An output stabilization of second order semilinear systems ⋆

Lahcen Ezzaki * **El Hassan Zerrik** **

* *Higher School of Education and Training, Ibn Zohr University, Agadir, Morocco (e-mail: l.ezzaki@uiz.ac.ma).*
** *MACS Laboratory, Faculty of Sciences, Moulay Ismail University, Meknes, Morocco (e-mail: zerrik3@yahoo.fr)*

Abstract: This paper considers regional stabilization for a class of second order semilinear systems on a subregion of the system evolution domain. Then under sufficient conditions we give controls that ensure regional exponential and strong stabilization.

Keywords: Distributed systems, Second order systems, Semilinear systems, Regional stabilization.

1. INTRODUCTION

In this work we study regional strong and exponential stabilization of the system defined on an open, regular and bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 0$ by

$$\begin{cases} y_{tt}(t) = Ay(t) + u(t)By_t(t),\ t \geq 0 \\ y(0) = y_0,\ y_t(0) = y_1, \end{cases} \tag{1}$$

where A is an unbounded dissipative operator on a functional space $H = H(\Omega)$ endowed with the inner product $\langle ., . \rangle$ and the corresponding norm $\| \ \|_H$, $B : H \to H$ is a bounded operator and possibly nonlinear, u denotes the control function. Let $V = D(A^{\frac{1}{2}}) = V(\Omega)$ endowed with the norm $\|v\|_V = \|A^{\frac{1}{2}}v\|_H$ and $X = V \times H$ be the state space.

The stability of such systems has been considered in many works : Haraux (1989), studied the exponential stability of a second order linear system which verifies the observability inequality. Bardos et al. (1992) showed that, when the system evolution domain and the damping term are of class C^∞, the exponential stability is guaranteed if and only if the "geometric control condition" is satisfied. In Chen et al. (1991), authors proved the exponential decay of solution of one dimensional wave equation under broad hypotheses on the damping term. In Kangsheng et al. (2002), authors studied the exponential decay of energy of wave equation with potential and indefinite damping. The strong stabilization of a second order bilinear system has been studied by Couchouroun (2011) with a diagonal control operator. Also, Tebou (2009) extended the result given in Haraux (2001) concerning linear systems to semilinear case, establishing an equivalence between the stabilization of a semilinear system and the observability of the corresponding undamped system. In Martinez et al. (2000), authors considered the wave equation with a nonlinear internal damping, and proved that the energy of solutions decays exponentially to zero. Also in Haraux (2001), author studied the weak stabilization of the wave equation perturbed by a non monotone damping term.

The notion of regional stabilization for semilinear distributed systems was developed by Zerrik and Ouzahra (2007), Zerrik and Ezzaki (2017), and it consists in studying the state behavior of such a system, not in its whole geometrical evolution domain Ω, but just in a subregion ω of Ω. Also, it makes sense to the usual concept of stabilization taking into account the spatial variable and then becomes closer to real applications, where one wishes to stabilize a system in a critical subregion. Furthermore, regional stabilization may be useful for systems which are not stabilizable but stabilizable only on a subregion ω and it may be realized by lower cost than the stabilization on the whole evolution domain Ω.

In Zerrik and Ezzaki (2019); Ezzaki and Zerrik (2020), we studied exponential and strong stabilization of bilinear and semilinear second order systems using nonlinear controls. In this paper, we extend the results of Ezzaki and Zerrik (2020) concerning exponential and strong stabilization of bilinear and semilinear second order systems to regional case.

This paper is organized as follows : in the second section, we give sufficient conditions for regional exponential and strong stabilization of bilinear systems. Third section is devoted to regional exponential and strong stabilization of semilinear systems.

