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Computation of Invariant Tubes for Robust Output Feedback Model Predictive Control

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Abstract: This paper presents an algorithm to calculate tightened invariant tubes for output feedback model predictive controllers (MPC). We consider discrete-time linear time-invariant (DLTI) systems with bounded state and input constraints and subject to bounded disturbances. In contrast to existing approaches which either use pre-defined control and observer gains or compute the control and observer gains that optimize the volume of the invariant sets for the estimation and control errors separately, we consider the problem of optimizing the volume of these sets simultaneously. The nonlinearities associated with computing the control and observer gains are circumvented by the application of Farkas' Theorem and an extended Elimination Lemma, to convert the nonconvex optimization problem into a convex semidefinite program. An update algorithm is then used to reduce the volume of the invariant tube through a finite number of iterations. Numerical examples are provided to illustrate the effectiveness of the proposed algorithm.

Keywords: Robust control invariant sets, linear matrix inequality, robust model predictive control, uncertain linear systems, optimization

1. INTRODUCTION

Robust control invariant (RCI) sets are fundamental tools in robust control synthesis for uncertain systems subject to disturbances. RCI sets play an integral part in establishing stability of Robust Model Predictive Control (RMPC) schemes (Tahir and Jaimoukha (2013)) and are also suitable for robust time-optimal control (Blanchini (1992); Mayne and Schroeder (1997)). Invariant set computation has been discussed widely in the past several decades (Blanchini and Miani (2008)), and important results are included in Kolmanovsky and Gilbert (1998); Dorea and Hennet (1999). In Kolmanovsky and Gilbert (1998), the authors show that the exact computation of polytopic RCI sets for systems subject to uncertainty is an intractable problem in general since it includes infinite Minkowski's sum terms. Therefore, most of the literature has been concerned with the efficient computation of inner/outer approximations to the maximal/minimal RCI sets, see (Raković et al. (2005a,b); Raković and Baric (2010); Trodden (2016)). More recently, an appealing approach is to consider both RCI set and feedback gain as decision variables. Tahir and Jaimoukha (2014) presents an algorithm to compute low complexity RCI sets for linear discrete-time systems involving additive disturbances and norm bounded uncertainty. Nevertheless, low-complexity polytopic RCI sets restrict the number of faces of the polytope. In the work of Liu and Jaimoukha (2015), the authors advocate a method to compute full-complexity polytopic RCI sets for linear systems subject to additive disturbances, which allows us to compute less conservative invariant approximations of RCI sets. This work has been extended to linear systems subject to additive disturbances and structured norm-bounded or polytopic uncertainties in Liu et al. (2019).

Due to the large computational burden of conventional on-line optimizations for RMPC, Langson et al. (2004) proposes the concept of Tube MPC, which uses a piecewise affine control law to maintain the controlled trajectories in the tube even in the presence of uncertainty. In addition, in many practical control problems, not all states are measurable and an observer is required to estimate the states. Mayne et al. (2006) proposes output MPC design by using a Luenberger observer, the difference between the actual and nominal states is the sum of the estimation and control errors bounded by two separate invariant sets, which are pre-computed along with pre-defined observer and feedback gains. Kögel and Findeisen (2017) proposes an idea to compute less conservative results on tighter constraints with respect to Mayne et al. (2006), they adopt a single tube to describe the sum of the estimation and control errors, but their observer and feedback gains still need to be pre-defined. In the work of Liu (2017), the author provides an algorithm to optimize the volume of the invariant set of the estimation error by treating the observer gain as a variable firstly, and then use this given set as an artificial disturbance and the associated observer gain L to optimize the volume of the invariant set of the control error along with the feedback gain K. However, this method is still somewhat conservative due to the fact that it takes L and K as variables separately.

In this paper, we focus on an extension of the approach in Liu (2017) for tube based robust output MPC of DLTI systems. The main contribution of this paper is to compute less conservative tightened constraints on the nominal system state and input. Two initial invariant sets for the estimation and control errors, together with the corresponding observer and feedback gains are computed separately. Then, the volume of these two sets is iteratively optimized by considering both the observer and feedback gains as variables simultaneously. This allows us to consider the interaction between the estimation and control errors. It will be shown, through numerical examples, that the total volume of two sets obtained by our algorithm is smaller than the approach in Liu (2017), and this results in less tightened constraints imposed on the nominal system.

Notation For integer $m \geq 1$, we define $\mathcal{N}_m := \{1, \cdots, m\}$. The set of positive semidefinite diagonal, positive definite symmetric, and symmetric matrices of dimension $m \times m$ are denoted as $\mathcal{D}_+^m$, $\mathcal{S}_+^m$, $\mathcal{S}^m$, respectively. The notation $A \succ 0$ or $A \prec 0$ denotes that matrix A is positive or negative definite. Given two sets $\mathcal{U}$ and $\mathcal{V}$, the Minkowski set addition and difference are defined as $\mathcal{U} \oplus \mathcal{V} = \{u + v \mid u \in \mathcal{U}, v \in \mathcal{V}\}$ and $\mathcal{U} \ominus \mathcal{V} = \{x \mid x \oplus \mathcal{V} \subseteq \mathcal{U}\}$, respectively. For $P \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, the notation $\mathcal{P}(P, b)$ describes a polytope $\{x \in \mathbb{R}^n : -b \leq Px \leq b\}$ and for $Q = Q^T \succ 0$ the notation $\mathcal{Q}(Q)$ denotes the ellipsoid $\left\{x \in \mathbb{R}^n : x^T Q x \leq 1\right\}$. A congruence transformation means effecting a congruence T that has full column rank, on a matrix inequality $A \succ 0$ which corresponds to pre- and post-multiplication by T^T and T, respectively, to deduce that $T^T A T \succ 0$. A Schur complement argument refers to the result that if $A = A^T$ and $C = C^T$ then $\begin{bmatrix} A & B \\ \star & C \end{bmatrix} \succ 0 \Leftrightarrow A \succ 0, C - B^T A^{-1} B \succ 0 \Leftrightarrow C \succ 0, A - BC^{-1}B^T \succ 0$, where $\star$ refers to a term easily inferred from symmetry.

