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On the Choice of Multiple Flat Outputs for Fault Detection and Isolation of a Flat System

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Abstract: This paper presents a rigorous definition of the isolability of a fault in a flat system whose flat outputs are measured by sensors that are subject to faults. In particular, if only one sensor or actuator is faulty at a time, we show that the isolation of faults can be achieved if a pair of flat outputs satisfies some independence condition. A detailed characterization of this condition is presented. Finally, the pertinence of the isolability concept is demonstrated on the example of a three tank system.

Keywords: nonlinear flat system, flat output, fault detection and isolation, three tank system.

1. INTRODUCTION

The fault detection and isolation (FDI) problem has been introduced in automatic control as a paradigm for designing algorithms able to detect the outbreak of faults and isolate their causes. Various FDI techniques have been developed and can be found in survey papers, see *e.g.* (Zhou et al., 2014; Thirumarimurugan et al., 2016). The first proposed method is the *hardware redundancy* in which multiple sensors and actuators are used to measure and control a particular variable (Chen et al., 2015). The drawbacks of this method are the extra equipment, maintenance cost and additional space required to accommodate the equipment. This approach was improved later on by the introduction of the *model-based analytical redundancy method*, based on the notion of *generating residual signals*. These residues are defined as the difference between the measured variables and the estimated ones. In the case of no fault, and in the ideal case of noise free observations, the values of the residues are equal to zero. In the non zero case, the estimation method must be specified, see *e.g.* the observer-based approach (Tousi and Khorasani, 2011), the parity-space approach (Diversi et al., 2002) or the Kalman-based approach (Izadian and Khayyer, 2010). However, in these approaches, a sensor may be wrongly declared faulty because of the lack of efficiency of the estimation algorithm, hence the importance of the notion of *detectability*.

Recently, the flatness property has been introduced into the repertoire of FDI techniques (Suryawan et al., 2010; Martínez-Torres et al., 2014). Here, residues are calculated using the differential flatness property. Roughly speaking, let us recall that a system is said to be flat if all the state and input variables can be expressed as functions of a particular variable, called flat output, and a finite number of its successive derivatives. The method presented in Suryawan et al. (2010) is dedicated to linear flat systems and uses the properties of B-spline parameterisation to estimate the time derivatives of the flat output, which may not be defined because of the presence of noise. This derivative estimation can take time and cause a delay in the reconfiguration process. In order to overcome these issues, a high-gain observer has been proposed in Martínez-Torres et al. (2014) to evaluate the time derivative of the noisy signals. The observer may be complemented by a low-pass filter to improve its performance. Note that the latter method can be applied to both, linear and nonlinear flat systems.

In the present flatness-based FDI approach, an effort is made to dissociate the theoretical *isolability* property, based on residue computation, and the estimation process. For this purpose, we compute the residues between the measurements and their expression exactly obtained from the measured flat outputs and their derivatives estimated online. The treatment of these residues slightly differs from the ones of the previous approaches (Kóscielny et al., 2016): every sensor and actuator admits a *fault alarm signature*, *i.e.* a number of residues affected by a fault on this sensor/actuator and a fault on a sensor/actuator is isolable if its corresponding fault alarm signature is distinct. In practice, the treatment of these residues is adapted, in the presence of noise, by introducing a threshold and an estimation process as in the previous approaches (Martínez-Torres et al., 2013). Moreover, we show that it is possible to increase the isolability of faults by considering several flat outputs, at the condition that they are independent, thus completing in a rigorous way some heuristic results of Martínez-Torres et al. (2013). These results are applied to a three tank FDI problem where we compute two independent flat outputs that allow the isolation of all possible simple faults (only one faulty sensor or actuator at a time).

The main contributions of this paper are the above mentioned rigorous definition of isolability of faults and the characterization of the flat outputs to be used in the fault isolation.

This paper is organized as follows: section 2 introduces the basic concepts of FDI for nonlinear differentially flat systems and their definitions. Section 3 discusses the conditions for independence between flat outputs. Section 4 deals with the application of this FDI approach to the three tank system. Finally, section 5 concludes the paper.

