqslant 1$, is a vector of the canonical basis of $\mathbb{R}^p$;
- U^T is the transposed matrix of U;
- U is a square matrix. $U > 0$ ($U < 0$) designates a positive definite (negative definite) matrix U;
- Blocks induced by symmetry in a matrix are represented by the symbol $(\star)$.

1.3 Preliminaries

Lemma 1. (Zemouche et al. (2017)). Consider a nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^q$. Then the following two items are equivalent :

- f is globally Lipschitz with respect to its argument, i.e.:
$$\| f(U) - f(V) \| \leqslant \gamma_f \| U - V \|, \quad \forall\, U,\, V \in \mathbb{R}^n. \quad (1)$$
- there exist finite and positive scalar constants $\underline{\gamma}_{\mathfrak{f}_{ij}}$ and $\overline{\gamma}_{\mathfrak{f}_{ij}}$ so that for all $\forall\ U, V \in \mathbb{R}^n$ there exist $z_i \in Co(U, V)$, $z_i \neq U$, $z_i \neq V$ and functions $\mathfrak{f}_{ij} : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the following equality:
$$f(U) - f(V) = \sum_{i,j=1}^{q,n} \mathfrak{f}_{ij}(z_i) H_{ij}(U - V) \quad (2)$$
and $\underline{\gamma}_{\mathfrak{f}_{ij}} \leqslant \mathfrak{f}_{ij}(z_i) \leqslant \overline{\gamma}_{\mathfrak{f}_{ij}}$ where $\mathfrak{f}_{ij}(z_i) = \dfrac{\partial f_i}{\partial x_j}(z_i)$ and $H_{ij} = e_q(i)e_n^T(j)$. ■

Lemma 2. (Zemouche et al. (2017)). Given two matrices U and V of appropriate dimensions, then the following inequality holds for any symmetric positive definite matrix S of appropriate dimensions:

$$U^T V + V^T U \leqslant \frac{1}{2}\left[U + SV\right]^T S^{-1} \left[U + SV\right] \quad (3)$$

■

Lemma 3. (Petersen (1987)). Given three matrices U, V and S of appropriate dimensions with $S^T S \leq I$. Then, the following inequality holds $\forall\ \eta > 0$:

$$USV + V^T S^T U^T \leq \eta U U^T + \frac{1}{\eta} V^T V \quad (4)$$

■

2. PROBLEM STATEMENT

The class of nonlinear uncertain systems under study is described by:

$$\begin{cases} x_{k+1} = \big(A + \Delta A_k\big)x_k + Bu_k + Df\big(x_k\big) + E_1\omega_k \\ y_k \quad\;\; = \big(C + \Delta C_k\big)x(k) + E_2\omega_k \end{cases} \quad (5)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $\omega_k \in \mathbb{R}^s$ and $y_k \in \mathbb{R}^p$ represent the state, the input, the disturbance and the output vectors, respectively. A, B, D, E_1, C and E_2 are constant matrices. $f : \mathbb{R}^n \to \mathbb{R}^q$ is a Lipschitz nonlinear vector. The unknown matrices ΔA_k and ΔC_k represent the time-varying parameter uncertainties with

$$\Delta A_k = M_1 F_k N_1 \quad (6a)$$
$$\Delta C_k = M_2 F_k N_2 \quad (6b)$$

where F_k is an unknown matrix satisfying the following condition :

$$F_k^T F_k \leq I. \quad (7)$$

For system (5), let us consider the following observer:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + Df\big(\hat{x}_k\big) + \mathcal{L}\begin{pmatrix} y_k - C\hat{x}_k \\ y_{k-1} - C\hat{x}_{k-1} \\ \vdots \\ y_{k-r+1} - C\hat{x}_{k-r+1} \end{pmatrix} \quad (8)$$

where $\hat{x}_k$ and $\mathcal{L}$ represent the estimated state and the observer gain matrix, respectively. r is the size of the sliding window.

