Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Periodic Set Invariance as a Tool for Constrained Reference Tracking

Martin Soyer [*,**] **Sorin Olaru** [*] **Konstantinos Ampountolas** [***]
Sheila Scialanga [***] **Zhou Fang** [**]

[*] *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, Gif-sur-Yvette, France (e-mail: martin.soyer/sorin.olaru@centralesupelec.fr).*
[**] *Renault Technocentre, AD Simulation & VR center, Guyancourt, France (e-mail: martin.soyer/zhou.fang@renault.com)*
[***] *James Watt School of Engineering, University of Glasgow, Glasgow G12 8QQ, United Kingdom, (e-mail: k.ampountolas@uth.gr, s.scialanga.1@research.gla.ac.uk)*

Abstract: This paper explores the concept of periodic invariance and its use for trajectory tracking problems subject to state and input constraints, offering important computational advantages. In principle, traditional techniques based on receding horizon optimization are computationally expensive due to long prediction and optimization horizons, and number of control and state constraints in a constrained control problem. Their complexity is further affected by additional constraints needed to ensure recursive feasibility via a controllable invariant set. Practically, such invariant sets are difficult to obtain off-line and use them on-line. To overcome this problem, this paper suggests to employ periodic invariant sets as a simple set-theoretic tool for constrained reference tracking problems.

Keywords: Predictive Control, Constrained Control, Periodic Invariance, Reference Tracking

1. INTRODUCTION

This paper presents an alternative approach to the traditional constrained tracking design methods (Bemporad, 1998). Essentially, we aim to develop an attractive framework from the computational point of view that:

- guarantees recursive feasibility for a pre-defined region in the state space;
- simplified constraints in the on-line optimization.

The recursive feasibility of constrained tracking can be guaranteed by the characterization of the *maximal controllable sets* (Blanchini and Miani, 2000). Any initial state within this set can generate feasible trajectories by exploiting the controlled invariance and subsequently optimized with respect to a tracking criterion (Blanchini and Miani (2000)). We note however, that the characterization of the *maximal controllable set* is a notorious complex problem both in terms of off-line effort and complexity of the representation, which subsequently affects the on-line computational effort. As an alternative to the explicit use of the maximal controllable set, model predictive control (MPC) has been widely used with its receding horizon formulation. Recursive feasibility in MPC is related to the existence of an invariant terminal set. In the tracking case, this terminal set is parameterized by a virtual feasible trajectory (Olaru and Dumur, 2005; Limon and Alamo, 2013; Falugi, 2015; Chisci and Zappa, 2003). This parametrization leads to high on-line computational effort and has been the subject of research in different studies. Reducing the complexity of the maximal controllable set or the terminal constraints in MPC by means of approximations can compromise the invariance property and consequently the recursive feasibility. The present paper revisits the concept of *periodic invariance* (Lee and Kouvaritakis, 2006) that allows the relaxation of constraints for a finite number of iterations before reinforcing into the constraints. This property preserves the recursive feasibility of optimization-based tracking control by maintaining a low computational effort.

Among constrained control techniques, the recent Interpolation Based Control (IBC) developed in (Nguyen et al., 2013; Scialanga and Ampountolas, 2017, 2018, 2019) is shown to be faster than traditional optimisation-based techniques such as MPC, while preserving stability and performance. However its extension to tracking problems is based on on-line operations involving invariant sets (Soyer et al., 2020). This paper explores how IBC can be enhanced via the periodic invariance concept to deal with the constrained reference tracking problem, and consequently to maintain a low computational effort, compared to optimization-based methods like MPC. By using the notion of *strong* and *weak periodic invariance*, we present efficient algorithms for the tracking problem that guarantee the recursive feasibility of IBC.

Notation: A *Polytope* P in the *H-Representation* is a set $P = \{x \in \mathbb{R}^n \mid Fx \leq g\}$ where $F \in \mathbb{R}^{q\times n}$ and $g \in \mathbb{R}^q$. A *Polyhedron* denotes a bounded polytope. A *Polyhedron* P in *V-Representation* is a set $P = \{\sum_i^n \lambda_i v_i / \forall i \in \{0,\ldots,n\}\ \lambda_i \geq 0\}$ where $\{v_1,...,v_n\}$ are the vertices. The weighted 2-norm $\|.\|_Q$ is defined as : $\forall x \in \mathbb{R}^n$ $\|x\|_Q^2 = x^T Q x$ where $Q \in \mathbb{R}^{n\times n}$ is positive definite. $[\![1, N]\!]$ is the set of integers between 1 and N.