2. FLATNESS-BASED FDI

2.1 Differentially Flat System

Consider the following nonlinear system

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \tag{1}$$

where x, the vector of states, evolves in a n-dimensional manifold X, $u \in \mathbb{R}^m$ is the vector of inputs, $y \in \mathbb{R}^p$ is the measured output, $m \leq n$, $\mathrm{rank}(\frac{\partial f}{\partial u}) = m$ and $m \leq p$. Let $(x, \overline{u}) \triangleq (x, u, \dot{u}, \ddot{u}, \ldots)$ be a prolongation of the coordinates (x,u) to the manifold of jets of infinite order $\mathfrak{X} \triangleq X \times \mathbb{R}^m_\infty$ (Fliess et al., 1999), (Levine, 2009, Chapter 5).

In the sequel, we systematically denote by $\overline{\xi} \triangleq (\xi, \dot{\xi}, \ddot{\xi}, \ldots)$ the sequence of infinite order jets of a vector ξ and $\overline{\xi}^{(\alpha)} \triangleq (\xi, \dot{\xi}, \ddot{\xi}, \ldots, \xi^{(\alpha)})$ the truncation at the finite order $\alpha \in \mathbb{N}$ of the previous sequence.

The system (1) is flat at a point $(x_0, \overline{u}_0) \in \mathfrak{X}$ if and only if there exist a vector $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, two integers ρ and ν and mappings ψ defined on a neighbourhood $\mathcal{V}$ of $(x_0, \overline{u}_0)$ in $\mathfrak{X}$ and $\varphi = (\varphi_0, \varphi_1, \ldots)$ defined on a neighbourhood $\mathcal{W} \subset \psi(\mathcal{V})$ of $\overline{z} \triangleq (z, \dot{z}, \ddot{z}, \ldots) \triangleq \psi(x_0, \overline{u}_0)$ in $\mathbb{R}^m_\infty$ such that:

(1) $z = \psi(x, \overline{u}^{(\nu)}) \in \mathcal{W}$
(2) $z_1, \ldots, z_m$ and their successive derivatives are linearly independent in $\mathcal{W}$
(3) The state x and the input u are functions of z and its successive derivatives:
$$(x,u) = (\varphi_0(\overline{z}^{(\rho)}), \varphi_1(\overline{z}^{(\rho+1)})) \in \mathrm{pr}_{X \times \mathbb{R}^m}(\mathcal{V}) \tag{2}$$
where $\mathrm{pr}_{X \times \mathbb{R}^m}(\mathcal{V})$ is the canonical projection from $\mathcal{V}$ to $X \times \mathbb{R}^m$
(4) The differential equation $\dot{\varphi}_0(\overline{z}) = f(\varphi_0(\overline{z}), \varphi_1(\overline{z}))$ is identically satisfied in $\mathcal{W}$.

The vector z is called flat output of the system. The mappings ψ and φ are called *Lie-Bäcklund isomorphisms* and are inverse of one another.

Remark 1. The property of flatness is not defined globally. The Lie-Bäcklund isomorphisms ψ and φ are non unique and only locally defined. Thus, there might exist points in $\mathfrak{X}$ where no such isomorphisms exist or, otherwise stated, where the system is not flat. It has been proven in Kaminski et al. (2018) that the set of intrinsic singularities contains the set of equilibrium points of the system that are not first order controllable.

2.2 Fault Detection and Isolation

For the flat system (1), we suppose that the vector $y^s = (y_1^s, \ldots, y_p^s)^T$ is measured by sensors $\mathsf{S}_1, \ldots, \mathsf{S}_p$ respectively. We also suppose that the flat output z is part of these measurements according, without loss of generality, to

$$z^s = (y_1^s, \ldots, y_m^s)^T. \tag{3}$$

Moreover, the value of the input vector $u = (u_1, \ldots, u_m)^T$, corresponding to the actuators $\mathsf{A}_1, \ldots, \mathsf{A}_m$, is assumed to be available at every time. We now propose a new definition of the notion of residue that generalizes the one introduced by Martínez-Torres et al. (2014).

According to (2), the state and input read:

$$x^z = \varphi_0(\overline{z^s}^{(\rho)}), \quad u^z = \varphi_1(\overline{z^s}^{(\rho+1)}) \tag{4}$$

where the superscript z indicates that they are evaluated as functions of the measurements z^s and, according to (1),

$$y_k^z \triangleq h_k(\varphi_0(\overline{z^s}^{(\rho)}), \varphi_1(\overline{z^s}^{(\rho+1)})) \tag{5}$$

is the virtual value of y_k computed via the measured flat output z^s.

