Flat Filtering Cascade Control of Fourth Order Systems *

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Abstract: In this article, the analysis and implementation of an alternative Flat Filtering Control for a class of partially known fourth order flat systems is given. The flat filtering control uses the cascade property of the system, which leads to a simplified control design in which high order time derivatives are algebraically simplified in terms of low order measurable states and the subsequent integral compensation. The control proposal is implemented and validated experimentally in a fourth order mechanical system (rotary flexible joint) with accurate tracking results.

Keywords: Disturbance Rejection, Flat filtering, ADRC, Under-actuated system, Rotary Flexible Joint.

1. INTRODUCTION

Active Disturbance Rejection Control (ADRC Han (2009)) is a robust control methodology against process parameter variations that has been extensively studied during the last years. Numerous recent approaches have been developed to address many academic and industrial problems such as motion control (Zhao and Gao (2013), Sen et al. (2019), Hernández-Melgarejo et al. (2019), Tonhami et al. (2019)), power electronics (Wu et al. (2017), Huangfu et al. (2019), Sun et al. (2019), Zheng and Gao (2018)), robotic tele-operation (Gutiérrez-Giles and Arteaga-Pérez (2019), Gutiérrez-Giles et al. (2019)), industrial processes (Zheng and Gao (2012), Zheng et al. (2018)), underactuated systems (Ramírez-Neria et al. (2016), Sira-Ramírez et al. (2018), Ramírez-Neria et al. (2019)), among others. These academic and industrial study cases have shown excellent performance in terms of accuracy, repeatability, energy efficiency, easy implementation and intuitiveness of each control term.

The close relation of ADRC and an accurate state and disturbance estimation has motivated the use of alternative schemes of disturbance approximation. From a practical perspective, the use of simple controllers with the least possible information of a complex system is a challenging control problem. In this sense, Generalized Proportional Integral (GPI) Control (Fließ et al. (2002)) has provided the path to obtain a technique of state approximation which, along with an iterative integral compensation has led an observer-free output based control. The link between ADRC and GPI control was stated through the Flat Filtering Control, which constitutes a reinterpretation of GPI in the form of classical compensation networks (CCN). (see Sira-Ramírez et al. (2018), Sira-Ramírez et al. (2016)).

As a result was developed a tool for output feedback control design in linear controllable systems. A controllable linear system exhibits a natural flat output (Brunovsky’s output) from which the system is also trivially observable. Flat filtering is based on the fact that a GPI controller is viewed as a dynamical linear system which exhibits as a natural flat output a filtered version of the plant output signal. This property is particularly helpful in the design of efficient output feedback stabilization schemes and in solving output reference trajectory tracking tasks. Flat Filtering approach is naturally extended to efficiently handling control tasks on significantly perturbed differentially flat SISO nonlinear systems, affected by unknown endogenous nonlinearities, in the presence of exogenous disturbances and un-modeled dynamics. Recently, in Sira-Ramírez et al. (2019) is shown that, both, ADRC and flat filtering control are equivalent by means of an algebraic procedure involving a reduced order observer structure. This relation allows to use typical procedures of ADRC based control in the context of flat filtering control design. One of the alternative approaches of flatness based ADRC design deals with the simplification of the controllers through the use of the cascade flat property (Ramírez-Neria et al. (2014)) which allows to reduce the high order time derivatives estimation though algebraic equivalence. This scheme is typically used in a class of fourth order underactuated systems whose approximate linearization is flat (controllable). This principle can be also extended to a general class of fourth order nonlinear feedback linearizable systems (typically underactuated), which also exhibit the cascade property, reducing the complexity of the controller and leading to an alternative control design with the essence of the flat filtering approach.

In this article, the robust control of a class of fourth order differentially flat systems with partially known structure is tackled by means of a flat filtering control in combination...
with the cascade property, leading to a simplified family of controllers of GPI class.

The article is organized as follows: Section 2 presents the class of systems of study and the general control design. Section III the mathematical model of a flexible link arm and controller design details is presented. Section IV presents the description of a laboratory prototype, as well as the corresponding experimental results are presented. A brief discussion of the results and the conclusions are given in Section V.