2. PROBLEM FORMULATION

The dynamical system will be represented in terms of a linear discrete-time model :

$$\begin{aligned} x_{k+1} = Ax_k + Bu_k, \quad x_0 \text{ is given,} \\ \text{subject to: } x_k \in \mathbb{X},\ u_k \in \mathbb{U}, \quad \forall\, k, \end{aligned} \tag{1}$$

where $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are compact convex sets containing the origin in their respective interior.

Definition 1. (Positive invariance). A set $C \subset \mathbb{R}^n$ is said to be positively invariant for the autonomous system $x_{k+1} = f(x_k)$ if for any $x_0 \in C \Rightarrow x_k \in C, \forall\, k \in \mathbb{Z}_+$.

Definition 2. (Controlled invariant set). A set $C \subset \mathbb{X}$ is said to be controlled invariant with respect to the constrained system (1) if for any state $x_0 \in C$ there exists a sequence $u_k \in \mathbb{U}$ such that $x_k \in C, \forall k \in \mathbb{Z}_+$.

Definition 3. (Maximal Controllable Set to Ω). Given a proper controlled invariant set $\Omega \subset \mathbb{X}$ we define the *Maximal controllable set to* Ω, denoted $\mathcal{C}(\Omega)$, to be the collection of initial states $x_0 \in \mathbb{X}$ for which there exists a finite admissible control sequence that brings x_0 to Ω.

Practically, the controlled invariant set is chosen to be as large as possible in $\mathbb{X}$ to avoid conservativeness. The maximal controllable set can be approached off-line with an iterative procedure which implies projections of polyhedrons that can lead to a complex representation (Nguyen et al. (2011)). The objective in the remaining of the paper is to introduce optimization-based tracking formulations building on a generalised invariance property with the goal to circumvent the off-line and on-line complexity of the maximal controllable set.

We will present the main contribution by revisiting the concept of periodic invariance (Lee and Kouvaritakis, 2006) that will be used to overcome the complexity induced by the construction of the maximal controllable set. A difference will be made in between a strong and weak version of the periodic invariance, the former implying the later. In the following, periodic invariance will be denote as p-invariance.

Definition 4. (Strong p-invariance). A set $\mathcal{B} \subset \mathbb{X}$ containing the origin is said to be *strongly p-invariant* with respect to the constrained system (1) if there exists $p \in \mathbb{N}^*$ such that for all state $x_k \in \mathcal{B}$, there exists a control sequence $(u_k, ..., u_{k+p-1}) \in \mathbb{U}^p$ such that $x_{k+p} \in \mathcal{B}$ and $(x_{k+1}, ..., x_{k+p-1}) \in \mathbb{X}$.

The notion of p-invariance which assumes the same periodicity index for any point in the set. We show next that this characteristics can be relaxed to a certain extent and the tracking control associated carry on with simple modifications.

Definition (Weak p-invariance) : Given a compact convex set $\mathcal{B} \subset \mathbb{R}^n$ and $p \in \mathbb{N}^*$, $\mathcal{B}$ is said to be *weakly p-invariant* with respect to the system (1) if for any state $x_k \in \mathcal{B}$ there exists $r \leq p$ and a control sequence $(u_k, ..., u_{k+r-1}) \in \mathbb{U}^r$ such that $x_{k+r} \in \mathcal{B}$ and $(x_{k+1}, ..., x_{k+r-1}) \in \mathbb{X}$.

In other words, any state in $\mathcal{B}$ returns into $\mathcal{B}$ in *at most* p number of steps.

3. STRONG p-INVARIANCE

The concept of strong p-invariance can be relevant to the considered systems and the tracking objective whenever maximal controllable set is replaced by a simpler approximation thanks to the following theorem.

Theorem 5. Let Ω be a controlled invariant set containing the origin and $C_N(\Omega)$ be the maximal controllable set to Ω. Given a set $\mathcal{B}$ such that $\Omega \subseteq \mathcal{B} \subset C_N$. There exists an integer $p \in \mathbb{N}^*$ such that $\mathcal{B}$ is strongly p-invariant.

Proof: $\mathcal{B} \subset C_N(\Omega)$, so for any initial state $x_k \in \mathcal{B}$ there exists a control sequence $(u_k, ..., u_{k+N-1}) \in \mathbb{U}^N$ such that $x_{k+N} \in \Omega \subset \mathcal{B}$. By fixing the periodicity index to the maximal number of time steps N to reach Ω from $\mathcal{B}$ the existence is proved. □

In other words, a simpler inner approximation of the maximal controllable set, while contains the attractive controlled invariant set, is necessarily strongly periodic invariant.