To integrate the delayed states into the state equation, (5) is rewritten as follows :

$$z_{k+1} = \big(\mathcal{A} + \Delta\mathcal{A}_k\big)z_k + \mathcal{B}u_k + \mathcal{D}f\big(\mathcal{I}^T z_k\big) + \mathcal{E}_1\nu_k \quad (9)$$

with

$$z_k = \begin{pmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-r+1} \end{pmatrix},\ \nu_k = \begin{pmatrix} \omega_k \\ \omega_{k-1} \\ \vdots \\ \omega_{k-r+1} \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A & 0 & \cdots & \cdots & 0 \\ I_n & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_n & 0 \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} B \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \mathcal{D} = \begin{pmatrix} D \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\Delta\mathcal{A}_k = \begin{pmatrix} \Delta A_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} = \mathcal{M}_1 F_k \mathcal{N}_1,\ \mathcal{M}_1 = \begin{pmatrix} M_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\mathcal{N}_1^T = \begin{pmatrix} N_1^T \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \mathcal{I} = \begin{pmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \mathcal{E}_1 = \begin{pmatrix} E_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Likewise, the sliding window observer (8) can be rewritten as follows:

$$\hat{z}_{k+1} = \mathcal{A}\hat{z}_k + \mathcal{B}u_k + \mathcal{D}f\big(\mathcal{I}^T\hat{z}_k\big) + \mathcal{I}\mathcal{L}\mathcal{C}\zeta_k + \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k z_k + \mathcal{I}\mathcal{L}\mathcal{E}_2\nu_k \quad (10)$$

where $\zeta_k = z_k - \hat{z}_k$ represents the estimation error,

$$\mathcal{L} = (L_1\ L_2\ \cdots\ L_r),$$

$$\mathcal{C} = \text{block-diag}\Big(\overbrace{C, \ldots, C}^{r\ \text{times}}\Big),$$

$$\Delta\mathcal{C}_k = \text{block-diag}\left(\overbrace{\Delta C_k, \ldots, \Delta C_k}^{r \text{ times}}\right) = \mathcal{M}_2\mathcal{F}_k\mathcal{N}_2,$$
$$\mathcal{M}_2 = \text{block-diag}\left(\overbrace{M_2, \ldots, M_2}^{r \text{ times}}\right),$$
$$\mathcal{F}_k = \text{block-diag}\big(F_k, \ldots, F_{k-r+1}\big),$$
$$\mathcal{N}_2 = \text{block-diag}\left(\overbrace{N_2, \ldots, N_2}^{r \text{ times}}\right),$$
$$\mathcal{E}_2 = \text{block-diag}\left(\overbrace{E_2, \ldots, E_2}^{r \text{ times}}\right).$$

Then, let us consider the following state estimate feedback controller:

$$u_k = \sum_{i=1}^{r} K_i \hat{x}_{k-i+1} \tag{11}$$

where K_i for $i \in \{1, \cdots, r\}$ are the control gain matrices.

The controller (11) can be rewritten as follows:

$$u_k = \mathcal{K}\hat{z}_k \tag{12}$$

with $\mathcal{K} = (K_1 \ K_2 \ \cdots \ K_r)$.

In the rest of the paper, the objective is to determine the observer gain $\mathcal{L}$ and the control gain $\mathcal{K}$ while ensuring the asymptotic stability of the closed-loop system and satisfying the $\mathcal{H}_\infty$ criterion.

Let $\zeta_k = z_k - \hat{z}_k$. The dynamic of this estimation error is given by

$$\begin{aligned}\zeta_{k+1} = &\big(\Delta\mathcal{A}_k - \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k\big)z_k + (\mathcal{A} - \mathcal{I}\mathcal{L}\mathcal{C})\zeta_k \\ &+\mathcal{D}\Big(f\big(\mathcal{I}^T z_k\big) - f\big(\mathcal{I}^T \hat{z}_k\big)\Big) + (\mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2)\nu_k.\end{aligned} \tag{13}$$