2. PROBLEM DESCRIPTION

We consider the following linear discrete-time system with additive disturbance:

$$\begin{aligned} x^+ &= Ax + Bu + B_d d, \\ y &= Cx + Du + D_v v, \end{aligned}$$

where $x, x^+ \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, $d \in \mathbb{R}^{n_d}$, $v \in \mathbb{R}^{n_v}$, $y \in \mathbb{R}^{n_y}$ are the current state, successor state, control input, process noise, measurement noise and current output, respectively; all other symbols denote the appropriate distribution matrices. We combine the input and output noises as one augmented variable w, yielding the following dynamics with some redefinitions:

$$\begin{aligned} x^+ &= Ax + Bu + B_w w, \\ y &= Cx + Du + D_w w, \\ B_w &:= [B_d \ 0], \quad D_w := [0 \ D_v], \quad w := \begin{bmatrix} d \\ v \end{bmatrix}. \end{aligned} \tag{1}$$

We assume that (A, B) is controllable and (A, C) is observable. The state and input constraint sets are assumed to have the form:

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid \underline{x} \leq V_x x \leq \overline{x}\}, \ V_x \in \mathbb{R}^{m\times n}, \ \underline{x}, \ \overline{x} \in \mathbb{R}^m,$$

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid \underline{u} \leq V_u u \leq \overline{u}\}, \ V_u \in \mathbb{R}^{m_u\times n_u}, \ \underline{u}, \ \overline{u} \in \mathbb{R}^{m_u}.$$

The augmented disturbance w belongs to the bounded and symmetric polytope:

$$\mathcal{W} = \{w \in \mathbb{R}^{n_w} \mid -\overline{w} \leq V_w w \leq \overline{w}\}, V_w \in \mathbb{R}^{m_w\times n_w}, \overline{w} \in \mathbb{R}^{m_w}.$$

Furthermore, a simple Luenberger observer is employed to estimate the state:

$$\begin{bmatrix} \hat{x}^+ \\ \hat{y} \end{bmatrix} = \begin{bmatrix} A & B & L \\ C & D & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ u \\ y - \hat{y} \end{bmatrix}, \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the current observer state, $\hat{x}^+ \in \mathbb{R}^n$ is the successor state of the estimated system, $\hat{y} \in \mathbb{R}^{n_y}$ is the current observer output, and $L \in \mathbb{R}^{n\times n_y}$ is the Luenberger observer gain. We define the state estimation error $\tilde{x} := x - \hat{x}$, whose dynamics from (1) and (2) are given by:

$$\tilde{x}^+ = (A - LC)\tilde{x} + (B_w - LD_w)w,$$

where L satisfies $\rho(A - LC) < 1$ and $\rho(\cdot)$ denotes the spectral radius. The tube based MPC controller is implemented on the associated nominal system (Mayne et al. (2006)), which is obtained from (1) by neglecting the disturbance w:

$$\overline{x}^+ = A\overline{x} + B\overline{u},$$

where $\overline{x}$, $\overline{x}^+ \in \mathbb{R}^n$, $\overline{u} \in \mathbb{R}^{n_u}$ are the current state, successor state, and the control input of the nominal system, respectively. The control input is given by:

$$u = \overline{u} + K(\hat{x} - \overline{x}),$$

where $K \in \mathbb{R}^{n_u\times n}$ is the feedback gain, which satisfies $\rho(A + BK) < 1$. The error between the observer and nominal states, called the control error, is defined as $\xi := \hat{x} - \overline{x}$; its dynamics are given by:

$$\xi^+ = (A + BK)\xi + LC\tilde{x} + LD_w w. \tag{3}$$

We follow the standard definitions (Blanchini (1999);Kolmanovsky and Gilbert (1998)) for robust positively invariant set.

Definition 1. A set $\Omega \subset \mathbb{R}^n$ is robust positively invariant for the system $x^+ = f(x, w)$ and the constraint set $(\mathcal{X}, \mathcal{W})$ if $\Omega \subseteq \mathcal{X}$ and $x^+ = f(x, w) \in \Omega$, $\forall w \in \mathcal{W}, \forall x \in \Omega$.