2. DYNAMICS OF FOURTH ORDER FLAT PERTURBED SYSTEMS

Let us denote the flat output as \( y_f \). Consider a class of disturbed fourth order differentially flat systems, whose dynamics is described by the following equation:

\[
y_f^{(4)} = \alpha \left( y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)} \right) u(t) + \Phi(t)\tag{1}
\]

The disturbance function \( \Phi(t) \), lumps all the endogenous forces such as internal, non modeled or nonlinear dynamics. The exogenous forces are represented by \( \varphi(t) \). The disturbance depend on the time, the flat output and its time derivatives. It is assumed that only flat output \( y_f \) variable is available for measurement. The control again \( \alpha(t) \) is a nonlinear unknown gain depending of the set of the phase variables associated with the flat output \( y_f \). The nonlinear gain it is represented as follows

\[
\alpha \left( y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)} \right) = K + \bar{\alpha} \left( y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)} \right)\tag{2}
\]

Where \( K \) is a known constant gain and \( \bar{\alpha} \left( y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)} \right) \) represents the high order terms (HOT). Following the methodology of the model free control (Fließ and Join (2013)), and using well established procedures in ADRC control, the original model (1) is simplified as follows

\[
y_f^{(4)} = Ku(t) + \Phi(t)\tag{3}
\]

where \( \Phi(t) \) is the total disturbance (see Han (2009)) described as follows:

\[
\Phi(t) = \bar{\alpha} \left( y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)} \right) u(t) + \Phi \left( t, y_f, \dot{y}_f, \ddot{y}_f, y_f^{(3)}, \varphi(t) \right)\tag{4}
\]

the total disturbance, in addition to endogenous and exogenous disturbances, it depends now of the control input itself. Let us define the nominal reference system which depends on the desired output \( y^*(t) \) and its dynamics (Sira-Ramirez and Agrawal (2004))

\[
y^{(4)} = Ku^*(t)\tag{5}
\]

Then, the tracking error is defined as

\[
e = y - y^*(t)\tag{6}
\]

whose dynamics is given by the following expression

\[
e^{(4)} = Ke_e(t) + \Phi(t)\tag{7}
\]

where \( e_e(t) = u(t) - u^*(t) \). The cascade representation of (7) is shown in Fig. 1. The integral reconstructors based on neglecting the total disturbance term \( \Phi(t) \) in the simplified model (7) are implemented as follows

\[
e = y - y^*(t)\tag{8}
\]

\[
\dot{e} = y - \dot{y}^*(t)\tag{9}
\]

\[
\ddot{e} = y - \ddot{y}^*(t)\tag{10}
\]

The integralg reconstructors are shown in the Fig. 2.

Since the total disturbance \( \Phi(t) \) is unknown, the flat output time derivatives \( e^{(3)}_e, \dot{e}_e \) and \( \ddot{e}_e \) depend only of the control input error \( e_e \) and the gain of the system \( \Phi \). So, the time derivatives can not be estimated accurately with (8)-(10) if they are corrupted by noise. In order to overcome this problem, the cascade flat property is used (see Ramírez-Neria et al. (2016), Ramírez-Neria et al. (2019)) typically found on underactuated mechanical systems. The error dynamics (7) are represented as a cascade connection of two independent blocks (see Fig 3): The first one, controlled by the input error \( e_e \), whose corresponding output is given as the acceleration error. For flat systems (1) using the cascade flat property we found that the acceleration error \( \ddot{e}_e \) is a function which depends on the measurable flat output \( y_f \) the available states of the system and the desired trajectory \( y^*_f(t) \). Let us define the acceleration error \( \ddot{e}_e = e_{2y} \), now \( e_{2y} \) acts as a measurable auxiliary input to the second block, which consists of a chain of two integrators associated, respectively, with the phase variables \( \dot{e}_e \) and \( e_e \). Since \( e_{2y} \) is measurable, the time derivatives of the tracking error (8)-(10) can be estimated as follows

\[
e^{(3)}_e = K \int e_e \tag{11}
\]

\[
\dot{e}_e = e_{2y} = \dot{y}_f - \dot{y}^*_f(t)\tag{12}
\]

\[
\ddot{e}_e = e_{2y} \tag{13}
\]