Using equation (12), the closed-loop system is given by:

$$z_{k+1} = \big(\mathcal{A} + \Delta\mathcal{A}_k + \mathcal{B}\mathcal{K}\big)z_k - \mathcal{B}\mathcal{K}\zeta_k + \mathcal{D}f\big(\mathcal{I}^T z_k\big) + \mathcal{E}_1\nu_k. \tag{14}$$

Then, by applying lemma 1 to the nonlinear vector $f(.)$, we obtain

$$f\big(\mathcal{I}^T z_k\big) = \sum_{i,j=1}^{q,n} \varphi_{ij} H_{ij}\mathcal{I}^T z_k \tag{15}$$

$$f\big(\mathcal{I}^T z_k\big) - f\big(\mathcal{I}^T \hat{z}_k\big) = \sum_{i,j=1}^{q,n} \phi_{ij} H_{ij}\mathcal{I}^T \zeta_k \tag{16}$$

with $\underline{\gamma}_{\mathfrak{f}_{ij}} \leqslant \varphi_{ij} \leqslant \overline{\gamma}_{\mathfrak{f}_{ij}}$, $\underline{\gamma}_{\mathfrak{f}_{ij}} \leqslant \phi_{ij} \leqslant \overline{\gamma}_{\mathfrak{f}_{ij}}$, $H_{ij} = e_q(i)e_n^T(j)$.

Using equations (13), (14), (15) and (16), the following augmented system is defined:

$$\tilde{z}_{k+1} = \big(\tilde{\mathcal{A}} + \Xi(\Theta)\big)\tilde{z}_k + \tilde{\mathcal{E}}\nu_k \tag{17}$$

with

$$\tilde{z}_k = \begin{pmatrix} z_k \\ \zeta_k \end{pmatrix}, \ \Xi(\Theta) = \begin{pmatrix} \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T & 0 \\ 0 & \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T \end{pmatrix},$$
$$\tilde{\mathcal{A}} = \begin{pmatrix} \mathcal{A} + \Delta\mathcal{A}_k + \mathcal{B}\mathcal{K} & -\mathcal{B}\mathcal{K} \\ \Delta\mathcal{A}_k - \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k & \mathcal{A} - \mathcal{I}\mathcal{L}\mathcal{C} \end{pmatrix}, \ \tilde{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2 \end{pmatrix},$$
$$\Xi_1(\Theta) = \sum_{i,j=1}^{q,n} \varphi_{ij} H_{ij} \text{ and } \Xi_2(\Theta) = \sum_{i,j=1}^{q,n} \phi_{ij} H_{ij}.$$

The parameter Θ belongs to the bounded convex set $\mathcal{H}_{qn}$ for which the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{qn}} = \{\varphi, \ \phi \in \mathbb{R}^{q\times n} \ \text{ et } \ \varphi_{ij}, \ \phi_{ij} \in \{\underline{\gamma}_{\mathfrak{f}_{ij}}, \overline{\gamma}_{\mathfrak{f}_{ij}}\}\}. \tag{18}$$

3. NEW $\mathcal{H}_\infty$ OBSERVER-BASED CONTROLLER DESIGN METHOD

In this section, the design procedure of the proposed observer-based controller is detailed.

3.1 Stability analysis

The observer and the controller gains are calculated while respecting the following $\mathcal{H}_\infty$ norm:

$$\|G\big(x - \hat{x}\big)\|_{l_2} \leq \lambda\|\omega\|_{l_2} \tag{19}$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized and G is a known and constant matrix.
Therefore, We must look for a Lyapunov function V_k such that

$$\Delta V_k + \tilde{z}_k^T\tilde{\mathcal{I}}\mathbb{G}\tilde{\mathcal{I}}^T\tilde{z}_k - \frac{\lambda^2}{r}\nu_k^T\nu_k < 0. \tag{20}$$

with $\tilde{\mathcal{I}} = \text{block-diag}(\mathcal{I}, \mathcal{I})$, $\mathbb{G} = \text{block-diag}(0, G^T G)$, $x_k = \mathcal{I}^T z_k$ and $\hat{x}_k = \mathcal{I}^T\hat{z}_k$.