2. REGIONAL STABILIZATION OF BILINEAR SYSTEMS

In this section we study regional stabilization of system (1) that can be written in the form

$$\begin{cases} \dfrac{d}{dt}\tilde{y}(t) = \tilde{A}\tilde{y}(t) + \tilde{B}\tilde{y}(t) \\ \tilde{y}(0) = \tilde{y}_0 \end{cases} \tag{2}$$

where $\tilde{y}(t) = (y, y_t)$ and $\tilde{A}$ given by

$$\tilde{A}\tilde{y}(t) = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tilde{y}(t)$$

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and the operator $\tilde{B}$ is given by

$$\tilde{B}\tilde{y}(t) = \begin{pmatrix} 0 \\ u(t)By_t(t) \end{pmatrix}$$

System (2) has a unique weak solution in $C([0,T],X)$ (see Pazy (1983)), given by the variation of constants formula

$$\tilde{y}(t) = S(t)\tilde{y}_0 + \int_0^t S(t-s)\tilde{B}\tilde{y}(s)ds. \tag{3}$$

where $S(t)$ is the semigroup generated by the operator $\tilde{A}$.

Let ω be an open subregion of Ω; $V(\omega)$ and $H(\omega)$ are respectively functional spaces of the restriction of the functions of V and H. We define by

$$\begin{array}{rcl} \chi^1_\omega : V(\Omega) & \longrightarrow & V(\omega) \\ z_1 & \longmapsto & z_1|_\omega \end{array}$$

and

$$\begin{array}{rcl} \chi^2_\omega : H(\Omega) & \longrightarrow & H(\omega) \\ z_2 & \longmapsto & z_2|_\omega \end{array}$$

the restrictions operators to ω while χ^{1*}_ω and χ^{2*}_ω denote respectively the adjoint operators of χ^1_ω and χ^2_ω. Let us denote $i_\omega = \chi^{1*}_\omega\chi^1_\omega$ and $j_\omega = \chi^{2*}_\omega\chi^2_\omega$.
In what follows, we assume that

$$\langle i_\omega By_t, y_t\rangle\langle By_t, y_t\rangle \geq 0, \quad \forall y_t \in H. \tag{4}$$

and

$$\langle j_\omega Ay, y\rangle \leq 0, \quad \forall y \in \mathcal{D}(A). \tag{5}$$

On the other hand, we consider the following uncontrolled system

$$\begin{cases} z_{tt}(t) = Az(t), \\ z(0) = y_0, z_t(0) = y_1, \end{cases} \tag{6}$$

2.1 Regional strong stabilization

Here we give sufficient conditions for regional strong stabilization of system (1) when B is linear.

Theorem 1.
Suppose that there exist some positive constants T and α such that the solution z of system (6) satisfies

$$\|\chi^1_\omega y_0\|^2_{V(\omega)} + \|\chi^2_\omega y_1\|^2_{H(\omega)} \leq \alpha \int_0^T |\langle i_\omega Bz_t(t), z_t(t)\rangle|dt \tag{7}$$

then the control

$$u(t) = -\langle i_\omega By_t(t), y_t(t)\rangle \tag{8}$$

stabilizes regionally strongly system (1).

Proof. From the relation
$\langle i_\omega Bz_t, z_t\rangle = \langle i_\omega B(z_t - y_t), z_t\rangle - \langle i_\omega By_t, y_t - z_t\rangle + \langle i_\omega By_t, y_t\rangle$, and since B and χ^1_ω are bounded, then there exists $\delta > 0$ such that

$$\begin{aligned} |\langle i_\omega Bz_t, z_t\rangle| &\leq \delta^2\|B\|_H\|z_t - y_t\|_H\|z_t\|_H \\ &+ \delta^2\|B\|_H\|y_t - z_t\|_H\|y_t\|_H + |\langle i_\omega By_t, y_t\rangle| \end{aligned} \tag{9}$$