Note that the first m components of y^z are equal to the corresponding components of z^s:

$$y^z = (z^s, \widetilde{h}(\varphi_0(\overline{z^s}), \varphi_1(\overline{z^s})))^T \tag{6}$$

with $\widetilde{h} = (h_{m+1}(\varphi_0(\overline{z^s}), \varphi_1(\overline{z^s})), \ldots, h_p(\varphi_0(\overline{z^s}), \varphi_1(\overline{z^s})))^T$.

Definition 1. The $\mathrm{k^{th}}$-sensor residue R_{S_k} and l^{th}-input residue R_{A_l}, for $k = 1, \ldots, p$ and $l = 1, \ldots, m$, are given by:

$$R_{\mathsf{S}_k} = y_k^s - y_k^z, \quad R_{\mathsf{A}_l} = u_l - u_l^z. \tag{7}$$

In total, we have $p+m$ residues for a single flat output z^s and we denote the full residue vector by:

$$\begin{aligned} r &= (R_{\mathsf{S}_1}, \ldots, R_{\mathsf{S}_m}, R_{\mathsf{S}_{m+1}}, \ldots, R_{\mathsf{S}_p}, R_{\mathsf{A}_1}, \ldots, R_{\mathsf{A}_m})^T \\ &= (r_1, \ldots, r_m, r_{m+1}, \ldots, r_p, r_{p+1}, \ldots, r_{p+m})^T \end{aligned} \tag{8}$$

and according to (6)

$$\begin{aligned} r &= (0, \ldots, 0, R_{\mathsf{S}_{m+1}}, \ldots, R_{\mathsf{S}_p}, R_{\mathsf{A}_1}, \ldots, R_{\mathsf{A}_m})^T \\ &= (0, \ldots, 0, r_{m+1}, \ldots, r_p, r_{p+1}, \ldots, r_{p+m})^T. \end{aligned} \tag{9}$$

Measured and calculated variables are illustrated in Fig. 1.

A residue who is always equal to zero indicates that it cannot be affected by faults on one of the sensors or actuators. Then, we eliminate it and truncate the residue vector to keep the last p components only. This truncated vector is denoted by r_τ:

$$\begin{aligned} r_\tau &= (R_{\mathsf{S}_{m+1}}, \ldots, R_{\mathsf{S}_p}, R_{\mathsf{A}_1}, \ldots, R_{\mathsf{A}_m})^T \\ &= (r_{\tau_1}, r_{\tau_2}, \ldots, r_{\tau_p})^T. \end{aligned} \tag{10}$$

Hypothesis: From now on, we assume that there is only one fault at a time affecting the sensors or actuators.

In practice, due to the presence of noises on sensors and actuators, the successive derivatives of z^s may not be

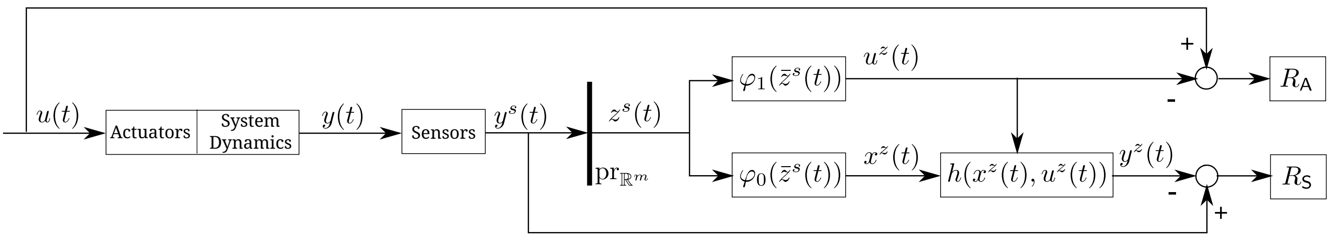

Fig. 1. *Flatness-based residual generation*

defined. We assume that they are computed via a high-gain observer, possibly completed by a low-pass filter as in Martínez-Torres et al. (2014) to improve its robustness. Moreover, a threshold is associated to each residue. In the non faulty case, the residues in (10) will not exceed their thresholds. If, otherwise, at least one of the residues exceeds its threshold then a fault alert is launched. If several residues in (10) trigger an alert at the same time, a fault alarm signature, defined bellow, is required to isolate the fault.