Then the polytopic invariant sets for the estimation error $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and the control error $\mathcal{P}(P_\xi, b_\xi)$ can be defined by:

$$\left.\begin{matrix} \tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ w \in \mathcal{W} \end{matrix}\right\} \Rightarrow \tilde{x}^+ \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}), \tag{4}$$

$$\left.\begin{matrix} \xi \in \mathcal{P}(P_\xi, b_\xi) \\ w \in \mathcal{W} \\ \tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \end{matrix}\right\} \Rightarrow \xi^+ \in \mathcal{P}(P_\xi, b_\xi), \tag{5}$$

where

$$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) = \{\tilde{x} \in \mathbb{R}^n : -b_{\tilde{x}} \leq P_{\tilde{x}}\tilde{x} \leq b_{\tilde{x}}\}, \tag{6}$$

$$\mathcal{P}(P_\xi, b_\xi) = \{\xi \in \mathbb{R}^n : -b_\xi \leq P_\xi \xi \leq b_\xi\}, \tag{7}$$

and $P_{\tilde{x}}, P_\xi \in \mathbb{R}^{m\times n}$ and $b_{\tilde{x}}, b_\xi \in \mathbb{R}^m$ are decision variables for the structure of the invariant set. By definition, the actual state differs from the nominal state by the estimation error $\tilde{x}$ and control error ξ, so that:

$$x = \overline{x} + \xi + \tilde{x}.$$

Similarly, the difference between the actual control input and nominal input is given by $K\xi$:

$$u = \overline{u} + K\xi.$$

We assume that the initial values of estimation and control errors belong to their respective RCI sets, $\xi(0) \in \mathcal{P}(P_\xi, b_\xi)$ and $\tilde{x}(0) \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$. The original state and control input constraints are satisfied for all $w \in \mathcal{W}$ if

$$\overline{x} \in \overline{\mathcal{X}} := \mathcal{X} \ominus \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \ominus \mathcal{P}(P_\xi, b_\xi),$$
$$\overline{u} \in \overline{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{P}(P_\xi, b_\xi).$$

Therefore, we can choose the initial nominal state $\overline{x}(0)$ and the nominal control input $\overline{u}$ to ensure that the actual (unknown) state and control input always satisfy the original constraints. In this way, the original constraints $\mathcal{X}$ and $\mathcal{U}$ are tightened by $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$. We next establish the conditions such that the sets $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$ are circumscribed by outer bounding ellipsoids $\mathcal{Q}(Q_{\tilde{x}})$ and $\mathcal{Q}(Q_\xi)$, respectively,

$$\exists Q_{\tilde{x}} \in \mathcal{S}_+^n : \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \subseteq \mathcal{Q}(Q_{\tilde{x}}), \tag{8}$$
$$\exists Q_\xi \in \mathcal{S}_+^n : \mathcal{P}(P_\xi, b_\xi) \subseteq \mathcal{Q}(Q_\xi). \tag{9}$$

Since the volume of $\mathcal{Q}(Q_{\tilde{x}})$ is proportional to the determinant of the matrix $Q_{\tilde{x}}^{-\frac{1}{2}}$, the term $log\ detQ_{\tilde{x}}^{-1}$ is adopted as the objective function to minimize the volume of the set $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$; similarly, we use $log\ detQ_\xi^{-1}$ as the objective function for $\mathcal{P}(P_\xi, b_\xi)$. Combining the invariance and outer bounding conditions for the invariant sets of the estimation and control errors, respectively, we can present the following problems to optimize the volume of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$, respectively:

$$\begin{aligned} \min_{P_{\tilde{x}}, b_{\tilde{x}}, L, Q_{\tilde{x}}} \quad & log\ detQ_{\tilde{x}}^{-1} \\ s.t. \quad & (4), (8). \end{aligned} \tag{10}$$

$$\begin{aligned} \min_{P_\xi, b_\xi, K, Q_\xi} \quad & log\ detQ_\xi^{-1} \\ s.t. \quad & (5), (9). \end{aligned} \tag{11}$$

3. INITIAL COMPUTATION FOR THE INVARIANT SET OF THE ESTIMATION ERROR

In this section, we first derive necessary and sufficient conditions, in the form of nonlinear matrix inequalities (NLMIs), for the existence of an admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$ for problem (10) by using Farkas' Theorem (Pólik and Terlaky (2007)). Subsequently, the corresponding sufficient conditions in the form of LMIs are given by the use of the following result, which is deduced from the Elimination Lemma.

Lemma 2. Liu and Jaimoukha (2015): Let $R \in \mathcal{S}^n$, $E \in \mathbb{R}^{n\times p}$, $F \in \mathbb{R}^{p\times m}$, and $Z \in \mathcal{S}^m$. Consider the following two statements:

$$(i)\ \begin{bmatrix} R & EF \\ \star & Z \end{bmatrix} \succ 0, \tag{12}$$

$$(ii)\ \exists Y \in \mathcal{Y} : \begin{bmatrix} R & EY & 0 \\ \star & Y + Y^T & F \\ \star & \star & Z \end{bmatrix} \succ 0. \tag{13}$$

Then $(ii) \Rightarrow (i)$ if $\mathcal{Y} \subseteq \mathbb{R}^{p\times p}$ and $(ii) \Leftrightarrow (i)$ if $\mathcal{Y} = \mathbb{R}^{p\times p}$.

Theorem 3. The invariance and outer bounding conditions for the invariant set of the estimation error are satisfied if and only if, $\forall i \in \mathcal{N}_m$, there exist $D_i \in \mathcal{D}_+^m$, $W_i \in \mathcal{D}_+^{m_w}$, $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$ and $Q_{\tilde{x}} \in \mathcal{S}_+^n$ such that

$$L_{\tilde{x}} := \begin{bmatrix} \Delta_{11}^i & e_i^T P_{\tilde{x}} B_w^L & e_i^T P_{\tilde{x}} A^L \\ \star & V_w^T W_i V_w & 0 \\ \star & \star & P_{\tilde{x}}^T D_i P_{\tilde{x}} \end{bmatrix} \succ 0, \tag{14}$$

$$P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}} - Q_{\tilde{x}} \succ 0, \quad 1 - b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}} > 0, \tag{15}$$

where $\Delta_{11}^i = 2e_i^T b_{\tilde{x}} - b_{\tilde{x}}^T D_i b_{\tilde{x}} - \overline{w}^T W_i \overline{w}$, $A^L = A - LC$, and $B_w^L = B_w - LD_w$.