Differentiating the energy $E_y(t) = \frac{1}{2}\{\|y_t\|^2_H + \|y\|^2_V\}$ and since A is dissipative, we get

$$\frac{d}{dt}E_y(t) \leq -f(y_t(t))$$

where $f(y_t(t)) = \langle i_\omega By_t, y_t\rangle\langle By_t, y_t\rangle$.
Then

$$E_y(t) \leq -\int_0^t f(y_s(s))ds + E_y(0)$$

It follows from (4), that

$$E_y(t) \leq E_y(0) \tag{10}$$

Moreover, we have

$$\|y_t\|^2_H \leq \|(y, y_t)\|^2_X \leq E_y(t) \quad \text{and} \quad \|z_t - y_t\|^2_H \leq \|\psi - \phi\|^2_X \tag{11}$$

where $\psi = (z, z_t)$ and $\phi = (y, y_t)$.
Using (9), (10) and (11), we obtain

$$|\langle i_\omega Bz_t, z_t\rangle| \leq 2\delta^2\|B\|_H\|\psi - \phi\|_X\sqrt{E_y(0)} + |\langle i_\omega By_t, y_t\rangle| \tag{12}$$

From (3) and since A is dissipative, we have

$$\|\psi - \phi\|_X \leq \sqrt{E_y(0)}\Big(\int_0^t \langle i_\omega By_s(s), y_s(s)\rangle ds\Big)$$

It follows from Schwartz's inequality, that

$$\|\psi - \phi\|_X \leq \sqrt{TE_y(0)}\Big(\int_0^T |\langle i_\omega By_s(s), y_s(s)\rangle|^2 ds\Big)^{\frac{1}{2}} \tag{13}$$

Integrating (12) over the interval $[0,T]$, and with (13) we obtain

$$\begin{aligned} \int_0^T |\langle i_\omega Bz_t, z_t\rangle|dt &\leq (2E_y(0)\|B\|_H + 1)\sqrt{T} \\ &\times \Big(\int_0^T |\langle i_\omega By_s(s), y_s(s)\rangle|^2 ds\Big)^{\frac{1}{2}} \end{aligned}$$

Condition (7) gives

$$\begin{aligned} &\|\chi^1_\omega y_0\|^2_{V(\omega)} + \|\chi^2_\omega y_1\|^2_{H(\omega)} \\ &\leq \beta\Big(\int_0^T |\langle i_\omega By_s(s), y_s(s)\rangle|^2 ds\Big)^{\frac{1}{2}} \end{aligned}$$

where $\beta = \alpha(2E_y(0)\|B\|_H + 1)\sqrt{T}$.

Now replacing (y_0, y_1) by (y, y_t), we get

$$\begin{aligned} &\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)} \\ &\leq \beta\Big(\int_t^{t+T} |\langle i_\omega By_s(s), y_s(s)\rangle|^2 ds\Big)^{\frac{1}{2}} \end{aligned}$$

Then $\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)} \longrightarrow 0,$ *as* $t \longrightarrow +\infty$, which completes the proof.

2.2 Regional exponential stabilization

The following result provides sufficient conditions for regional exponential stabilization of system (1) when B is linear.

Theorem 2.
Assume that the solution z of system (6) satisfies the condition

$$\|\chi^1_\omega y_0\|_{V(\omega)} + \|\chi^2_\omega y_1\|_{H(\omega)} \leq \alpha\int_0^T |\langle i_\omega Bz_t(t), z_t(t)\rangle|dt \quad (\text{for some } T, \alpha > 0) \tag{14}$$

then the control

$$u(t) = \begin{cases} -\dfrac{\langle i_\omega By_t(t), y_t(t)\rangle}{\|(y, y_t)\|^2_X}, & (y, y_t) \neq (0,0) \\ 0, & (y, y_t) = (0,0) \end{cases} \tag{15}$$

stabilizes regionally exponentially system (1), in other word there exist $M > 0$ and $\lambda > 0$ such that for every $(y_0, y_1) \in X$

$$\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)} \leq Me^{-\lambda t}\|(y_0, y_1)\|^2_X.$$

Proof. Let us consider the solution z of system (6) with the natural energy associated with (1)

$$E_y(t) = \frac{1}{2}\{\|y(t)\|_V^2 + \|y_t(t)\|_H^2\}$$

Since A is dissipative, we have

$$\frac{d}{dt}E_y(t) \leq -\frac{\langle i_\omega B y_t, y_t\rangle}{2E_y(t)}\langle By_t, y_t\rangle \tag{16}$$