For this purpose, we introduce the so-called *signature matrix*:

Definition 2. (Signature matrix). Given the vector of residues r_τ defined in (10) and $\zeta = (y_1^s, \ldots, y_p^s, u_1, \ldots, u_m)^T \in \mathbb{R}^{p+m}$ the vector of available measurements. We define by the *signature matrix* associated to z^s, the matrix $\mathbf{S}$ given by:

$$\mathbf{S} = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \ldots & \sigma_{1,p+m} \\ \vdots & \vdots & \ldots & \vdots \\ \sigma_{p,1} & \sigma_{p,2} & \ldots & \sigma_{p,p+m} \end{pmatrix} \tag{11}$$

with

$$\sigma_{i,j} \triangleq \begin{cases} 0 & \text{if} \quad \dfrac{\partial r_{\tau_i}}{\partial \zeta_j^{(\varrho)}} = 0 \quad \forall \varrho \in \{0, 1, \ldots\} \\ 1 & \text{if} \quad \exists\, \varrho \in \{0, 1, \ldots\} \text{ s.t. } \dfrac{\partial r_{\tau_i}}{\partial \zeta_j^{(\varrho)}} \neq 0 \end{cases} \tag{12}$$

Remark 1. Each column Σ_j of the signature matrix $\mathbf{S}$ indicates whether a residue r_{τ_i} is or is not functionally affected by a fault on the measurement ζ_j. So in (12), $\sigma_{i,j} = 0$ means that the residue r_{τ_i} is not affected by a fault on the measurement ζ_j and $\sigma_{i,j} = 1$ means that the residue may be affected.

Definition 3. A column Σ_j of the signature matrix $\mathbf{S}$ is called *fault alarm signature* or simply *signature*, associated to the sensor/actuator ζ_j.

From the signature matrix $\mathbf{S}$ we propose the following definitions of detectability and isolability in the flatness context:

Definition 4. (Detectability). A fault on a sensor/actuator ζ_j is *detectable* if, and only if there exists at least one $i \in \{1, \ldots, p\}$ such that $\sigma_{i,j} = 1$.

Definition 5. (Isolability). A fault on a sensor S_k, $k = 1, \ldots, p$, is said *isolable* if, and only if, its corresponding fault alarm signature Σ_k in the signature matrix $\mathbf{S}$ is distinct from the others, *i.e.*

$$\Sigma_k \neq \Sigma_j, \quad \forall j = 1, \ldots, p+m,\ j \neq k. \tag{13}$$

An isolable fault on the actuator A_l, for $l = 1, \ldots, m$, is defined analogously:

$$\Sigma_{p+l} \neq \Sigma_j, \quad \forall j = 1, \ldots, p+m,\ j \neq p+l. \tag{14}$$

We define μ as the number of distinct signatures of the signature matrix $\mathbf{S}$ associated to z^s. Then, μ is the number of isolable faults associated to z^s.

A more general, but much more complicated, definition of isolability in the structured residual context of polynomial systems has been introduced in Staroswiecki and Comtet-Varga (2001), based on elimination techniques.

Definition 5 means that if the signature matrix $\mathbf{S}$ has two identical signatures, *i.e.* $\Sigma_i = \Sigma_j$, for two different sensors/actuators $\zeta_i \neq \zeta_j$, then we cannot make a decision on the faulty device, hence the fault is detected but cannot be isolated. Thus, the number of isolated faults is equal to the number of distinct signatures in the matrix $\mathbf{S}$.

2.3 The Example of the three tank System

We consider a three tank system made up with three cylindrical tanks of cross-sectional area S, connected to each other by means of cylindrical pipes of section S_n, and two pumps P_1 and P_2 that supply tanks T_1 and T_2. These three tanks are also connected to a central reservoir through pipes (see Fig. 2).

The model is given by:

$$\dot{x}_1 = -Q_{10}(x_1) - Q_{13}(x_1, x_3) + u_1 \tag{15}$$
$$\dot{x}_2 = -Q_{20}(x_2) + Q_{32}(x_2, x_3) + u_2 \tag{16}$$
$$\dot{x}_3 = Q_{13}(x_1, x_3) - Q_{32}(x_2, x_3) - Q_{30}(x_3) \tag{17}$$

where the state variables x_i, $i = 1, 2, 3$ represent the water level of each tank, Q_{i0}, $i = 1, 2, 3$ the outflow between each tank and the central reservoir, Q_{13} is the outflow between tanks T_1 and T_3 and Q_{32} the outflow between tanks T_3 and T_2, u_1 and u_2 are the incoming flows by unit of surface of each pump.