3.1 Practical Construction of Strong p-Invariance

Given $\mathcal{B} \subset \mathbb{X}$ a convex polyhedron containing the origin with $V = \{v_1, ..., v_{N_v}\}$ its vertices and their cardinality $N_v \in \mathbb{N}$. The strong p-invariance of the set $\mathcal{B}$ with respect to (1) can be computed by Algorithm 1 below, which considers all the vertices of the candidate set $\mathcal{B}$ and tests the minimal contraction factor that can be obtained jointly along a time window of length p_s. The search for this contraction factor leads practically to a simple Linear Programming (LP) problem.

Under the assumption that the candidate set is contained in a controllable set $\mathcal{B} \subset \mathcal{C}_N(\Omega)$, the procedures ends in finite time. If this assumption does not hold, the condition $\lambda_s\mathcal{B} \supset \mathbb{X}$ guarantees that the algorithm will terminate in a finite number of steps. In this framework, it is important to observe that an explicit description of a controllable set $\mathcal{C}_N(\Omega)$ is not necessary in the above construction.

Algorithm 1 Strong p-Invariance

Input: The pair (A, B), the sets $\mathbb{X}, \mathbb{U}$ and $\mathcal{B}$
Output: The periodicity index p
$p_s = 0$, $\lambda_s = 1$
while $\lambda_s \geq 1$ **do**
  $p_s = p_s + 1$
  Solve:

$$\begin{aligned} &\underset{\lambda_s, u_i}{\text{minimize}} \quad \lambda_s \\ &\text{subject to} \\ &A^{k-1}v_i + \sum_{j=0}^{k-2} A^j B u_{i,k-2-j} \in \mathbb{X} \\ &\qquad \forall k \in [\![2, p_s]\!], i \in [\![1, N_v]\!] \\ &A^{p_s}v_i + \sum_{j=0}^{p_s-1} A^j B u_{i,p_s-1-j} \in \lambda_s\mathcal{B}, i \in [\![1, N_v]\!] \\ &(u_{v_i})_{i\in[\![1,N_v]\!]} \in \mathbb{U}^{p_s} \end{aligned}$$

end while $\{\lambda_s\mathcal{B} \supset \mathbb{X}\}$

3.2 Strong p-Invariance Based Reference Tracking

Based on the construction of strong p-invariance, let us now consider that the tracking control problem can use the existence of a p-invariant set $\mathcal{B}$ satisfying $\mathcal{B} \subset \mathcal{C} \subset \mathbb{X}$.

A prototype receding-horizon optimization for reference-tracking, which employs the periodic invariance for a linear prediction model, will be denoted $\mathcal{O}(N_h, p, x_k)$ and formulated as follows :

$$\begin{aligned} J_p(x_k) = \underset{(u_k,...,u_{k+N_h-1})}{\text{minimize}} \quad & \sum_{i=1}^{N_h} \|x_{k+i}^{ref} - x_{k+i}\|_Q^2 \\ \text{subject to} \quad & \\ u_{k+i} \in \mathbb{U}\,, x_{k+i} & \in \mathbb{X}\ \forall i \in [\![1, \ldots, M]\!], \\ x_{k+p} \in \mathcal{B}&, \end{aligned} \tag{2}$$

with $M = \max(N_h, p)$.

Proposition 6. $\mathcal{O}(N_h, p, x_k)$ is feasible for all $x_k \in \mathcal{B}$.

Proof: For any state $x_k \in \mathcal{B}$, there exists a control sequence $(u_k, ..., u_{k+p-1}) \in \mathbb{U}^p$ such that $(x_{k+1}, ..., x_{k+p-1}) \in \mathbb{X}$ and $x_{k+p} \in \mathcal{B}$. Thus, in the case of a prediction horizon with $N_h \leq p$, the problem is feasible.

If $N_h > p$, the argument is slightly more elaborated and needs to rely on the invariance property of the controlled invariant superset $\mathcal{C}$. Indeed by construction $\mathcal{B} \subset \mathcal{C} \subset \mathbb{X}$, which implies the existence of a sequence $(u_k, ..., u_{k+N_h-1}) \in \mathbb{U}^{N_h}$ such that $(x_{k+1}, ..., x_{k+N_h-1}) \in C \subset \mathbb{X}$. □

Despite the result stated in Proposition 6, one cannot guarantee the recursive feasibility of the control strategy that implements the first part of the optimum control argument. This is because the *one-step invariance* property on $\mathcal{B}$ is not certified, and thus the feasibility of the optimization (2) on $\mathcal{B}$ does not imply the feasibility at iteration $k+1, \ldots$ as long as $x_{k+1+p} \in \mathcal{B}$ does not hold.

To overcome the absence of recursive feasibility, a simple procedure can be constructed to enhance the periodic invariance property. If $\mathcal{O}(N_h, p, x_k)$ designates the optimization (2), the main idea is to monitor the result of this optimization and to switch to a *safe return strategy* within the set $\mathcal{B}$ whenever the closed-loop trajectory leaves $\mathcal{B}$.