Integrating this inequality, gives

$$E_y(t) \leq -\int_0^t \frac{\langle i_\omega B y_s(s), y_s(s)\rangle}{2E_y(s)}\langle By_s(s), y_s(s)\rangle ds + E_y(0)$$

Then

$$E_y(t) \leq E_y(0) \tag{17}$$

Furthermore, we have

$$\|y_t\|_H^2 \leq \|(y, y_t)\|_X^2 \leq 2E_y(t) \quad and \quad \|z_t - y_t\|_H^2 \leq \|\psi - \phi\|_X^2 \tag{18}$$

where $\psi = (z, z_t)$ and $\phi = (y, y_t)$.
Using (9) and combining (17) and (18), we obtain

$$|\langle i_\omega B z_t, z_t\rangle| \leq 2\delta^2\|B\|_H\|\psi - \phi\|_H\sqrt{2E_y(0)} + |\langle i_\omega B y_t, y_t\rangle| \tag{19}$$

Furthermore, from (3), we have

$$\|\psi - \phi\|_X \leq \|B\|_H\Big\{\int_0^t \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|}{2E_y(s)}\sqrt{2E_s(s)}ds\Big\}$$

Schwartz's inequality, gives

$$\|\psi - \phi\|_X \leq \|B\|_H\sqrt{2TE_y(0)}\Big\{\int_0^T \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds\Big\}^{\frac{1}{2}} \tag{20}$$

Using (17), we have

$$|\langle i_\omega B y_t, y_t\rangle| \leq \frac{|\langle i_\omega B y_t, y_t\rangle|}{2E_y(t)}\sqrt{2E_y(t)}\sqrt{2E_y(0)} \tag{21}$$

Integrating (19) over the interval $[0, T]$ and taking into account (20) and (21), we obtain

$$\int_0^T |\langle i_\omega B z_t, z_t\rangle| dt \leq (2\|B\|_H + 1)\|B\|_H\sqrt{T2E_y(0)} \Big\{\int_0^T \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds\Big\}^{\frac{1}{2}}$$

It follows from (14) that

$$\|\chi_\omega^1 y_0\|_{V(\omega)} + \|\chi_\omega^2 y_1\|_{H(\omega)} \leq \beta \Big\{\int_0^T \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds\Big\}^{\frac{1}{2}}$$

where $\beta = \alpha(2\|B\|_H + 1)\|B\|_H\sqrt{2TE_y(0)}$.
Now replacing (y_0, y_1) by (y, y_t), we get

$$\|\chi_\omega^1 y_t(t)\|_{V(\omega)} + \|\chi_\omega^2 y(t)\|_{H(\omega)} \leq \beta\Big\{\int_t^{t+T} \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds\Big\}^{\frac{1}{2}}$$

Then

$$\|\chi_\omega^1 y(t)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(t)\|_{H(\omega)}^2 \leq \beta^2\int_t^{t+T} \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds \tag{22}$$

Moreover, we have

$$\frac{d}{dt}(\|\chi_\omega^1 y(t)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(t)\|_{H(\omega)}^2) \leq -\frac{|\langle i_\omega B y_t, y_t\rangle|^2}{2E_y(t)} \tag{23}$$

Integrating inequality (23) from nT to $(n+1)T$, and since $E_y(t)$ decreases, we obtain

$$\|\chi_\omega^1 y(nT)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(nT)\|_{H(\omega)}^2 - \|\chi_\omega^1 y((n+1)T)\|_{V(\omega)}^2 - \|\chi_\omega^2 y_t((n+1)T)\|_{H(\omega)}^2 \geq \int_{nT}^{(n+1)T} \frac{|\langle i_\omega B y_s(s), y_s(s)\rangle|^2}{2E_y(s)}ds$$