Proof. The proof of (14) is an application of Farkas' Theorem. Follow the definition of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ in (6), the invariance condition (4) is equivalent to

$$\left.\begin{matrix} -b_{\tilde{x}} \le P_{\tilde{x}}\tilde{x} \le b_{\tilde{x}} \\ -\overline{w} \le V_w w \le \overline{w} \end{matrix}\right\} \Rightarrow -b_{\tilde{x}} \le P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) \le b_{\tilde{x}}. \tag{16}$$

Considering the symmetry of the sets $\mathcal{W}$ and $\mathcal{P}$, the last inequality in (16) can be written as

$$2e_i^T(P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) - b_{\tilde{x}}) \le 0, \forall i \in \mathcal{N}_m.$$

For any $D_i \in \mathcal{D}_+^m$ and $W_i \in \mathcal{D}_+^{m_w}$, $\forall i \in \mathcal{N}_m$, it can be verified that

$$\begin{aligned} 2e_i^T(P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) - b_{\tilde{x}}) = & -(V_w w + \overline{w})^T W_i (\overline{w} - V_w w) \\ & -(b_{\tilde{x}} - P_{\tilde{x}}\tilde{x})^T D_i (P_{\tilde{x}}\tilde{x} + b_{\tilde{x}}) \\ & - g^T L_{\tilde{x}} g, \end{aligned} \tag{17}$$

where $L_{\tilde{x}}$ is defined in (14) and $g^T := \begin{bmatrix} -1 & w^T & \tilde{x}^T \end{bmatrix}$. Since the first and second terms on the RHS of (17) are nonpositive for all $\tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $w \in \mathcal{W}$, the invariance condition is satisfied if $L_{\tilde{x}} \succ 0$, which gives (14). If the LHS of (17) is nonpositive $\forall i \in \mathcal{N}_m$, then it follows from Farkas' Theorem that $L_{\tilde{x}} \succ 0$ is satisfied, which proves necessity.

Similarly, the outer bounding condition (8) is equivalent to

$$-b_{\tilde{x}} \le P_{\tilde{x}}\tilde{x} \le b_{\tilde{x}} \Rightarrow \tilde{x}Q_{\tilde{x}}\tilde{x} \le 1. \tag{18}$$

For any $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$ and $Q_{\tilde{x}} \in \mathcal{S}_+^n$, we have

$$\begin{aligned} \tilde{x}Q_{\tilde{x}}\tilde{x} - 1 = & -(b_{\tilde{x}} - P_{\tilde{x}}\tilde{x})^T \overline{D}_i (P_{\tilde{x}}\tilde{x} + b_{\tilde{x}}) \\ & - \begin{bmatrix} -1 & \tilde{x}^T \end{bmatrix} \underbrace{\begin{bmatrix} 1 - b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}} & 0 \\ 0 & P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}} - Q_{\tilde{x}} \end{bmatrix}}_{\overline{L}_{\tilde{x}}} \begin{bmatrix} -1 \\ \tilde{x} \end{bmatrix}. \end{aligned}$$

It is clear that since the first term on the RHS of the above equality is nonpositive for all $\tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$, the outer bounding condition is satisfied if $\overline{L}_{\tilde{x}} \succ 0$, which gives (15). If $\tilde{x}Q_{\tilde{x}}\tilde{x} - 1 \le 0$, then it follows from Farkas' Theorem that $\overline{L}_{\tilde{x}} \succ 0$, which proves necessity. □

As can be seen from (14) and (15), the nonlinearity terms include $P_{\tilde{x}}B_w^L$, $P_{\tilde{x}}A^L$, $P_{\tilde{x}}^T D_i P_{\tilde{x}}$, $b_{\tilde{x}}^T D_i b_{\tilde{x}}$, $P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}}$, $b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}}$. In order to deal with these nonlinearities, we next propose an initial full-complexity outer approximation to the minimal RCI set, such that

$$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) = \mathcal{P}(P_r X_{\tilde{x}}, b_r) = \{x \in \mathbb{R}^n : -b_r \le P_r X_{\tilde{x}} x \le b_r\},$$

where P_r and b_r are given, and $X_{\tilde{x}} \in \mathbb{R}^{n\times n}$ is a variable to rotate and scale the polyhedral set defined by P_r (see the work of Liu and Jaimoukha (2015) for details).

The next result uses Lemma 2 and a congruence transformation to derive sufficient conditions, in the form of LMIs, for computing an admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$.

Theorem 4. With all variables as defined in Theorem 3, let $P_{\tilde{x}} = P_r X_{\tilde{x}}$ and $b_{\tilde{x}} = b_r$ and define $\hat{L} = X_{\tilde{x}}L$. The NLMIs of (14) and (15) are satisfied if, $\forall i \in \mathcal{N}_m$, there exist $\hat{D}_i \in \mathcal{D}_+^m$, $\hat{W}_i \in \mathcal{D}_+^{m_w}$, $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$, $Q_{\tilde{x}} \in \mathcal{S}_+^n$ and $\lambda_i > 0$, such that

$$\begin{bmatrix} \Gamma_{11}^i & e_i^T P_r \hat{B} & e_i^T P_r \hat{A} & 0 & 0 \\ \star & 2I_{n_w} & 0 & \lambda_i I_{n_w} & 0 \\ \star & \star & X_{\tilde{x}} + X_{\tilde{x}}^T & 0 & \lambda_i I_n \\ \star & \star & \star & V_w^T \hat{W}_i V_w & 0 \\ \star & \star & \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0, \quad (19)$$

$$\begin{bmatrix} X_{\tilde{x}} + X_{\tilde{x}}^T - Q_{\tilde{x}} & I_n \\ \star & P_r^T \overline{D}_{\tilde{x}} P_r \end{bmatrix} \succ 0, \ 1 - b_r^T \overline{D}_{\tilde{x}} b_r > 0, \quad (20)$$

where $\Gamma_{11}^i = 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r - \overline{w}^T \hat{W}_i \overline{w}$, $\hat{A} = X_{\tilde{x}} A - \hat{L} C$, and $\hat{B} = X_{\tilde{x}} B_w - \hat{L} D_w$.