In order to simplify the switching criterion, the cost functions $J_i(x_k)$ and $J_p(x_k)$ will be compared for any $x_k \in \mathcal{B}$. As long as the first one is less costly the procedure apply the first component of its control law. The following proposition provides a criterion that separates the case where the system remains in $\mathcal{B}$ from the case it leaves $\mathcal{B}$.

Proposition 7. Given the optimized costs $J_1(x_k)$ (resp $J_p(x_k)$) of optimizations $\mathcal{O}(N_h, 1, x_k)$ (resp $\mathcal{O}(N_h, p, x_k)$). If $J_p(x_k) < J_1(x_k)$ then $x_{k+1} \notin \mathcal{B}$.

Proof: If U_p^* is the optimal solution of $\mathcal{O}(N_h, p, x_k)$, assume $x_{k+1} \in \mathcal{B}$, then, every constraints of $\mathcal{O}(N_h, 1, x_k)$ are satisfied, U_p^* is a feasible solution of $\mathcal{O}(N_h, 1, x_k)$. Thus, the optimization of $\mathcal{O}(N_h, 1, x_k)$ provides a better solution, and $J_1(x_k) \leq J_p(x_k)$. □

With this preliminary result the description of the procedure using strong p-invariance for tracking with recursive feasibility properties is presented in Algorithm 2.

Algorithm 2 Strong p-Invariance Reference Tracking

```
Input x_0 ∈ B ⊂ X, N_simu, N_h, (A, B), X, U, B
Output (x_k)_{k=0,...,N_simu}, (u_k)_{k=0,...,N_simu-1}
k = 1, i = 1
repeat for each k
    if i = 1 then
        Solve O(N_h, 1, x_k) → (J_1^*, (u_k^1, ..., u_{k+N_h-1}^1))^*
        if J_1^* < J_p^* then
            x_{k+1} = A x_k + B u_k^1
        else
            x_{k+1} = A x_k + B u_k^p
            i = p
        end if
    else
        Solve O(N_h, i, x_k) ← (J_i^*, (u_k^i, ..., u_{k+N_h-1}^i))^*
        if J_1 < J_i then
            x_{k+1} = A x_k + B u_k^1
            i = 1
        else
            x_{k+1} = A x_k + B u_k^i
            i = i - 1
        end if
    end if
    k = k + 1
until k = N_simu
```

Proposition 8. The control law resulting from the recursive implementation of the first input of the optimal control sequence according to Algorithm 2 is recursively feasible for any initial state $x_0 \in \mathcal{B}$.

Proof: Assume the procedure is feasible at step k. Then the current state becomes $x_{k+1} \in \mathbb{X}$ and two cases have to be considered :

- If $x_{k+1} \in \mathcal{B}$, then $\mathcal{O}(N_h, p, x_{k+1})$ is feasible thanks to Proposition 6.
- If $x_{k+1} \notin \mathcal{B}$, then x_{k+1} is part of a state sequence that began in $\mathcal{B}$. There exists an integer $q \leq p-1$ such that the state $x_{k+1-q} \in \mathcal{B}$. So there exists a control sequence $(u_{k+1-q}, ..., u_{k+1}, ..., u_{k-q+p-1}) \in \mathbb{U}^p$ such that $x_{k+1-q+p} \in \mathcal{B}$ and $(x_{k+2-q}, ..., x_{k-q+p}) \in \mathbb{X}^{p-1}$. Ignoring the tail, we conclude on the existence of a control sequence $(u_{k+1}, ..., u_{k+1-q+p}) \in \mathbb{U}^{p-q}$ such that $x_{k+2-q+p} \in \mathcal{B}$. This concludes the proof as long as $\mathcal{O}(N_h, p-q, x_{k+1})$ is feasible. □

4. WEAK p-INVARIANCE

The next result formalize the relationship between *strong* and *weak* version of the periodic invariance.

Theorem 9. If $\mathcal{B}$ is strongly p-invariant, then $\mathcal{B}$ is weakly p-invariant. Alternatively, if p_s (resp p_w) denotes the result of the computation of strong periodic invariance (resp weak periodic invariance), then $p_w \leq p_s$.

Proof: The proof is direct by observing that in the definition of weak invariance $q = p$ is a feasible choice for the number of steps for the return sequence. □

Construction of Weak p-invariance: Given a polyhedron $\mathcal{B} \subset \mathbb{X}$ with set of vertices $V = \{v_1, ..., v_{N_v}\}$ and cardinality $N_v \in \mathbb{N}$. The weak p-invariance index can be computed by Algorithm 3.