From (22), we get

$$\|\chi_\omega^1 y((n+1)T)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t((n+1)T)\|_{H(\omega)}^2 \leq r(\|\chi_\omega^1 y(nT)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(nT)\|_{H(\omega)}^2)$$

where $r = \frac{1}{\beta^2}$.
By recurrence, we show that

$$\|\chi_\omega^1 y(nT)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(nT)\|_{H(\omega)}^2 \leq r^n(\|\chi_\omega^1 y_0\|_{V(\omega)}^2 + \|\chi_\omega^2 y_1\|_{H(\omega)}^2)$$

Since $E_y(t)$ decreases and taking n the integer part of $\frac{t}{T}$, it follows that

$$\|\chi_\omega^1 y(t)\|_{V(\omega)}^2 + \|\chi_\omega^2 y_t(t)\|_{H(\omega)}^2 \leq Me^{-\lambda t}\|(y_0, y_1)\|_X^2$$

where $M = (2\|B\|_H + 1)\delta\|B\|_H\sqrt{T}$ and
$\lambda = \dfrac{\ln((2\|B\|_H + 1)\|B\|_H\sqrt{T})}{T}$.
In other word system (1) is regionally exponentially stabilizable by the control (15).

3. REGIONAL STABILIZATION OF SEMILINEAR SYSTEMS

In this section we study regional strong and exponential stabilization of system (1) when B is nonlinear.
Let us consider the system

$$\begin{cases} z_{tt}(t) = Az(t), \\ z(0) = y_0, z_t(0) = y_1, \end{cases} \tag{24}$$

The following result gives sufficient conditions for strong stabilization of system (1).

Theorem 3.
Assume that B is locally Lipschitz and there exist $T > 0$ and $\delta > 0$ such that the solution (z, z_t) of system (24) verifies the inequality

$$\int_0^T |\langle i_\omega B z_t(t), z_t(t)\rangle| dt \geq \delta\{\|\chi_\omega^2 y_t(0)\|_{H(\omega)}^2 + \|\chi_\omega^1 y(0)\|_{V(\omega)}^2\}, \tag{25}$$

then the control

$$u(t) = -\langle i_\omega B y_t(t), y_t(t)\rangle \tag{26}$$

stabilizes regionally strongly system (1).

Proof. We show first that $h : y_t \mapsto \langle i_\omega B y_t, y_t\rangle By_t$ is locally Lipschitz.
Since B is locally Lipschitz, then for all $R > 0$, there exists $L > 0$ such that

$$\|By_t - Bz_t\|_H \leq L\|y_t - z_t\|_H, \ \forall y_t, z_t \in H : \quad 0 < \|y_t\|_H \leq \|z_t\|_H \leq R. \tag{27}$$

Let remark that

$$\begin{aligned}&\langle i_\omega B z_t, z_t\rangle Bz_t - \langle i_\omega B y_t, y_t\rangle By_t \\ &= \langle i_\omega B z_t, z_t\rangle(Bz_t - By_t) \\ &+(\langle i_\omega B z_t, z_t\rangle - \langle i_\omega B y_t, y_t\rangle)By_t \\ &= (\langle i_\omega B z_t, z_t - y_t\rangle - \langle i_\omega B z_t - i_\omega B y_t, y_t\rangle)By_t \\ &+\langle i_\omega B z_t, z_t\rangle(Bz_t - By_t)\end{aligned}$$

Using (27), we have

$$\begin{aligned}\|h(z_t)-h(y_t)\|_H &\leq L^2\|z_t\|_H^2\|z_t-y_t\|_H\\ &+ L^2\|z_t-y_t\|_H(\|z_t\|_H+\|y_t\|_H)\|y_t\|_H\\ &\leq \mathcal{L}\|z_t-y_t\|_H\end{aligned}$$

where $\mathcal{L}=3LR^2$.
It follows that system (1) has a unique weak solution (y, y_t) (see Pazy (1983)).
Using (9) and (27), we obtain

$$\begin{aligned}|\langle i_\omega Bz_t, z_t\rangle| &\leq L\|z_t-y_t\|_H\|z_t\|_H\\ &+L\|y_t-z_t\|_H\|y_t\|_H+|\langle i_\omega By_t, y_t\rangle|\end{aligned} \quad (28)$$