Proof. Substituting $P_{\tilde{x}} = P_r X_{\tilde{x}}$ and $b_{\tilde{x}} = b_r$ shows that (14) can be rewritten as (12) with

$$\left[\begin{array}{c|c} R & E \\ \hline F & Z \end{array}\right] = \left[\begin{array}{c|c} 2e_i^T b_r - b_r^T D_i b_r - \overline{w}^T W_i \overline{w} & e_i^T P_r X_{\tilde{x}} \begin{bmatrix} B_w^L & A^L X_{\tilde{x}}^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} I_{n_w} & 0 \\ 0 & X_{\tilde{x}} \end{bmatrix} & \begin{bmatrix} V_w^T W_i V_w & 0 \\ 0 & X_{\tilde{x}}^T P_r^T D_i P_r X_{\tilde{x}} \end{bmatrix} \end{array}\right].$$

Applying Lemma 2 with $Y = \lambda_i^{-1} \begin{bmatrix} I_{n_w} & 0 \\ 0 & X_{\tilde{x}} \end{bmatrix}$, then effecting the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X_{\tilde{x}}^{-T})$ implies that (19) is a sufficient condition of (14), with the following redefinitions:

$$\hat{D}_i = \lambda_i D_i, \quad \hat{W}_i = \lambda_i W_i.$$

For the first inequality of (15), substituting $P_{\tilde{x}}$ with $P_r X_{\tilde{x}}$, followed by applying the congruence $X_{\tilde{x}}^{-T}$ and applying a Schur complement argument gives the following equivalent inequality

$$\begin{bmatrix} X_{\tilde{x}} Q_{\tilde{x}}^{-1} X_{\tilde{x}}^T & I_n \\ \star & P_r^T \overline{D}_{\tilde{x}} P_r \end{bmatrix} \succ 0. \quad (21)$$

Using the identity

$$X_{\tilde{x}} Q_{\tilde{x}}^{-1} X_{\tilde{x}}^T = X_{\tilde{x}} + X_{\tilde{x}}^T - Q_{\tilde{x}} + (X_{\tilde{x}} - Q_{\tilde{x}})^T Q_{\tilde{x}}^{-1} (X_{\tilde{x}} - Q_{\tilde{x}}),$$

the (1,1) block of (21) can be replaced with the first three terms on the right of the above identity since its last term is nonnegative. This gives the first inequality of (20). For the second inequality in (15), replacing $b_{\tilde{x}}$ by b_r gives (20) directly. □

Remark 5. While the feasibility of the LMI problem is not guaranteed by using arbitrary P_r and b_r, in practice, we found that using the vector of ones for b_r and the regular polytope with $2m$ faces for P_r can usually result in a feasible solution, although this may introduce some conservatism to Theorem 4. Note also that the degree of freedom in the choice of m provides flexibility in the shape of the RCI set, which provides additional accuracy of expressing the set. In general, guaranteeing the existence of an initial feasible RCI set is difficult (Blanchini (1999)). However, Theorem 4 in Liu et al. (2019) provides a choice of the initial RCI set that is guaranteed to be feasible under certain conditions.

In conclusion, the initial computation for the invariant set of the estimation error can be posed as the convex semidefinite program

$$\begin{aligned} \min_{X_{\tilde{x}}, \hat{L}, \hat{D}_i, \hat{W}_i, \overline{D}_{\tilde{x}}, Q_{\tilde{x}}, \lambda_i} \quad & \log \det Q_{\tilde{x}}^{-1} \\ s.t. \quad & (19), (20). \end{aligned} \quad (22)$$

4. INITIAL COMPUTATION FOR INVARIANT SET OF THE CONTROL ERROR

In the last section, an initial admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$ for the invariant set of the estimation error has been obtained. For the given invariant set $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$, we propose to re-parameterize the estimation error $\tilde{x}$ as an artificial disturbance by augmenting the dynamics of control error ξ in (3), such that

$$\xi^+ = \underbrace{(A + BK)}_{A^K} \xi + \underbrace{[LD_w \ \ LC]}_{B_\eta} \underbrace{\begin{bmatrix} w \\ \tilde{x} \end{bmatrix}}_{\eta}. \quad (23)$$

The new disturbance η belongs to be an extended polytope,

$$\eta \in \mathcal{W}^\eta := \left\{ \eta \in \mathbb{R}^{n_w + n} \mid -\overline{\eta} \leq V_\eta \eta \leq \overline{\eta} \right\},$$

with the following redefinitions:

$$V_\eta = \begin{bmatrix} V_w & 0 \\ 0 & P_{\tilde{x}} \end{bmatrix}, \quad \overline{\eta} = \begin{bmatrix} \overline{w} \\ b_{\tilde{x}} \end{bmatrix}.$$

We next propose the corresponding conditions for the initial computation of the admissible triple (P_ξ, b_ξ, K) by using Farkas' Theorem.