Notice that only \( e^{(3)}_e \) is estimated using the control input as is shown in Fig 3. We propose the application of the following feedback control law

\[
e = -\frac{1}{K} \left( k_3 e^{(3)}_e + k_2 \dot{e}_e + k_1 \ddot{e}_e + k_0 y_e \right)\tag{14}
\]

if we apply the control law (14) to the simplified error dynamics (7) may not lead to accurate tracking results since
Fig. 3. Flat output property

the integral reconstructors $\hat{e}^{(3)}y$, $\hat{\ddot{e}}y$ and $\hat{\dot{e}}y$ from equations (8)-(10) are inaccurate due to the total disturbance $\Phi(t)$ is not compensated. To overcome these facts, as stated in flat filtering approach, let proceed to compensate both effects with a finite number $m$ of iterated integrals of the tracking error (see Sira-Ramírez et al. (2016) and Sira-Ramírez et al. (2018)). The proposed controller (14) is now expressed as:

$$e_u = -\frac{1}{K} \left( k_{m+3} \hat{e}^{(3)}y + k_{m+2} \hat{\dot{e}}y + k_{m+1} \hat{\ddot{e}}y + k_m e_y \right)$$
$$+ k_{m-1} \int_{1}^{(1)} e_y + k_{m-2} \int_{2}^{(2)} e_y + \cdots + k_0 \int_{m}^{(m)} e_y$$

using the advantage of cascade property, the estimated states (11)-(13) are substituted in (15)

$$e_u(t) = -\frac{1}{K} \left( k_{m+3} K \int e_u + k_{m+2} e_{2y} + k_{m+1} \int e_{2y} ight)$$
$$+ k_m e_y + k_{m-1} \int_{1}^{(1)} e_y + k_{m-2} \int_{2}^{(2)} e_y + \cdots + k_0 \int_{m}^{(m)} e_y$$

The proposed controller (16) can be represented in the frequency domain as follows:

$$e_u(s) = -\frac{1}{K} \left( k_{m+3} K e_u(s) + k_{m+2} + k_{m+1} \right) e_{2y}(s)$$
$$- \frac{1}{K} \left( k_m + k_{m-1} + k_{m-2} + \cdots + k_0 \right) e_y(s)$$

After some algebraic manipulations, the following parallel controller structure is obtained

$$e_u(s) = -\frac{1}{K} \left( \frac{k_{m+2} s + k_{m+1}}{s + k_{m+3}} \right) e_{2y}(s)$$
$$- \frac{1}{K} \left( \frac{k_m s^m + k_{m-1} s^{m-1} + k_{m-2} s^{m-2} + \cdots + k_0}{s^{m-1} + k_{m+3}} \right) e_y(s)$$

The closed loop tracking error system, expressed in the frequency domain, leads, after using $s^2 e_y = e_{2y}(s)$ to the following closed loop dynamics for the simplified perturbed system (7)

$$s^{m+1} + k_{m+3} s^{m+3} + k_{m+2} s^{m+2} + k_{m+1} s^{m+1} + k_m s^m + k_{m-1} s^{m-1} + k_{m-2} s^{m-2} + \cdots + k_0 \right) e_y(s) = s^m \Phi(s)$$

We select the set of gains $[k_{m+3}, k_{m+2}, \ldots, k_1, k_0]$ matching the close loop polynomial

$$P(s) = s^{m+1} + k_{m+3} s^{m+3} + k_{m+2} s^{m+2} + k_{m+1} s^{m+1} + k_m s^m + k_{m-1} s^{m-1} + k_{m-2} s^{m-2} + \cdots + k_0$$

with a Hurwitz polynomial: if $m + 4$ is even

$$P(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^{(m+4)/2}$$

In order to implement the proposed controller in the time domain, from (18) it is defined a filtered version of the tracking error $e_{yf}(s)$ and the acceleration $e_{2yf}(s)$.

$$e_{yf}(s) = \frac{e_y(s)}{s^{m-1} + k_{m+3}}$$
$$e_{2yf}(s) = \frac{e_{2y}(s)}{s + k_{m+3}}$$

we define the following states $e_0 = e_{yf}$, $e_1 = \dot{e}_{yf}$, $e_2 = \ddot{e}_{yf}$, $\cdots, e_{m-1} = \frac{e_{yf}}{s^{m-1}}$. The filtered error dynamics (23) and (24) can be represented in state space as follows:

$$\dot{e}_i = e_{i+1}, \quad i = 0, 1, \ldots, m - 2$$
$$e_{m-1} = e_0 - k_{m+3} e_{m-1}$$
$$e_m = e_{2y} - k_{m+3} e_m$$