Now, we differentiate $E_y(t)$

$$\frac{d}{dt}E_y(t)\leq -\langle i_\omega By_t, y_t\rangle\langle By_t, y_t\rangle$$

Then

$$E_y(t)\leq -\int_0^t \langle i_\omega By_s(s), y_s(s)\rangle\langle By_s(s), y_s(s)\rangle ds + E_y(0)$$

It follows from (4), that

$$E_y(t)\leq E_y(0) \quad (29)$$

Moreover, we have

$$\|y_t\|_H^2\leq \|(y,y_t)\|_X^2\leq 2E_y(t) \quad \textit{and} \quad \|z_t-y_t\|_H^2\leq\|\psi-\phi\|_X^2 \quad (30)$$

where $\psi=(z,z_t)$ and $\phi=(y,y_t)$.
From (28) and combining (29) with (30), we get

$$|\langle i_\omega Bz_t, z_t\rangle|\leq 2L\|\psi-\phi\|_X\sqrt{2E_y(0)}+|\langle i_\omega By_t, y_t\rangle| \quad (31)$$

From (3), we have

$$\|\psi-\phi\|_X\leq\sqrt{2E_y(0)}\int_0^t|\langle i_\omega By_t, y_t\rangle|^2 ds$$

For a fixed $T>0$, Schwarz inequality, yields

$$\|\psi-\phi\|_X\leq\sqrt{2TE_y(0)g(0)} \quad (32)$$

where $g(t)=\displaystyle\int_t^{t+T}|\langle i_\omega By_s(s), y_s(s)\rangle|^2 ds$.
Integrating (31) on the interval $[0,T]$ and using (32), we obtain

$$\int_0^T|\langle i_\omega Bz_t, z_t\rangle|dt\leq(2E_y(0)+1)L\sqrt{Tg(0)}$$

Condition (25) allows

$$\|\chi_\omega^2 y_1\|_{H(\omega)}^2+\|\chi_\omega^1 y_0\|_{V(\omega)}^2\leq\frac{(2E_y(0)+1)L\sqrt{T}}{\delta}\sqrt{g(0)}$$

Now, replacing (y_0, y_1) by (y, y_t), we obtain

$$\|\chi_\omega^2 y_t\|_{H(\omega)}^2+\|\chi_\omega^1 y\|_{V(\omega)}^2\leq\beta\sqrt{g(t)}$$

where $\beta=\dfrac{(2E_y(0)+1)L\sqrt{T}}{\delta}$.
It follows that $\|\chi_\omega^1 y(t)\|_{V(\omega)}^2+\|\chi_\omega^2 y_t(t)\|_{H(\omega)}^2\longrightarrow 0$ as $t\longrightarrow+\infty$, we deduce that the control (26) stabilizes regionally strongly system (1).

The following result gives sufficient conditions for regional exponential stabilization of system (1).

Theorem 4.
Assume that B is locally Lipschitz and there exist $T>0$ and $C>0$ such that the solution (z, z_t) of system (24) satisfy the following condition

$$\int_0^T|\langle i_\omega Bz_t(t), z_t(t)\rangle|dt\geq C\{\|\chi_\omega^2 y_1\|_{H(\omega)}+\|\chi_\omega^1 y_0\|_{V(\omega)}\}, \quad (33)$$

then the control

$$u(t)=\begin{cases}-\dfrac{\langle i_\omega By_t(t), y_t(t)\rangle}{\|(y,y_t)\|_X^2}, & (y,y_t)\neq(0,0)\\ 0, & (y,y_t)=(0,0)\end{cases} \quad (34)$$

stabilizes exponentially system (1) on ω.