The selected candidate Lyapunov function is defined as follows:

$$V_k = \tilde{z}_k^T P\tilde{z}_k \tag{21}$$

with $P = P^T > 0$ is the matrix of Lyapunov.

Usually, to solve this kind of problem, most existing works on observer-based controller design for linear and nonlinear systems, use a particular form of the matrix $P = \text{block-diag}(P_1, P_2)$. To improve the existing results, we choose to use the following non-diagonal Lyapunov matrix in order to get a more relaxed LMI conditions:

$$P = \begin{pmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{pmatrix}. \tag{22}$$

Let $\Delta V_k = V_{k+1} - V_k$. Then, by developing the inequality (20), we obtain (23):

$$\begin{pmatrix} \tilde{z}_k \\ \nu_k \end{pmatrix}^T \Omega \begin{pmatrix} \tilde{z}_k \\ \nu_k \end{pmatrix} < 0. \tag{23}$$

with

$$\Omega = \begin{pmatrix} (1,1) & \big(\tilde{\mathcal{A}} + \Xi(\Theta)\big)^T P\tilde{\mathcal{E}} \\ (\star) & \tilde{\mathcal{E}}^T P\tilde{\mathcal{E}} - \frac{\lambda^2}{r}I_{s\times r} \end{pmatrix}.$$

and $(1,1) = \tilde{\mathcal{A}} + \Xi(\Theta)\big)^T P\big(\tilde{\mathcal{A}} + \Xi(\Theta)\big) - P + \tilde{\mathcal{I}}\mathbb{G}\tilde{\mathcal{I}}^T$.
Note that the inequality (23) is satisfied if $\Omega < 0$. Then, using Schur's lemma, the following inequality is obtained:

$$\begin{pmatrix} -P^{-1} & \tilde{\mathcal{A}} + \Xi(\Theta) & \tilde{\mathcal{E}} \\ (\star) & -P + \tilde{\mathcal{I}}\mathbb{G}\tilde{\mathcal{I}}^T & 0 \\ (\star) & (\star) & -\frac{\lambda^2}{r}I_{s\times r} \end{pmatrix} < 0. \tag{24}$$

The inequality (24) still unresolved due to its bilinear nature caused by the existence of an unknown matrix with its inverse (P and P^{-1}). To overcome this major obstacle, a slack variable technique is used with a judicious reformulation of Young's lemma. Thus, we obtain sufficient and less conservative conditions ensuring the asymptotic stability of the considered system. Corollary 1 introduce the main contribution of the paper.

Corollary 1. For a disturbance attenuation level $\lambda > 0$, the robust $\mathcal{H}_\infty$ observer-based stabilization design problem, corresponding to the system (5), the controller (11) and the observer (8), is solvable if for positive scalars η_2 and η_3, there exist positive scalar η_1 and matrices $\hat{P}_1$, P_2, $\hat{P}_3$, $\hat{Q}_1 > 0$, $Q_2 \in \mathbb{R}^{nM\times nM}$, $\mathcal{L} \in \mathbb{R}^{n\times pM}$ and $\mathcal{K} \in \mathbb{R}^{m\times nM}$ such that the following BMI is feasible for all $\Theta \in \mathcal{V}_{\mathcal{H}_{qn}}$:

$$\min \;\; \lambda \;\; \text{subject to}$$

$$\begin{pmatrix} \begin{pmatrix} \Pi_{11} & \hat{P}_3 & \Pi_{13} & 0 & \mathcal{E}_1 & \Pi_{16} \\ (\star) & \Pi_{22} & 0 & \Pi_{24} & \Pi_{25} & 0 \\ (\star) & (\star) & -\hat{P}_1 & -\hat{P}_3 & 0 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 & I_{n\times r} \\ (\star) & (\star) & (\star) & (\star) & \Pi_{55} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{2}\hat{Q}_1 \end{pmatrix} & \Upsilon_1 \\ (\star) & \Upsilon_2 \end{pmatrix} < 0 \qquad (25)$$