Theorem 6. The invariance and outer bounding conditions for the invariant set of the control error are satisfied if and only if, $\forall i \in \mathcal{N}_m$, there exist $D_\xi^i \in \mathcal{D}_+^m$, $W_\eta^i \in \mathcal{D}_+^{m_w+m}$, $\overline{D}_\xi \in \mathcal{D}_+^m$ and $Q_\xi \in \mathcal{S}_+^n$, such that

$$\begin{bmatrix} 2e_i^T b_\xi - b_\xi^T D_\xi^i b_\xi - \overline{\eta}^T W_\eta^i \overline{\eta} & e_i^T P_\xi B_\eta & e_i^T P_\xi A^K \\ \star & V_\eta^T W_\eta^i V_\eta & 0 \\ \star & \star & P_\xi^T D_\xi^i P_\xi \end{bmatrix} \succ 0, \quad (24)$$

$$P_\xi^T \overline{D}_\xi P_\xi - Q_\xi \succ 0, \quad 1 - b_\xi^T \overline{D}_\xi b_\xi > 0. \quad (25)$$

Proof. The proof is also an application of Farkas' Theorem that is similar to the proof in Theorem 3, thus it is omitted here for brevity. □

We also use an initial outer approximation to the minimal RCI set to convert the NLMIs of (24) and (25) into LMIs using Lemma 2 and a congruence transformation.

Theorem 7. With all variables as in Theorem 6 let $P_\xi = P_r X_\xi$ and $b_\xi = b_r$ and define $\tilde{X} = X_\xi^{-1}$ and $\hat{K} = K X_\xi^{-1}$. The NLMIs of (24) and (25) are satisfied if, $\forall i \in \mathcal{N}_m$, there exist $\hat{D}_\xi^i \in \mathcal{D}_+^m$, $\hat{W}_\eta^i \in \mathcal{D}_+^{m_w+m}$, $\overline{D}_\xi \in \mathcal{D}_+^m$, $Q_\xi^{-1} \in \mathcal{S}_+^n$ and $\gamma_i > 0$, such that

$$\begin{bmatrix} \Lambda_{11}^i & \gamma_i e_i^T P_r & 0 & 0 \\ \star & \tilde{X} + \tilde{X}^T & B_\eta & A\tilde{X} + B\hat{K} \\ \star & \star & V_\eta^T \hat{W}_\eta^i V_\eta & 0 \\ \star & \star & \star & P_r^T \hat{D}_\xi^i P_r \end{bmatrix} \succ 0, \quad (26)$$

$$\begin{bmatrix} Q_\xi^{-1} & \tilde{X} \\ \star & P_r^T \overline{D}_\xi P_r \end{bmatrix} \succ 0, \quad 1 - b_r^T \overline{D}_\xi b_r > 0, \quad (27)$$

where $\Lambda_{11}^i = 2\gamma_i e_i^T b_r - b_r^T \hat{D}_\xi^i b_r - \overline{\eta}^T \hat{W}_\eta^i \overline{\eta}$.

Proof. Substituting $P_\xi = P_r X_\xi$ and $b_\xi = b_r$ shows that (24) can be rewritten as (12) with

$$\left[\begin{array}{c|c} R & E \\ \hline F & Z \end{array}\right] = \left[\begin{array}{c|c} 2e_i^T b_r - b_r^T D_\xi^i b_r - \overline{\eta}^T W_\eta^i \overline{\eta} & e_i^T P_r X_\xi \\ \hline \begin{bmatrix} B_\eta & A^K \end{bmatrix} & \begin{bmatrix} V_\eta^T W_\eta^i V_\eta & 0 \\ 0 & X_\xi^T P_r^T D_\xi^i P_r X_\xi \end{bmatrix} \end{array}\right].$$

Applying Lemma 2 with $Y = \gamma_i X_\xi^{-1}$ where $0 < \gamma_i \in \mathbb{R}$, then effecting the congruence $diag(\gamma_i^{\frac{1}{2}}, \gamma_i^{-\frac{1}{2}} I_n, \gamma_i^{\frac{1}{2}} I_{n_w}, \gamma_i^{\frac{1}{2}} X_\xi^{-T})$ implies that (26) is a sufficient condition for (24) with the redefinitions:

$$\hat{D}_\xi^i = \gamma_i D_\xi^i, \;\; \hat{W}_\eta^i = \gamma_i W_\eta^i.$$

For the first inequality in (25), substituting P_ξ with $P_r X_\xi$ followed by applying the congruence X_ξ^{-T} and applying a Schur complement argument gives the first inequality in (27). For the second inequality in (25), replacing b_ξ by b_r gives the second term in (27) directly. □

To summarize, the initial computation for the invariant set of the control error can be posed as the convex semidefinite program

$$\begin{aligned} \min_{\tilde{X}, \hat{K}, \hat{D}_\xi^i, \hat{W}_\eta^i, \overline{D}_\xi, Q_\xi^{-1}, \gamma_i} \quad & trace\,(Q_\xi^{-1}) \\ s.t. \quad & (26), (27). \end{aligned} \tag{28}$$

Remark 8. Since the function $log\ det(Q_\xi^{-1})$ is concave, we minimize an upper bound on $log\ det(Q_\xi^{-1})$ by replacing it with trace(Q_ξ^{-1}).

Remark 9. Theorems 4 and 7 give sufficient condition only; the conservatism comes from restricting the structure of Y in Lemma 2 to obtain a tractable solution. Necessary and sufficient conditions could be obtained if the structure of Y is free, however, this will result in an intractable solution.