We assume the following inequalities to avoid singularities [1] :

$$x_1 > x_3 > x_2.$$

We consider that the valves connecting tanks T_1 and T_3 with the central reservoir are closed, *i.e.* $Q_{10} \equiv 0$ and $Q_{30} \equiv 0$. The expressions of Q_{13}, Q_{32} and Q_{20} are given by:

$$Q_{13}(x_1, x_3) = a_{z1}\sqrt{2g(x_1 - x_3)} \tag{18}$$
$$Q_{20}(x_2) = a_{z2}\sqrt{2g(x_2)} \tag{19}$$
$$Q_{32}(x_2, x_3) = a_{z3}\sqrt{2g(x_3 - x_2)} \tag{20}$$

[1] According to the *Remark 1*, the point $\overline{x} \in \mathfrak{X}$ s.t. $x_1 = x_2 = x_3$ is an equilibrium point which is not first order controllable, then it is a point of intrinsic flatness singularity.

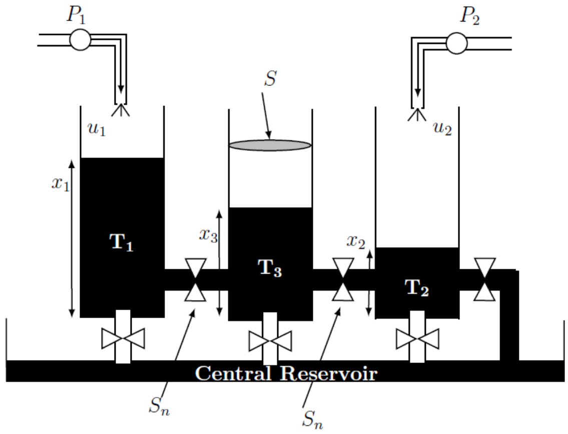

Fig. 2. *Three Tank System, Source: (Noura et al., 2009)*

where a_{zr}, $r = 1, 2, 3$, is the flow coefficient and g the gravitational force. Each tank T_i is equipped with a sensor S_i to measure its level x_i. Hence, the measured output is:

$$y^s = (y_1^s, y_2^s, y_3^s)^T = (x_1^s, x_2^s, x_3^s)^T \tag{21}$$

The system (15)-(16)-(17) is flat with $z = (x_1, x_3)^T = (z_1, z_2)^T$ as flat output. The measured flat output is then given by $z^s = (y_1^s, y_3^s)^T = (z_1^s, z_2^s)^T$. In order to construct the vector of residues, using (4) and (5), we set:

$$\begin{aligned}
y_1^z &= z_1^s \\
y_2^z &= z_2^s - \frac{1}{2g}\Big(\frac{a_{z1}\sqrt{2g(z_1^s - z_2^s)} - \dot{z}_2^s}{a_{z3}}\Big)^2 \\
y_3^z &= z_2^s \\
u_1^z &= \dot{z}_1^s + a_{z1}\sqrt{2g(z_1^s - z_2^s)} \\
u_2^z &= \dot{y}_2^z - a_{z3}\sqrt{2g(z_2^s - y_2^z)} + a_{z2}\sqrt{2gy_2^z}.
\end{aligned}$$

According to (7), the vector of residues, associated to z^s, is then given by:

$$r = \begin{pmatrix} R_{\mathsf{S}_1} \\ R_{\mathsf{S}_2} \\ R_{\mathsf{S}_3} \\ R_{\mathsf{A}_1} \\ R_{\mathsf{A}_2} \end{pmatrix} = \begin{pmatrix} y_1^s \\ y_2^s \\ y_3^s \\ u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} y_1^z \\ y_2^z \\ y_3^z \\ u_1^z \\ u_2^z \end{pmatrix}. \tag{22}$$

However, residues R_{S_1} and R_{S_3} are identically zero:

$$\begin{aligned}
R_{\mathsf{S}_1} &= y_1^s - y_1^z = z_1^s - z_1^s = 0 \\
R_{\mathsf{S}_3} &= y_3^s - y_3^z = z_2^s - z_2^s = 0
\end{aligned} \tag{23}$$

hence, according to (10), the vector r is truncated to:

$$r_\tau = (R_{\mathsf{S}_2}, R_{\mathsf{A}_1}, R_{\mathsf{A}_2})^T = (r_{\tau_1}, r_{\tau_2}, r_{\tau_3})^T. \tag{24}$$