Algorithm 3 Weak p-invariance

Input: The pair (A, B), the sets $\mathbb{X}, \mathbb{U}$ and $\mathcal{B}$
Output: the periodicity index p
for $v_i \in V$ **do**
  $p_w^i = 0$; $\lambda_w = 1$
  while $\lambda_w \geq 1$ **do**
    $p_w^i = p_w^i + 1$
    Solve:

$$\min_{\lambda_w, (u_i)_{i \in [\![1, p_w^i - 1]\!]}} \quad \lambda_w$$

subject to

$$A^{k-1} v_i + \sum_{j=0}^{k-2} A^j B u_{k-2-j} \in \mathbb{X}, \ \forall k \in [\![2, p_w^i]\!]$$

$$A^{p_w^i} v_i + \sum_{j=0}^{p_w^i - 1} A^j B u_{p_w^i - 1 - j} \in \lambda_w \mathcal{B}, \ i \in [\![1, p_w^i - 1]\!]$$

  end while
end for
$p_w = \max(p_w^1, ..., p_w^{N_v})$

Proposition 10. Given a convex compact polyhedron $\mathcal{B}$ containing the origin, and $\{p_1, ..., p_{N_v}\}$ and the periodicity of vertices $\{v_1, ..., v_{N_v}\}$ computed by Algorithm 3 is thus inherited by the entire set. Then, for any state $x_k \in \mathcal{B}$,one can use the same periodicity as its vertex component with highest periodicity.

A tracking procedure using weak periodic invariance and based on the optimization (2) can be proposed as in Algorithm 4. The principle behind the properties of recursive feasibility of the tracking procedure is summarized next, the proofs following the arguments in Section 3:

- If $x_k \in \mathcal{B}$, the cost functions for an optimization problem with 1-step invariance constraints and p-step invariance constriants are compared.
- If the system leaves $\mathcal{B}$, then every costs from 1-step to p-steps return to $\mathcal{B}$ are compared. In other words, the validation of the weak periodic invariance is tested at each iteration in order to find the shortest return path into $\mathcal{B}$.

5. TRACKING IBC WITH p-INVARIANCE

Interpolation-Based Control (IBC) was originally developed in Nguyen et al. (2011, 2013); Nguyen (2014) as an enhancement of Vertex Control established in Gutman and Cwikel (1986). It guarantees the same region of attraction as predictive control as well as structural property (smoothness) for set-point regulation problems. The IBC adaptation for tracking is provided in Soyer et al. (2020) and we point the reader to this reference for the details.

The IBC design is built on two convex compact controlled-invariant sets containing the origin Ω^o and Ω^v, where $\Omega^o \subset \Omega^v \subset \mathbb{X}$. Ω^o is called the *inner* set and Ω^v is called *outer* set. These inner and outer sets have to be re-scaled and translated in order to contain the origin of the dynamical system governing the tracking error. The periodic invariance property is preserved by homogeneous transformations.

Algorithm 4 Weak p-Invariance Reference Tracking

Input $x_0 \in \mathcal{B} \subset \mathbb{X}$, N_{simu},N_h, (A, B) ,$\mathbb{X}, \mathbb{U}$,$\mathcal{B}$
Output $(x_k)_{k=0,...,N_{simu}}$,$(u_k)_{k=0,...,N_{simu}-1}$
$k = 1$, $i = 1$
repeat for each k
  if $i = 1$ **then**
    Solve $\mathcal{O}(N_h, 1, x_k) \rightarrow (J_1^*, (u_k^1, ..., u_{k+N_h-1}^1))^*$
    Solve $\mathcal{O}(N_h, p, x_k) \rightarrow (J_p^*, (u_k^p, ..., u_{k+N_h-1}^p))^*$
    if $J_1^* < J_p^*$ **then**
      $x_{k+1} = Ax_k + Bu_k^1$
    else
      $x_{k+1} = Ax_k + Bu_k^p$
      $i = p$
    end if
  else
    Solve $\mathcal{O}(N_h, i, x_k) \rightarrow (J_i^*, (u_k^i, ..., u_{k+N_h-1}^i))^*$
    Solve $\mathcal{O}(N_h, i-1, x_k)$
      $\rightarrow (J_{i-1}^*, (u_k^{i-1}, ..., u_{k+N_h-1}^{i-1}))^*$
      $\vdots$
    Solve $\mathcal{O}(N_h, 1, x_k) \rightarrow (J_1^*, (u_k^1, ..., u_{k+N_h-1}^1))^*$
    $q^* = \arg \min_{q \in [\![1, i]\!]} (J_q)$
    $x_{k+1} = Ax_k + Bu_k^{q^*}$
    $i = i - 1$
  end if
until $k = N_{Simu}$

Theorem 11. (Homogeneity of periodic invariance). Given $\mathcal{B}$ strongly p-invariant ($p \in \mathbb{N}^*$) with respect to $(\mathbb{X}, \mathbb{U})$, then for all $\alpha \in [0, 1]$, $\alpha\mathcal{B}$ is strongly p-invariant with respect to $(\alpha\mathbb{X}, \alpha\mathbb{U})$.