5. UPDATE COMPUTATION ALGORITHM

In the previous two sections, we proposed the initial computations of the invariant sets of the estimation and control errors by considering L and K as variables separately. Since the linearization algorithm resulting from using Lemma 2 gives sufficient condition only, this conservatism leads to the RCI sets being unlikely to be minimal. Therefore, in this section, we propose an update computation algorithm based on the following Newton-like update to obtain approximate minimal RCI sets.

Lemma 10. Liu et al. (2019): Let L, $L_0 \in \mathbb{R}^{m\times n}$ and D, $D_0 \in \mathcal{S}_+^m$. Denote

$$\mathcal{L}_{L,D}^{L_0,D_0} := L^T D_0^{-1} L_0 + L_0^T D_0^{-1} L - L_0^T D_0^{-1} D D_0^{-1} L_0$$
$$\mathcal{N}_{L,D} := L^T D^{-1} L$$

Then $\mathcal{N}_{L,D} \succeq \mathcal{L}_{L,D}^{L_0,D_0}$ and $\mathcal{N}_{L_0,D_0} = \mathcal{L}_{L_0,D_0}^{L_0,D_0}$. Therefore,

$$\left\{\exists L_0 \in \mathbb{R}^{m\times n}, D_0 \in \mathcal{S}_+^m : \mathcal{N}_{L_0,D_0} \succ 0\right\} \Rightarrow$$
$$\left\{\exists L \in \mathbb{R}^{m\times n}, D \in \mathcal{S}_+^m : \mathcal{N}_{L,D} \succeq \mathcal{L}_{L,D}^{L_0,D_0} \succ 0\right\}.$$

Theorem 11. Let the initial solutions of the invariant sets of the estimation and control errors be denoted as $(P_{\tilde{x}}^0, b_{\tilde{x}}^0, L_0, D_{\tilde{x}}^{i0}, W_{\tilde{x}}^{i0}, Q_{\tilde{x}}^0, \overline{D}_{\tilde{x}0})$ and $(P_\xi^0, b_\xi^0, K_0, D_\xi^{i0}, D_{\tilde{x}\xi}^{i0}, W_\xi^{i0}, Q_\xi^0, \overline{D}_{\xi 0})$, which satisfy conditions (14), (15), (25) and (34). Then these solutions can be updated if there exist $P_{\tilde{x}} \in \mathbb{R}^{m\times n}$, $b_{\tilde{x}} \in \mathbb{R}^m$, $L \in \mathbb{R}^{n\times n_y}$, $(D_{\tilde{x}}^i)^{-1} \in \mathcal{D}_+^m$, $W_{\tilde{x}}^i \in \mathcal{D}_+^{m_w}$, $Q_{\tilde{x}} \in \mathcal{S}_+^n$, $(\overline{D}_{\tilde{x}})^{-1} \in \mathcal{D}_+^m$, $P_\xi \in \mathbb{R}^{m\times n}$, $b_\xi \in \mathbb{R}^m$, $K \in \mathbb{R}^{n_u\times n}$, $(D_{\tilde{x}\xi}^i)^{-1} \in \mathcal{D}_+^m$, $(D_\xi^i)^{-1} \in \mathcal{D}_+^m$, $W_\xi^i \in \mathcal{D}_+^{m_w}$, $Q_\xi \in \mathcal{S}_+^n$ and $(\overline{D}_\xi)^{-1} \in \mathcal{D}_+^m$, $\forall i \in \mathcal{N}_m$ such that

$$\begin{bmatrix} \mathcal{M}_{\tilde{x}} + \mathcal{L}_{L_{\tilde{x}}^i, F_{\tilde{x}}^i}^{L_{\tilde{x}}^{i0}, F_{\tilde{x}}^{i0}} & \star \\ E_{\tilde{x}} L_{\tilde{x}}^i & I_n \end{bmatrix} \succ 0, \tag{29}$$

$$\mathcal{L}_{P_{\tilde{x}}, \overline{D}_{\tilde{x}}^{-1}}^{P_{\tilde{x}}^0, \overline{D}_{\tilde{x}0}^{-1}} - Q_{\tilde{x}} \succ 0, \qquad \begin{bmatrix} \overline{D}_{\tilde{x}}^{-1} & b_{\tilde{x}} \\ \star & 1 \end{bmatrix} \succ 0, \tag{30}$$

$$\begin{bmatrix} \mathcal{M}_\xi + \mathcal{L}_{L_\xi^i, F_\xi^i}^{L_\xi^{i0}, F_\xi^{i0}} & \star \\ E_\xi L_\xi^i & I_n \end{bmatrix} \succ 0, \tag{31}$$

$$\mathcal{L}_{P_\xi, \overline{D}_\xi^{-1}}^{P_\xi^0, \overline{D}_{\xi 0}^{-1}} - Q_\xi \succ 0, \qquad \begin{bmatrix} \overline{D}_\xi^{-1} & b_\xi \\ \star & 1 \end{bmatrix} \succ 0, \tag{32}$$

where,

$$E_{\tilde{x}} = [-I_n \;\; I_n \;\; 0], \quad F_\xi^i = diag(I_n, I_n, (D_{\tilde{x}\xi}^i)^{-1}, (D_\xi^i)^{-1}),$$
$$E_\xi = [-I_n \;\; I_n \;\; 0 \;\; 0], F_{\tilde{x}}^i = diag(I_n, I_n, (D_{\tilde{x}}^i)^{-1}),$$