Finally the flat filtering control is synthesized using (18) and (25) as follows

$u(t) = -\frac{1}{K} \left( k_{m+2} e_{2y} + (k_{m+1} - k_{m+2} k_{m+3}) e_m + k_m e_y \right)$
$$+ (k_{m-1} - k_{m+3} k_{m-3}) e_{m-1} + k_{m-2} e_{m-2} + \cdots + k_0 e_0$$
$$+ \frac{1}{K} y^{(4)}(t)$$

3. ROTARY FLEXIBLE JOINT CASE OF STUDY

Consider the Rotary Flexible Joint studied in the work Ramirez-Neria et al. (2016), it consists of a free arm attached to two identical springs. The springs are mounted to an aluminum chassis which is fastened to the rotary base which is actuated by a DC motor, the Figure 4 shows a diagram of the system with the following parameters:

$\theta_1$ Angular position of the rotating base
$\theta_2$ Angular position of the arm
$J_a$ Moment of inertia of the rotating base
$J_b$ Moment of inertia of the arm
$m$ Mass of the arm
$l$ Arm length
$k_s$ Spring stiffness
$\tau$ Torque applied to the system

The angular position of the arm tip is defined by the relation

$$\vartheta = \theta_1 + \theta_2$$

In order to determine the dynamic model of the flexible joint robot, the Euler-Lagrange approach was followed.
Using the Euler Lagrange methodology with generalized coordinates $\theta_1, \theta_2$, we obtain the dynamic of the system
\[
\tau = J_b \ddot{\theta}_1 + J_a \ddot{\theta}_2 + \frac{mg}{2} \sin(\theta)
\]
the motor torque is represented with $\tau_m$ and the torque applied to the system are related as: $\tau = N \tau_m$, being $N$ the mechanical advantage of the pulley system. The voltage applied to the motor $V(t)$, and the torque generated by the motor $\tau_m$, are obtained using the following simplified relation
\[
\tau_m = k_r V(t) - \frac{N^2 J_b}{R_m} \dot{\theta}_1
\]
where $R_m$ denotes the armature resistance, $k_r$ represents the torque constant of the motor. Using last relation, the dynamics of the rotary flexible joint, are given by
\[
\frac{Nk_r}{R_a} V(t) = (J_b + J_a) \ddot{\theta}_1 + J_a \ddot{\theta}_2 + \frac{mg}{2} \sin(\theta)
\]
where $a_1 = k_r^2 J_a$, $a_2 = k_r^2 J_b$, $a_3 = \frac{k_r^2 J_a + J_b}{J_a}$, $a_4 = \frac{mg}{2 J_a}$, $b = \frac{k_r N}{R_a}$. Now, performing the variable change $x := [x_1, x_2, x_3, x_4]^T = [\theta_1, \theta_1, \theta_2, \theta_2]^T$ then system (29)-(30) can be rewritten as
\[
\dot{x}(t) = f(x) + g(x) V(t),
\]
where
\[
f(x) = \begin{pmatrix}
x_2 \\
-x_2 x_2 + x_2 x_3 \\
x_4(a_2 x_3 - a_3 x_3) - a_4 \sin(x_1 + x_3)
\end{pmatrix},
\]
\[
g(x) = \begin{pmatrix}
0 \\
b \\
0
\end{pmatrix}
\]
Now, the control law design via feedback linearization is presented. Let define the output of the system as follows
\[
h(x) = x_1 + x_3
\]
We compute $L_f h(x)$ the Lie derivatives of $h(x)$ with respect to $f(x)$ are given as
\[
\dot{h}(x) = L_f h(x)
\]
\[
\dot{h}(x) = L_f h(x) + L_g L_j h(x) V(t)
\]
where
\[
L_f h(x) = x_2 + x_4
\]
\[
L_j h(x) = \frac{a_2 - a_3}{a_4} x_3 - a_4 \sin(x_1 + x_3)
\]
\[
L_j h(x) = a_4 x_2 + x_4 \cos(x_1 + x_3) + (a_2 - a_3) x_4
\]
\[
L_j h(x) = a_4 x_3 + a_3 x_3 \cos(x_1 + x_3) + (a_2 - a_3) x_4
\]
\[
L_j h(x) = -(a_2 - a_3) b
\]
Since $L_g L_j h(x) \neq 0$ in the whole state space, the relative degree of the system is four. Notice that the flat output coincides with $\tilde{y}_f = h(x)$, so we have
\[
y_f = x_1 + x_3
\]
\[
y_f = x_2 + x_4
\]
\[
y_f = (a_2 - a_3) x_3 - a_4 \sin(x_1 + x_3)
\]
\[
y_f^{(3)} = a_4 x_2 + x_4 \cos(x_1 + x_3) + (a_2 - a_3) x_4
\]
and the tracking error perturbed dynamics (7) for the Flexible link arm system (37)-(41)
\[
\dot{e}_y(t) = K e_C V(t) + \dot{\Phi}(t) + \Phi(t)
\]
with $e_v(t) = V(t) - V^*(t)$. We propose the following flat filtering controller using (25) and (26) with $m = 5$
\[
i_1 = c_1, i = 1, 2, 3
\]
\[
i_5 = c_5 - k_8 e_5
\]
\[
V(t) = -\frac{1}{K} [k_7 c_2 + k_6 - k_7 k_8 e_5 + k_5 e_y] + (k_4 - k_5 k_8) e_4 + k_3 e_3 + k_2 e_2 + k_1 e_1 + k_0 e_0
\]
the set of gains is chosen matching close loop characteristic polynomial with a Hurwitz polynomial
\[
P(s) = s^9 + k_8 s^8 + k_7 s^7 + k_6 s^6 + k_5 s^5 + k_4 s^4 + \cdots + k_0
\]
\[
P(s) = (s + \omega_c)(s^2 + 2 \zeta \omega_c s + \omega_c^2)^4
\]
4. EXPERIMENTAL RESULTS