with

$$\begin{aligned} \Pi_{11} &= \hat{P}_1 - \hat{Q}_1 - \hat{Q}_1^T + \eta_1 \mathcal{M}_1 \mathcal{M}_1^T \\ \Pi_{13} &= \big(\mathcal{A} + \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T + \mathcal{B}\mathcal{K}\big)\hat{Q}_1^T \\ \Pi_{16} &= -\mathcal{B}\mathcal{K}\hat{Q}_1^T \\ \Pi_{22} &= P_2 - Q_2 - Q_2^T \\ \Pi_{24} &= Q_2\big(\mathcal{A} + \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T - \mathcal{I}\mathcal{L}\mathcal{C}\big) \\ \Pi_{25} &= Q_2(\mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2) \\ \Pi_{44} &= -P_2 + \mathcal{I}G^T G\mathcal{I}^T \\ \Pi_{55} &= -\frac{\lambda^2}{r} I_{s\times r} \end{aligned}$$

$$\Upsilon_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ Q_2\mathcal{M}_1 & Q_2\mathcal{I}\mathcal{L}\mathcal{M}_2 & 0 & 0 & 0 \\ 0 & 0 & \hat{Q}_1\mathcal{N}_1^T & \hat{Q}_1\mathcal{N}_1^T & \hat{Q}_1\mathcal{N}_2^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Upsilon_2 = \begin{pmatrix} -\frac{1}{\eta_2} I & 0 & 0 & 0 & 0 \\ (\star) & -\frac{1}{\eta_3} I & 0 & 0 & 0 \\ (\star) & (\star) & -\eta_1 I & 0 & 0 \\ (\star) & (\star) & (\star) & -\eta_2 I & 0 \\ (\star) & (\star) & (\star) & (\star) & -\eta_3 I \end{pmatrix}.$$

■

Proof. A slack variable Q is introduced to eliminate P^{-1}. Pre-multiplying (24) by block-diag(Q, I, I) and post-multiplying it by block-diag(Q^T, I, I). Then, using the inequality $-QP^{-1}Q^T T \leq P - Q - Q^T$, (24) is equivalent to

$$\begin{pmatrix} P - Q - Q^T & Q\big(\tilde{\mathcal{A}} + \Xi(\Theta)\big) & Q\tilde{\mathcal{E}} \\ (\star) & -P + \tilde{\mathcal{I}}\mathbb{G}\tilde{\mathcal{I}}^T & 0 \\ (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} \end{pmatrix} < 0. \qquad (26)$$

The slack variable Q is defined by:

$$Q = \begin{pmatrix} Q_1 & Q_3 \\ Q_4 & Q_2 \end{pmatrix}. \qquad (27)$$

Using (22) and (27), the inequality (26) can be written as follows:

$$\begin{pmatrix} \Pi_{11} & P_2 - Q_3 - Q_4^T & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ (\star) & P_2 - Q_2 - Q_2^T & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -P_1 & -P_3 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 \\ (\star) & (\star) & (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} \end{pmatrix} < 0 \qquad (28)$$

with

$$\begin{aligned} \Pi_{11} &= P_1 - Q_1 - Q_1^T \\ \Pi_{13} &= Q_1\big(\mathcal{A} + \Delta\mathcal{A}_k + \mathcal{B}\mathcal{K} + \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T\big) \\ &\quad + Q_3\big(\Delta\mathcal{A}_k - \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k\big) \\ \Pi_{14} &= -Q_1\mathcal{B}\mathcal{K} + Q_3\big(\mathcal{A} - \mathcal{I}\mathcal{L}\mathcal{C} + \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T\big) \\ \Pi_{15} &= Q_1\mathcal{E}_1 + Q_3(\mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2) \\ \Pi_{23} &= Q_4\big(\mathcal{A} + \Delta\mathcal{A}_k + \mathcal{B}\mathcal{K} + \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T\big) \\ &\quad + Q_2\big(\Delta\mathcal{A}_k - \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k\big) \\ \Pi_{24} &= -Q_4\mathcal{B}\mathcal{K} + Q_2\big(\mathcal{A} - \mathcal{I}\mathcal{L}\mathcal{C} + \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T\big) \\ \Pi_{25} &= Q_4\mathcal{E}_1 + Q_2(\mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2) \\ \Pi_{44} &= -P_2 + \mathcal{I}G^T G\mathcal{I}^T. \end{aligned}$$