Proof. Let $\phi=(y,y_t)$ et $\psi=(z,z_t)$. We show first that the operator $f:\phi\mapsto\dfrac{\langle i_\omega By_t, y_t\rangle}{\|(y,y_t)\|_X^2}By_t$ is locally Lipschitz.
Since B is locally Lipschitz, then for all $R>0$, there exists $K>0$ such that

$$\begin{aligned}&\|By_t-Bz_t\|_H\leq K\|y_t-z_t\|_H,\ \forall y_t, z_t\in H:\\ &0<\|y_t\|_H\leq\|z_t\|_H\leq R.\end{aligned} \quad (35)$$

Then

$$\begin{aligned}f(\psi)-f(\phi) &= \frac{\|\phi\|_X^2(\langle i_\omega Bz_t, z_t\rangle Bz_t-\langle i_\omega By_t, y_t\rangle By_t)}{\|\psi\|_X^2\|\phi\|_X^2}\\ &-\frac{(\|\psi\|_X^2-\|\phi\|_X^2)\langle i_\omega By_t, y_t\rangle By_t}{\|\psi\|_X^2\|\phi\|_X^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\|f(\psi)-f(\phi)\|_X &\leq \frac{\|\langle i_\omega Bz_t, z_t\rangle Bz_t-\langle i_\omega By_t, y_t\rangle By_t\|_H}{\|\psi\|_X^2}\\ &+K^2|\|\psi\|_X^2-\|\phi\|_X^2|.\frac{\|\phi\|_X}{\|\psi\|_X^2}\\ &\leq \frac{\|\langle i_\omega Bz_t, z_t\rangle(Bz_t-By_t)}{\|\psi\|_X^2}\\ &+\frac{(\langle i_\omega Bz_t, z_t-y_t\rangle}{\|\psi\|_X^2}\\ &-\frac{\langle i_\omega By_t-i_\omega Bz_t, y_t\rangle)By_t\|_H}{\|\psi\|_X^2}\\ &+2K^2\|\psi-\phi\|_X\end{aligned}$$

Furthermore, we have

$$\|z_t-y_t\|_H^2\leq\|(\psi-\phi)\|_X^2 \quad (36)$$

Using (36), we get $\|f(\psi)-f(\phi)\|_X\leq L\|\psi-\phi\|_X$ where $L=5K^2$.
We conclude that system (1) has a unique weak solution (y, y_t) (see Pazy (1983)).
Using (35) and (9), then there exists $\alpha>0$ such that

$$\begin{aligned}|\langle i_\omega Bz_t, z_t\rangle| &\leq K\alpha^2\|z_t-y_t\|_H\|z_t\|_H\\ &+K\alpha^2\|y_t-z_t\|_H\|y_t\|_H+|\langle i_\omega By_t, y_t\rangle|\end{aligned} \quad (37)$$

Moreover, we have

$$\|y_t\|_H^2\leq\|(y,y_t)\|_X^2\leq 2E_y(t)$$

Since A is dissipative, then

$$\frac{d}{dt}E_y(t)\leq-\frac{\langle i_\omega By_t, y_t\rangle\langle By_t, y_t\rangle}{\|y_t\|_H^2+\|y\|^2}$$

Thus

$$E_y(t)=-\int_0^t\frac{\langle i_\omega By_s(s), y_s(s)\rangle\langle By_s(s), y_s(s)\rangle}{\|y_s(s)\|_H^2+\|y(s)\|^2}ds+E_y(0)$$

It follows from (4), that

$$E_y(t)\leq E_y(0) \quad (38)$$

Using (36), (37) and (38), we obtain

$$|\langle i_\omega Bz_t, z_t\rangle|\leq 2K\alpha^2\|\psi-\phi\|_X\sqrt{2E_y(0)}+|\langle i_\omega By_t, y_t\rangle| \quad (39)$$

From (3), we deduce that

$$\|\psi - \phi\|_X \leq K \int_0^t |f(s)|\sqrt{2E_y(s)}ds$$

where $f(s) = \dfrac{\langle i_\omega B y_s(s), y_s(s)\rangle}{2E_y(s)}$.