The experimental device is shown in Figure 5. It consists of a DC motor NISCA: model NC5475, which drives a rotating base through a synchronous belt and pulley system with a 16:1 ratio. The main arm is attached to the rotating base by two identical springs, resulting in a flexible joint. Both, the angular positions of the rotating base and the arm were measured with incremental optical encoders of 1000 pulses/revolution. The data acquisition is carried out through a data card Sensoray model 626. This board, is responsible to read the signals from the optical incremental encoders and supplies control voltage to the power amplifier. The proposed control law was implemented in the Matlab-Simulink environment. Finally, the sampling time was set to be 0.001[s] with consideration of the dominating dynamics in the mechanical system under study. The rotary flexible joint parameters, used to implement the control laws (44) are: $l = 0.5[m]$, $m = 0.1635[Kg]$, $J_b = 0.0136[Kg-m^2]$, $J_a = 0.002405[Kg-m^2]$, $k_s = 4[N-m/\text{rad}]$ and $N = 16$, while the motor parameters are: $k_r = 0.0724[N-m/A]$ and $R_m = 2.983 [\Omega]$. The initial conditions for the test were $[\theta_1(0), \theta_2(0)]$ this implies that the flat output $y_f(0) = 0$, the controller gains were selected using $\zeta = 3.3$ and $\omega_c = 18$. Fig. 6 depicts the flat output trajectory performance, the tracking error is showed in Fig. 7 we can observed it is bounded in a region approximately of $[-0.025, 0.025] \text{ rad}$. The control voltage is presented in Fig. 8.
4.1 External disturbance test

Fig. 9 shows the trajectory tracking performance of the proposed controller when it is affected with a set impulsive external force applied directly on the arm, the proposed controller reject the external disturbance with bounded error showed in Fig. 10, we can notice that when the external force is applied on the arm the magnitude of the controller increase canceling out the external disturbance.

5. CONCLUSIONS AND REMARKS

In this article, a Flat Filtering controller scheme was proposed for a fourth order nonlinear feedback linearizable system which exhibits the cascade property. The effectiveness of the proposed controller was proved on a rotary flexible joint prototype exhibited an excellent behavior and an remarkable trajectory tracking performance and robustness in presence of unknown external disturbance inputs. As a suggestion for further research, an extension of the flat filtering controller may be addressed for a suitable high order nonlinear systems.

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