The IBC scheme is applied within a (constrained) admissible framework, thus the reference has to be feasible. In the following procedure, a virtual reference will be considered, generated within an optimization-based reference governor (Gilbert et al., 1995). Given the outer set Ω^v and assume its strong p-invariance, the virtual reference and the scaling factor are solutions of the optimization problem $\mathcal{R}(x_k^{ref}, x_k)$:

$$\begin{aligned} \min_{(\bar{x}_k, \bar{u}_k, \alpha_k)} \quad & \|x_k^{ref} - \bar{x}_k\|^2 \\ \text{s.t.} \quad & \bar{x}_k = A\bar{x}_k + B\bar{u}_k, \ \bar{u}_k \in (1 - \alpha_k)\mathbb{U}, \\ & x_k \in \{\bar{x}_k\} \oplus \alpha_k \Omega^v \subset \Omega^v. \end{aligned} \quad (3)$$

The main idea is to solve, for the current state x_k in Ω^v, the optimization problem $\mathcal{R}(x_k^{ref}, x_k)$ and once the admissible $\bar{x}_k$ and $\bar{u}_k$ are computed to regulate the tracking error by solving a IBC problem. This tracking error will be defined as $\epsilon_k = \bar{x}_k - x_k \in \alpha_k \Omega^v$ and two cases have to be considered:

- if $\epsilon_{k+1} \notin \Omega^v$ as a result of the IBC at step k then we hold $\bar{x}_k$ and $\bar{u}_k$ for maximum p steps in order to allow $\epsilon_{k+i} \in \Omega^v$ for some $0 < i \leq p$. Practically, at each iteration i, we check if $\epsilon_{k+i} \in B$ and if this is the case we release $\bar{x}_{k+i}$ and $\bar{u}_{k+i}$
- if $\epsilon_{k+1} \in \Omega^v$ as a result of the IBC at step k we start from the beginning the procedure with virtual reference design and IBC.

Practically, the optimization (3) generates a virtual trajectory which are fixed points of the system (1). Whenever the reference to be tracked x_k^{ref} converges to a fixed point, it can be shown that $\bar{x}_k$ converges to a feasible fixed point $\bar{x} \in \mathbb{X}$ with respect to an admissible control $\bar{u} \in \mathbb{U}$ and the the tracking error ϵ_k asymptotically decreases to zero using the IBC. The formal proofs are adapting the IBC properties for the case of p-invariance similar to the previous sections and are not presented for brevity.

6. ILLUSTRATIVE NUMERICAL EXAMPLES

Consider the linear double integrator :

$$x(k+1) = \begin{bmatrix} 1 & 0.08 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.0032 \\ 0.08 \end{bmatrix} u(k) \qquad (4)$$

subject to constraints:

$$-2.6 \leq x_1(k) \leq 2.6, -3 \leq x_2(k) \leq 3, -5 \leq u(k) \leq 5. \quad (5)$$

The two techniques presented in the paper are considered for simulation and comparison: Strong and weak p-invariant reference tracking and periodic IBC. Then two types of trajectories are presented:

- A dynamically generated trajectory with the goal to test and illustrate the recursive feasibility and the reactivity of both tracking algorithms;
- A sequence of switching of fixed points which aims to test (aside the recursive feasibility) the convergence properties.

The inner polyhedral set $\Omega^o = \{x \in \mathbb{R}^n | F_o x \leq g_o\}$ considered is the Maximal Admissible Set (MAS) with respect to a linear feedback law $u_k = -[16.65 \; 6.23]\, x_k$ and has been constructed based on the procedure given in Gilbert and Tan (1991) in polyhedral form. In order to illustrate the complexity of the the maximal controllable set, a controlled invariant approximation $\mathcal{C}(\Omega^o)$ is computed as a N-step controllable set ($N \in \mathbb{N}^*$), thanks to the iterative procedure proposed in Nguyen et al. (2011) leading to the polyhedral form: $C_N(\Omega^o) = \{x \in \mathbb{R}^n | F_N x \leq g_N\}$.

The candidate p-invariant set is represented by a simple inner approximation of $C_N(\Omega^o)$, represented in blue in Fig.1. The same set, denoted Ω^v , is used as an outer set for the p-invariant MPC and IBC strategies.