As can be seen from (28), the observer gain is coupled with the unknown matrices Q_3 and Q_2. While, the gain $\mathcal{K}$ is coupled with the unknown matrices Q_1 and Q_4. To overcome this problem, we can choose $Q_3 = Q_4 = 0$. Then, the new structure of the matrix Q is given by:

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}. \qquad (29)$$

Pre-multiplying (28) by block-diag$(\hat{Q}_1, I, I, \hat{Q}_1, I)$ and post-multiplying it by block-diag$(\hat{Q}_1^T, I, \hat{Q}_1^T, I, I)$ with $\hat{Q}_1 = Q_1^{-1}$. Then, using the notations $\hat{P}_3 = \hat{Q}_1 P_3$ and $\hat{P}_1 = \hat{Q}_1 P_1 \hat{Q}_1^T$, we get the following inequality:

$$\begin{pmatrix} \hat{P}_1 - \hat{Q}_1 - \hat{Q}_1^T & \hat{P}_3 & \Pi_{13} & -\mathcal{B}\mathcal{K} & \mathcal{E}_1 \\ (\star) & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -\hat{P}_1 & -\hat{P}_3 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 \\ (\star) & (\star) & (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} \end{pmatrix} < 0 \qquad (30)$$

with

$$\begin{aligned} \Pi_{13} &= \big(\mathcal{A} + \Delta\mathcal{A}_k + \mathcal{B}\mathcal{K} + \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T\big)\hat{Q}_1^T \\ \Pi_{22} &= P_2 - Q_2 - Q_2^T \\ \Pi_{23} &= Q_2(\Delta\mathcal{A}_k - \mathcal{I}\mathcal{L}\Delta\mathcal{C}_k)\hat{Q}_1^T \\ \Pi_{24} &= Q_2\big(\mathcal{A} - \mathcal{I}\mathcal{L}\mathcal{C} + \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T\big) \\ \Pi_{25} &= Q_2(\mathcal{E}_1 - \mathcal{I}\mathcal{L}\mathcal{E}_2) \\ \Pi_{44} &= -P_2 + \mathcal{I}G^T G\mathcal{I}^T. \end{aligned}$$

To couple the gain $\mathcal{K}$, in Π_{14}, with the matrix $\hat{Q}_1^T$, we rewrite the inequality (30) as follows:

$$\begin{aligned} &\begin{pmatrix} \hat{P}_1 - \hat{Q}_1 - \hat{Q}_1^T & \hat{P}_3 & \Pi_{13} & 0 & \mathcal{E}_1 \\ (\star) & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -\hat{P}_1 & -\hat{P}_3 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 \\ (\star) & (\star) & (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} \end{pmatrix} \\ &+ \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ I_{n\times r} \\ 0 \end{pmatrix}}_{U^T} \underbrace{\begin{pmatrix} -(\mathcal{B}\mathcal{K})^T & 0 & 0 & 0 & 0 \end{pmatrix}}_{V} + V^T U < 0. \end{aligned} \qquad (31)$$

Then, by applying lemma 2 on (31) with $S = \hat{Q}_1$, the following inequality is obtained:

$$\begin{pmatrix} \hat{P}_1 - \hat{Q}_1 - \hat{Q}_1^T & \hat{P}_3 & \Pi_{13} & 0 & \mathcal{E}_1 & -\mathcal{BK}\hat{Q}_1^T \\ (\star) & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & 0 \\ (\star) & (\star) & -\hat{P}_1 & -\hat{P}_3 & 0 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 & I_{n\times r} \\ (\star) & (\star) & (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{2}\hat{Q}_1 \end{pmatrix} < 0 \quad (32)$$

Note that in order to be able to apply lemma 2 on (31), the matrix $\hat{Q}_1$ should be symmetric and positive.