Therefore, the signature matrix $\mathbf{S}$, associated to z^s, is constructed as follows:

– All the residues in (24) depend on the measurement of $z^s = (y_1^s, y_3^s)^T$ then the first and the third columns of the signature matrix contain only ones:
$$\sigma_{i,1} = \sigma_{i,3} = 1, \forall i = 1, 2, 3$$
– Only residue r_{τ_1} depends on y_2^s and its successive derivatives, then the second column will be such that:
$$\sigma_{1,2} = 1 \quad \text{and} \quad \sigma_{i,2} = 0,\ i = 2, 3$$
– Since r_{τ_2} depends only on u_1 and r_{τ_3} depends only on u_2, then column 4 and column 5 of $\mathbf{S}$ are such that:
$$\sigma_{2,4} = 1 \quad \text{and} \quad \sigma_{i,4} = 0\ \forall i = 1, \ldots, 3,\ i \neq 2$$
and
$$\sigma_{3,5} = 1 \quad \text{and} \quad \sigma_{i,5} = 0\ \forall i = 1, \ldots, 3,\ i \neq 3$$
respectively.

Hence, the signature matrix, associated to r_τ, is given by:

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{25}$$

According to definition 4, all faults on the three tank system's sensors and actuators are detectable. Since fault alarm signatures Σ_2, Σ_4 and Σ_5 are distinct, then, according to definition 5, faults on sensor S_2 and actuators A_1 and A_2 are isolable. This reflects the fact that if, at some point during system operation, a fault alarm is launched with the signature Σ_2 then we conclude that the sensor S_2 is faulty. However, if we obtain a signature like Σ_1, the fault could be on the sensor S_1 or S_3, since signatures Σ_1 and Σ_3 are identical. Then, a fault on S_1 or S_3 cannot be isolated. To conclude, this example shows that the isolability property is strongly conditioned by the dependence of the flat output with respect to the measured variables. This motivates the study of the choice of flat outputs of the next section.

Remark 2. In Nagy et al. (2009), it has been shown that system (15)-(16)-(17) is observable through x_1 only and that x_2 and x_3 can be estimated using x_1 given the measurements of u_1 and u_2, leading to different isolability results. The reader may refer to this article for more details. Note that, here, the measurements of u_1 and u_2 are not necessary to guarantee the x_2-isolability.

3. FLAT OUTPUT SELECTION

In order to get more isolabilty on systems sensor and actuator, the authors in Martínez-Torres et al. (2014) propose to increase the number of residues by using several flat outputs. These flat outputs must be *independent* in the sense that when we use them together we gain more isolability of faults. In this section, we propose a characterization of the relation between different flat outputs using a so-called *augmented signature matrix*. This characterization leads to a decision concerning the choice of flat outputs that are useful for the isolability.

According to definition 5, the number μ of isolated faults by a flat output z is equal to the number of distinct signatures Σ_k of its signature matrix. Then, in order to get more isolability of faults, we need to increase the number of distinct signatures. This is possible when different projections of the system's output y are available that are flat outputs. For this purpose, we introduce definitions 6 and 7.

In the following, we denote the i$^{\text{th}}$ element of the set of q flat output vectors Z_i by $Z_i = (z_{i1}, \ldots, z_{im})^T$.

Definition 6. (Augmented signature matrix). Let $Z_1, \ldots, Z_q$ be q different flat output vectors of the flat system (1), such that $Z_i = \mathrm{pr}_{\mathbb{R}^m}(y)$. The *augmented signature matrix* $\widetilde{\mathbf{S}}$ associated to $Z_1, \ldots, Z_q$ is defined by:

$$\widetilde{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_q \end{pmatrix} \tag{26}$$

where $\mathbf{S}_i$ is the signature matrix associated to the flat output vector Z_i.