Schwarz inequality, gives

$$\|\psi - \phi\|_X \leq K\sqrt{2T}\Big(\int_0^T |f(s)|^2 E_y(s)ds\Big)^{\frac{1}{2}} \qquad (40)$$

Using (38), we get

$$|\langle i_\omega B y_t, y_t\rangle| \leq |f(t)|\sqrt{2E_y(t)}\sqrt{2E_y(0)} \qquad (41)$$

Integrating (39) on the interval $[0, T]$ and using (40) and (41), we obtain

$$\int_0^T |\langle i_\omega B z_s(s), z_s(s)\rangle|ds \leq 2(2K^2\alpha^2 + 1)\sqrt{TE_y(0)} \Big(\int_0^T |f(s)|^2 E_y(s)ds\Big)^{\frac{1}{2}}$$

It follows from (33), that

$$\|\chi^2_\omega y_t(0)\|_{H(\omega)} + \|\chi^1_\omega y(0)\|_{V(\omega)} \leq \frac{2(2K\alpha^2+1)}{C}\sqrt{TE_y(0)} \Big(\int_0^T |f(s)|^2 E_y(s)ds\Big)^{\frac{1}{2}}$$

Replacing (y_0, y_1) by (y, y_t), we get

$$\|\chi^2_\omega y_t(t)\|_{H(\omega)} + \|\chi^1_\omega y(t)\|_{V(\omega)} \leq \frac{2(2K\alpha^2+1)}{C}\sqrt{TE_y(t)} \Big(\int_t^{t+T} |f(s)|^2 E_y(s)ds\Big)^{\frac{1}{2}}$$

It follows that

$$\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)} \leq \gamma \int_t^{t+T} |f(s)|^2 E_y(s)ds \qquad (42)$$

where $\gamma = \dfrac{2(2K\alpha^2+1)}{C}\sqrt{TE_y(0)}$.

Integrating from kT to $(k+1)T$ the inequality

$$\frac{d}{dt}(\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)}) \leq -\frac{|\langle i_\omega B y_t, y_t\rangle|^2}{2E_y(t)}$$

and since $(\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)})$ decreases, we obtain

$$\begin{aligned}&\|\chi^1_\omega y(kT)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(kT)\|^2_{H(\omega)} \\ &-\|\chi^1_\omega y((k+1)T)\|^2_{V(\omega)} - \|\chi^2_\omega y_t((k+1)T)\|^2_{H(\omega)} \\ &\geq \int_{kT}^{(k+1)T} |f(s)|^2 E_y(s)ds\end{aligned}$$

From (42), we get

$$\begin{aligned}&\|\chi^1_\omega y((k+1)T)\|^2_{V(\omega)} + \|\chi^2_\omega y_t((k+1)T)\|^2_{H(\omega)} \\ &\leq r(\|\chi^1_\omega y(kT)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(kT)\|^2_{H(\omega)})\end{aligned}$$

where $r = \dfrac{1}{\gamma}$.

This implies that

$$\begin{aligned}&\|\chi^1_\omega y((k+1)T)\|^2_{V(\omega)} + \|\chi^2_\omega y_t((k+1)T)\|^2_{H(\omega)} \\ &\leq r^k(\|\chi^1_\omega y_0\|^2_{V(\omega)} + \|\chi^2_\omega y_1\|^2_{H(\omega)})\end{aligned} \qquad (43)$$

Applying (43) to the integer part $k = E(\dfrac{t}{T})$ of $\dfrac{t}{T}$, we deduce that

$$\|\chi^1_\omega y(t)\|^2_{V(\omega)} + \|\chi^2_\omega y_t(t)\|^2_{H(\omega)} \leq Me^{-\lambda t}\|(y_0, y_1)\|^2_X$$

where $M = \frac{2\alpha\sqrt{T}}{C}(2K+1)$ and $\lambda = \dfrac{\ln(\frac{2\sqrt{T}}{C}(2K+1))}{T}$.

Then control (34) stabilizes exponentially system (1) on ω.

4. CONCLUSION

In this work we have considered the problem of regional stabilization of second order semilinear systems. Under sufficient conditions, we gave controls that ensures exponential and strong stabilization of such systems. This work gives an opening to others questions : this is the case of regional boundary stabilization.

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