Using lemma 3 and equations (6), we obtain

$$\begin{pmatrix} \hat{P}_1 - \hat{Q}_1 - \hat{Q}_1^T & \hat{P}_3 & \Pi_{13} & 0 & \mathcal{E}_1 & -\mathcal{BK}\hat{Q}_1^T \\ (\star) & \Pi_{22} & 0 & \Pi_{24} & \Pi_{25} & 0 \\ (\star) & (\star) & -\hat{P}_1 & -\hat{P}_3 & 0 & 0 \\ (\star) & (\star) & (\star) & \Pi_{44} & 0 & I_{n\times r} \\ (\star) & (\star) & (\star) & (\star) & -\frac{\lambda^2}{r} I_{s\times r} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{2}\hat{Q}_1 \end{pmatrix}$$
$$+\eta_1 U_1 U_1^T + \frac{1}{\eta_1} V_1^T V_1 + \eta_2 U_2 U_2^T + \frac{1}{\eta_2} V_1^T V_1 + \eta_3 U_3 U_3^T + \frac{1}{\eta_3} V_2^T V_2 < 0 \quad (33)$$

with

$$\begin{aligned} \Pi_{13} &= \big(\mathcal{A} + \mathcal{BK} + \mathcal{D}\Xi_1(\Theta)\mathcal{I}^T\big)\hat{Q}_1^T \\ \Pi_{24} &= Q_2\big(\mathcal{A} - \mathcal{ILC} + \mathcal{D}\Xi_2(\Theta)\mathcal{I}^T\big) \\ U_1 &= \big(\mathcal{M}_1^T\ 0\ 0\ 0\ 0\ 0\big)^T \\ U_2 &= \big(0\ \mathcal{M}_1^T Q_2^T\ 0\ 0\ 0\ 0\big)^T \\ U_3 &= \big(0\ \mathcal{M}_2^T \mathcal{L}^T \mathcal{I}^T Q_2^T\ 0\ 0\ 0\ 0\big)^T \\ V_1 &= \big(0\ 0\ \mathcal{N}_1 \hat{Q}_1^T\ 0\ 0\ 0\big) \\ V_2 &= \big(0\ 0\ \mathcal{N}_2 \hat{Q}_1^T\ 0\ 0\ 0\big). \end{aligned}$$

Finally, by applying Schur's lemma, we obtain the constraint (25). This ends the proof of corollary 1.

3.2 Converting BMI into LMI

To linearize the BMI given by (25), the following change of variables for the terms coupled with the control gain $\mathcal{K}$ is defined:

$$\hat{\mathcal{K}} = \mathcal{K}\hat{Q}_1^T, \quad (34)$$

but this is not the case for the terms coupled with the observer gain $\mathcal{L}$ because of the matrix $\mathcal{I}$. Therefore, a particular solution is proposed to linearize this BMI.
We consider the following particular form of the matrix Q_2:

$$Q_2 = \begin{pmatrix} Q_2^{11} & \alpha_1 Q_2^{11} & \cdots & \cdots & \alpha_1 Q_2^{11} \\ \beta_1 Q_2^{11} & Q_2^{22} & \alpha_2 Q_2^{22} & \cdots & \alpha_2 Q_2^{22} \\ \vdots & \beta_2 Q_2^{22} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Q_2^{r-1,r} \\ \beta_1 Q_2^{11} & \beta_2 Q_2^{22} & \cdots & Q_2^{r,r-1} & Q_2^{rr} \end{pmatrix} \quad (35)$$

with $Q_2^{r-1,r} = \alpha_{r-1} Q_2^{r-1,r-1}$, $Q_2^{r,r-1} = \beta_{r-1} Q_2^{r-1,r-1}$, $0 \le \alpha_i < 1$ and $0 \le \beta_i < 1$ for $i \in \{1, \cdots, r-1\}$.
Then, the following changes of variables is defined:

$$\hat{L}_i = Q_2^{11} L_i, \quad \text{for } i \in \{1, \cdots, r\} \quad (36)$$

So, the BMI (25) is transformed into a convex problem.