The choice of flat output vectors is not arbitrary. They must be independent in the sense given by the following definition:

Definition 7. (Independence). Let $\widetilde{\mathbf{S}}$ be the augmented signature matrix associated to Z_1 and Z_2:

$$\widetilde{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{pmatrix},$$

μ_i, $i = 1, 2$, the number of distinct signatures of the matrix $\mathbf{S}_i$ and $\widetilde{\mu}$ the number of distinct signatures of the augmented matrix $\widetilde{\mathbf{S}}$. We say that Z_1 and Z_2 are *independent* if, and only if

$$\widetilde{\mu} > \mu_1 \quad \text{and} \quad \widetilde{\mu} > \mu_2. \tag{27}$$

Definition 7 means that two flat outputs are independent if, by using them together, the number of distinct signatures increases which corresponds to the number of isolated faults. If the condition (27) is not satisfied then the combination of Z_1 and Z_2 is not helpful for the isolability, and we have to find another combination by calculating more flat outputs. To conclude, the condition of full isolability is given by the following proposition:

Proposition 2. Let $Z_1, \ldots, Z_q$ be q different flat output vectors of the system (1). A full isolability of faults on sensors and actuators is achieved if the augmented matrix

$$\widetilde{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_q \end{pmatrix}$$

has $p + m$ distinct signatures, *i.e.* $\widetilde{\mu} = p + m$.

4. APPLICATION TO THE THREE TANK SYSTEM

Back to the three tank system presented in section 2.3, we denote by Z_1 the flat output vector $Z_1 = (z_{11}, z_{12})^T = (x_1, x_3)^T$. The corresponding vector of residues is given by (24). We recall the signature matrix associated to Z_1, and we denote it by $\mathbf{S}_1$:

$$\mathbf{S}_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{28}$$

We also recall that, according to definition 5, faults on sensors S_1 and S_3 cannot be isolated. The number of distinct signatures of $\mathbf{S}_1$ is $\mu_1 = 3$.

In order to increase the number of isolable faults, we consider $Z_2 = (z_{21}, z_{22})^T = (x_2, x_3)^T$ another flat output vector of the three tank system. It is measured by sensors S_2 and S_3, *i.e.* $Z_2^s = (z_{21}^s, z_{22}^s)^T = (y_2^s, y_3^s)^T$. To construct the vector of residues associated to Z_2^s and its signature matrix, we set, using (4) and (5):

$$\begin{aligned} y_1^{Z_2} &= z_{22}^s + \frac{1}{2g}\Big(\frac{a_{z3}\sqrt{2g(z_{22}^s - z_{21}^s)} + \dot{z}_{22}^s}{a_{z1}}\Big)^2 \\ y_2^{Z_2} &= z_{21}^s \\ y_3^{Z_2} &= z_{22}^s \\ u_1^{Z_2} &= \dot{z}_{22}^s + a_{z1}\sqrt{2g(z_{21}^s - z_{22}^s)} \\ u_2^{Z_2} &= \dot{y}_2^{Z_2} - a_{z3}\sqrt{2g(z_{22}^s - y_2^{Z_2})} + a_{z2}\sqrt{2gy_2^{Z_2}}. \end{aligned}$$

Therefore, as shown for the flat output Z_1, residues $R_{\mathsf{S}_2}^{Z_2}$ and $R_{\mathsf{S}_3}^{Z_2}$ are identically zero and the truncated vector of residues (10) reads:

$$r_\tau^{Z_2} = \begin{pmatrix} R_{\mathsf{S}_1}^{Z_2} \\ R_{\mathsf{A}_1}^{Z_2} \\ R_{\mathsf{A}_2}^{Z_2} \end{pmatrix} = \begin{pmatrix} y_2^s \\ u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} y_2^{Z_2} \\ u_1^{Z_2} \\ u_2^{Z_2} \end{pmatrix}. \tag{29}$$

Hence, the signature matrix associated to Z_2 is given by:

$$\mathbf{S}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}. \tag{30}$$

Signatures Σ_1, Σ_4 and Σ_5 in the matrix $\mathbf{S}_2$ are distinct, then, according to definition 5, faults on sensor S_1 and actuators A_1 and A_2 are isolable by the flat output Z_2. Moreover, the number of distinct signatures of $\mathbf{S}_2$ is $\mu_2 = 3$. However, since signatures Σ_2 and Σ_3 are identical, then faults on sensors S_2 and S_3 cannot be isolated.

It remains to be verified whether the two flat outputs Z_1 and Z_2 are independent.