6.1 Time-Varying (Dynamic) Trajectory

In the first simulation scenario, the reference is generated using the dynamical model (4) based on a excitation signal which violates drastically the imposed constraints. As a result, the time-varying reference trajectory leaves the state constraints set as illustrated by dashed trajectories in Fig.1. The initial state of the system is selected on the frontier of Ω^v on a extreme state (corresponding to zero speed if the state is interpreted in terms of position-speed coordinates). For the comparative study, strong and weak p-invariant sets have been constructed and the p index has been computed using Algorithms 1 and 3 to be $p = 14$. Using this low complexity set (4 vertices) the reference tracking optimization has been solved according to the Algorithms 2 and 4 and the results confirm the recursive feasibility of both receding-horizon reference-tracking algorithms.

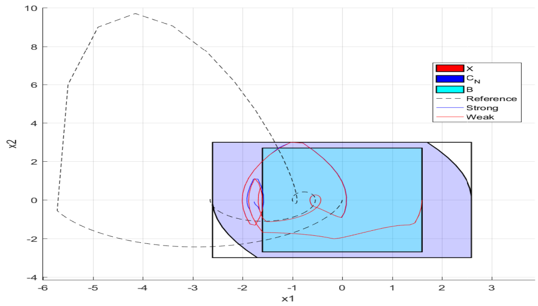

Fig. 1. Trajectories in the state space of reference (dashed), Strong p-invariance procedure (blue) and Weak p-invariance procedure (red).

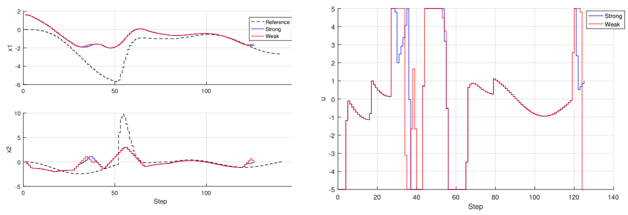

Fig. 2. Temporal trajectories of reference (dashed), Strong p-invariance (blue) and Weak p-invariance (red). Right: Control action of Strong p-invariance (blue) and Weak p-invariance (red).

Even if globally the behaviour is similar, a slight difference can be observed when the state leaves $\mathcal{B}$ (the signals are depicted with respect to the time in Fig. 2).

6.2 Static Reference Trajectory

In this case, the reference is a fixed-point out of the admissible set and this reference commutes to a symmetric fixed point out of the admissible set at a regular time interval. The invariant set Ω^v is the strongly p-invariant set with index $p = 8$. Figs 3 and 4, compare strong and weak p-invariant based reference tracking which provide a recursive feasible control law and good performances.

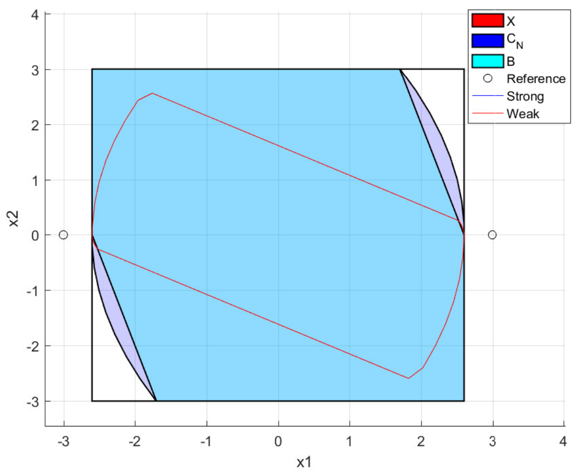

Fig. 3. Trajectories in the state space of reference (dashed), Strong p-invariance (blue) & Weak p-invariance (red).

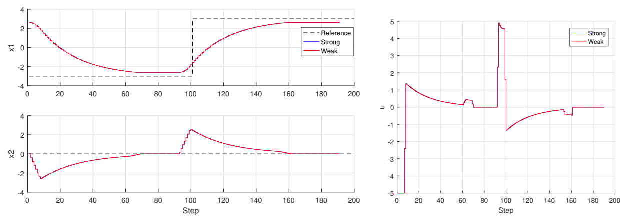

Fig. 4. Temporal trajectories of reference (dashed), Strong p-invariance (blue) and Weak p-invariance (red). Left: states; right: control.

In order to complete the design with stability (convergence) guarantees for such piece-wise constant references, an IBC for tracking is implemented using the periodic invariance notion. The closed-loop performance is depicted in Fig. 5, 6. It is important to observe that although the virtual reference is a fixed point, it is updated at each iteration, and thus leads to a sequence of feasible set-points for the IBC procedure.