Finally, the controller and the observer gains can be computed through the following equations:

$$\mathcal{K} = \hat{\mathcal{K}}^T \hat{Q}_1^{-T} \quad (37)$$

$$L_i = (Q_2^{11})^{-1} \hat{L}_i, \quad \text{for } i \in \{1, \cdots, r\} \quad (38)$$

4. NUMERICAL EXAMPLE

Let us consider the system studied in Kheloufi et al. (2016). This system is described by the state model (5) with

$$A = \begin{pmatrix} 0.2 & 0.1 & 0.4 \\ 0.6 & 1 & 0.5 \\ -0.3 & 0 & 0.3 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ -0.4 & 0.5 \\ 0.6 & -0.4 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$E_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 0 \\ 0.1 & 0.3 \\ 0 & 0.2 \end{pmatrix},$$
$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0.3 \\ 0 & 0.8 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$
$$f(x_k) = \begin{pmatrix} 0.1\sin(x_{2k}) \\ 0.2\sin(x_{3k}) \\ 0.3\sin(x_{1k}) \end{pmatrix}, F_k = \sin(k^4).$$

$f(x_k)$ satisfy lemma 1 such that the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{33}} = \left\{ \begin{pmatrix} 0 & \pm 0.1 & 0 \\ 0 & 0 & \pm 0.2 \\ \pm 0.3 & 0 & 0 \end{pmatrix} \right\}.$$

The obtained gain matrices using the sliding window approach with $r = 2$ and $\alpha = \beta = 0.01$ are as follows:

$$L_1 = \begin{pmatrix} 0.4181 & -0.1258 \\ 1.1749 & -0.3818 \\ 0.2278 & -0.0960 \end{pmatrix}, K_1^T = \begin{pmatrix} 0.3357 & -0.2507 \\ -0.0392 & -0.1028 \\ -0.1523 & -0.1311 \end{pmatrix},$$
$$L_2 = \begin{pmatrix} 0.0055 & -0.0030 \\ 0.0036 & -0.0025 \\ -0.0012 & 0.0003 \end{pmatrix}, K_2^T = \begin{pmatrix} 0.0000 & 0.0000 \\ 0.0000 & -0.0000 \\ 0.0005 & -0.0042 \end{pmatrix},$$

and $\lambda = 1.4551$.
The simulation results using the proposed approach are given by figure 1 such that $x_0 = (10\ 7\ {-5})^T$, $\hat{x}_0 = (-1\ 4\ 1.5)^T$ and $\omega = 1$ for $t \in [2, 3]$.

5. CONCLUSION

A robust sliding window controller for Lipschitz nonlinear systems with parametric uncertainties in a noisy context is proposed in this paper. The proposed strategy introduces two sliding windows of delayed measurements and states, respectively, into the Luenberger observer and the control law in order to get less restrictive LMI conditions. A judicious use of Young's lemma combined with a particular slack variable allows to enhance the obtained optimization problem. Numerical example is presented to validate the proposed LMI conditions.

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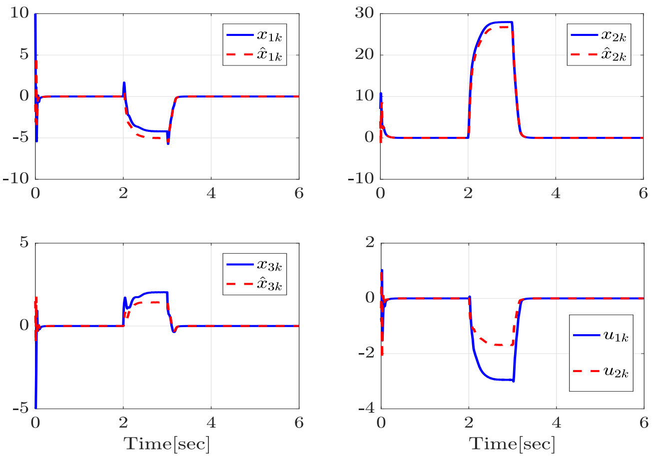

Fig. 1. Results of the sliding window approach

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