The augmented signature matrix associated to Z_1 and Z_2 is given by:

$$\widetilde{\mathbf{S}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}. \tag{31}$$

The number of distinct fault alarm signatures of $\widetilde{\mathbf{S}}$ is $\widetilde{\mu} = 5$, and we have

$$\widetilde{\mu} > \mu_1 \quad \text{and} \quad \widetilde{\mu} > \mu_2.$$

Then, according to definition 6, the flat output vectors Z_1 and Z_2 are independent. Moreover, since $\widetilde{\mu} = p + m$, then flat output vectors Z_1 and Z_2 ensure full isolability of faults on the three tank system.

Simulation results that confirm the effectiveness of this approach can be found in Martínez-Torres et al. (2013).

5. CONCLUSION

The current paper introduces a novel and rigorous definition of the isolability of faults affecting a system's sensors and actuators, using the flatness-based FDI approach. The described condition of isolability provides an efficient way to select flat outputs that are useful for fault isolation. Our results are tested and validated using the three tank system. Future work should focus on the development of a method that calculates independent flat outputs directly.

REFERENCES

Chen, J., Li, H., Sheng, D., and Li, W. (2015). A hybrid data-driven modeling method on sensor condition monitoring and fault diagnosis for power plants. *International Journal of Electrical Power & Energy Systems*, 71, 274–284.

Diversi, R., Simani, S., and Soverini, U. (2002). Robust residual generation for dynamic processes using decoupling technique. In *Proceedings of the International Conference on Control Applications*, volume 2, 1270–1275. IEEE.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1999). A lie-backlund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on automatic control*, 44(5), 922–937.

Izadian, A. and Khayyer, P. (2010). Application of kalman filters in model-based fault diagnosis of a dc-dc boost converter. In *IECON 2010-36th Annual Conference on IEEE Industrial Electronics Society*, 369–372. IEEE.

Kaminski, Y.J., Lévine, J., and Ollivier, F. (2018). Intrinsic and apparent singularities in differentially flat systems, and application to global motion planning. *Systems & Control Letters*, 113, 117–124.

Kóscielny, J.M., Syfert, M., Rostek, K., and Sztyber, A. (2016). Fault isolability with different forms of the faults–symptoms relation. *International Journal of Applied Mathematics and Computer Science*, 26(4), 815–826.

Levine, J. (2009). *Analysis and control of nonlinear systems: A flatness-based approach.* Springer Science & Business Media.

Martínez-Torres, C., Lavigne, L., Cazaurang, F., Alcorta-García, E., and Díaz-Romero, D.A. (2013). Fault detection and isolation on a three tank system using differential flatness. In *2013 European Control Conference (ECC)*, 2433–2438. IEEE.

Martínez-Torres, C., Lavigne, L., Cazaurang, F., Alcorta-García, E., and Díaz-Romero, D.A. (2014). Flatness-based fault tolerant control. *Dyna*, 81(188), 131–138.

Nagy, A.M., Marx, B., Mourot, G., Schutz, G., and Ragot, J. (2009). State estimation of the three-tank system using a multiple model. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, 7795–7800. IEEE.

Noura, H., Theilliol, D., Ponsart, J.C., and Chamseddine, A. (2009). *Fault-tolerant control systems: Design and practical applications.* Springer Science & Business Media.

Staroswiecki, M. and Comtet-Varga, G. (2001). Analytical redundancy relations for fault detection and isolation in algebraic dynamic systems. *Automatica*, 37(5), 687–699.

Suryawan, F., De Doná, J., and Seron, M. (2010). Fault detection, isolation, and recovery using spline tools and differential flatness with application to a magnetic levitation system. In *2010 Conference on Control and Fault-Tolerant Systems (SysTol)*, 293–298. IEEE.

Thirumarimurugan, M., Bagyalakshmi, N., and Paarkavi, P. (2016). Comparison of fault detection and isolation methods: A review. In *2016 10th International Conference on Intelligent Systems and Control (ISCO)*, 1–6. IEEE.

Tousi, M. and Khorasani, K. (2011). Robust observer-based fault diagnosis for an unmanned aerial vehicle. In *2011 IEEE International Systems Conference*, 428–434. IEEE.

Zhou, Y., Xu, G., and Zhang, Q. (2014). Overview of fault detection and identification for non-linear dynamic systems. In *2014 IEEE International Conference on Information and Automation (ICIA)*, 1040–1045. IEEE.