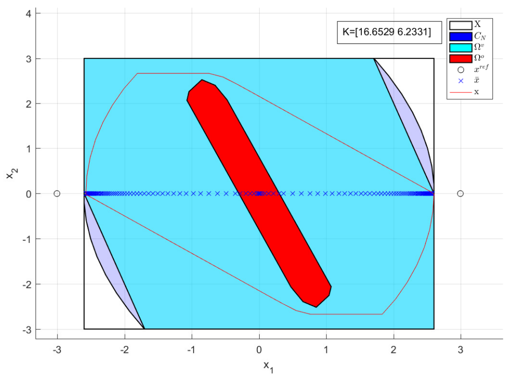

Fig. 5. Trajectories in the state space of reference (dashed), virtual $\bar{x}$ (blue cross) and state trajectory (red).

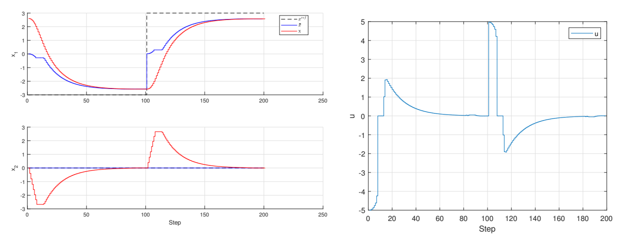

Fig. 6. Left: time trajectories of reference (dashed), virtual trajectory $\bar{x}$ (blue) and and state trajectory (red). Right: Control action of periodic invariant IBC.

7. CONCLUSIONS

The paper presented a novel scheme for the constrained reference tracking control problem. The proposed optimization framework guarantees recursive feasibility through the use of low complexity periodic invariant sets that replace the costly (in terms of computation and complexity) approximations of the maximal controlled invariant sets. Moreover it has been shown that the receding horizon optimization can be further enhanced by an interpolation-based control scheme, and thus it guarantees the convergence in the case of piecewise constant reference signals.

REFERENCES

Bemporad, A. (1998). Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 43(3), 415–419.

Blanchini, F. and Miani, S. (2000). Any domain of attraction for a linear constrained system is a tracking domain of attraction. *SIAM Journal on Control and Optimization*, 38(3), 971–994.

Chisci, L. and Zappa, G. (2003). Dual mode predictive tracking of piecewise constant references for constrained linear systems. *International Journal of Control*, 76(1), 61–72.

Falugi, P. (2015). Model predictive control for tracking randomly varying references. *International Journal of Control*, 88(4), 745–753.

Gilbert, E.G., Kolmanovsky, I., and Tan, K.T. (1995). Discrete-time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of robust and nonlinear control*, 5(5), 487–504.

Gilbert, E.G. and Tan, K.T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic control*, 36(9), 1008–1020.

Gutman, P.. and Cwikel, M. (1986). Admissible sets and feedback control for discrete-time linear dynamical systems with bounded controls and states. *IEEE Transactions on Automatic Control*, 31(4), 373–376. doi: 10.1109/TAC.1986.1104270.

Lee, Y.I. and Kouvaritakis, B. (2006). Constrained robust model predictive control based on periodic invariance. *Automatica*, 42(12), 2175–2181.

Limon, D. and Alamo, T. (2013). Tracking model predictive control. *Encyclopedia of Systems and Control*, 1–12.

Nguyen, H., Gutman, P.O., Olaru, S., Hovd, M., and Colledani, F. (2011). Improved vertex control for time-varying and uncertain linear discrete-time systems with control and state constraints. In *Proceedings of the 2011 American Control Conference*, 4386–4391. IEEE.

Nguyen, H.N. (2014). Constrained control of uncertain, time-varying, discrete-time systems. *Lecture Notes in Control and Infomration Sciences*, 451, 17.

Nguyen, H.N., Gutman, P.O., Olaru, S., and Hovd, M. (2013). Implicit improved vertex control for uncertain, time-varying linear discrete-time systems with state and control constraints. *Automatica*, 49(9), 2754–2759.

Olaru, S. and Dumur, D. (2005). Compact explicit mpc with guarantee of feasibility for tracking. In *Proceedings of the 44th IEEE Conference on Decision and Control*, 969–974. IEEE.

Scialanga, S. and Ampountolas, K. (2017). Decentralised control of interconnected systems: An interpolation based approach.

Scialanga, S. and Ampountolas, K. (2018). Interpolating constrained control of interconnected systems. *IFAC-PapersOnLine*, 51(9), 7–12.

Scialanga, S. and Ampountolas, K. (2019). Interpolating control toolbox (ict). In *2019 18th European Control Conference (ECC)*, 2510–2515. IEEE.

Soyer, M., Olaru, S., and Fang, Z. (2020). Interpolation based control for reference tracking under constraints. In *European Control Conference 2020.*