$$\mathcal{M}_{\tilde{x}} = \begin{bmatrix} (D_{\tilde{x}}^i)^{-1} & b_{\tilde{x}} & 0 & 0 \\ \star & 2e_i^T b_{\tilde{x}} - \overline{w}^T W_{\tilde{x}}^i \overline{w} & 0 & 0 \\ \star & \star & V_w^T W_{\tilde{x}}^i V_w & 0 \\ \star & \star & \star & 0 \end{bmatrix},$$

$$\mathcal{M}_\xi = \begin{bmatrix} (D_{\tilde{x}\xi}^i)^{-1} & 0 & b_{\tilde{x}} & 0 & 0 & 0 \\ \star & (D_\xi^i)^{-1} & b_\xi & 0 & 0 & 0 \\ \star & \star & 2e_i^T b_\xi - \overline{w}^T W_\xi^i \overline{w} & 0 & 0 & 0 \\ \star & \star & \star & V_w^T W_\xi^i V_w & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

$$L_{\tilde{x}}^i = \begin{bmatrix} 0 & P_{\tilde{x}}^T e_i & 0 & 0 \\ 0 & 0 & B_w^L & A^L \\ 0 & 0 & 0 & P_{\tilde{x}} \end{bmatrix}, L_\xi^i = \begin{bmatrix} 0 & 0 & P_\xi^T e_i & 0 & 0 & 0 \\ 0 & 0 & 0 & LD_w & LC & A^K \\ 0 & 0 & 0 & 0 & P_{\tilde{x}} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_\xi \end{bmatrix}.$$

Proof. Applying an upper Schur complement on $b_{\tilde{x}}^T D_{\tilde{x}}^i b_{\tilde{x}}$ in (14), the following identity can be verified

$$(14) \Leftrightarrow \mathcal{M}_{\tilde{x}} + \mathcal{N}_{L_{\tilde{x}}^i, F_{\tilde{x}}^i} - (E_{\tilde{x}} L_{\tilde{x}}^i)^T (E_{\tilde{x}} L_{\tilde{x}}^i) \succ 0. \tag{33}$$

A subsequent application of Lemma 10 on $\mathcal{N}_{L_{\tilde{x}}^i, F_{\tilde{x}}^i}$ in (33), followed by a Schur complement on the third term gives (29). For the first inequality in (15), it can be noted that $\mathcal{N}_{P_{\tilde{x}}, \overline{D}_{\tilde{x}}^{-1}} = P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}}$. Then using Lemma 10 on this equality gives the first inequality in (30). The second inequality in (15) and (30) are equivalent by effecting Schur complement directly.

The invariant set of the estimation error is unknown if we want to update these two sets simultaneously. Hence, the invariance condition (24) for the invariant set of the control error in Theorem 6 needs to be modified by Farkas' Theorem. The invariance condition in (5) is equivalent to

$$\left.\begin{matrix} -b_{\tilde{x}} \le P_{\tilde{x}} \tilde{x} \le b_{\tilde{x}} \\ -b_\xi \le P_\xi \xi \le b_\xi \\ -\overline{w} \le V_w w \le \overline{w} \end{matrix}\right\} \Rightarrow -b_\xi \le P_\xi (A^K \xi + LC\tilde{x} + LD_w w) \le b_\xi.$$

For any $D_{\tilde{x}\xi}^i \in \mathcal{D}_+^m$, $D_\xi^i \in \mathcal{D}_+^m$, $W_\xi^i \in \mathcal{D}_+^{m_w}$, $\forall i \in \mathcal{N}_m$,

$$\begin{aligned} & 2e_i^T (P_\xi (A^K \xi + LC\tilde{x} + LD_w w) - b_\xi) \\ = & -(V_w w + \overline{w})^T W_\xi^i (\overline{w} - V_w w) - (b_{\tilde{x}} - P_{\tilde{x}} \tilde{x})^T D_{\tilde{x}\xi}^i (P_{\tilde{x}} \tilde{x} + b_{\tilde{x}}) \\ & - (b_\xi - P_\xi \xi)^T D_\xi^i (P_\xi \xi + b_\xi) \\ & - \left[-1 \;\; w^T \;\; \tilde{x}^T \;\; \xi^T\right] L_\xi \left[-1 \;\; w^T \;\; \tilde{x}^T \;\; \xi^T\right]^T \le 0 \end{aligned}$$

if and only if

$$L_\xi := \begin{bmatrix} \Phi_{11}^i & e_i^T P_\xi L D_w & e_i^T P_\xi L C & e_i^T P_\xi A^K \\ \star & V_w^T W_\xi^i V_w & 0 & 0 \\ \star & \star & P_{\tilde{x}}^T D_{\tilde{x}\xi}^i P_{\tilde{x}} & 0 \\ \star & \star & \star & P_\xi^T D_\xi^i P_\xi \end{bmatrix} \succ 0, \quad (34)$$

where $\Phi_{11}^i = 2e_i^T b_\xi - b_\xi^T D_\xi^i b_\xi - b_{\tilde{x}}^T D_{\tilde{x}\xi}^i b_{\tilde{x}} - \overline{w}^T W_\xi^i \overline{w}$. Note that (34) is equivalent to (24) with the definition $W_\eta^i = diag(W_\xi^i, D_{\tilde{x}\xi}^i)$. Subsequently, applying a Schur complement on $b_{\tilde{x}}^T D_{\tilde{x}\xi}^i b_{\tilde{x}}$ and $b_\xi^T D_\xi^i b_\xi$ of (34) successively shows that it is equivalent to the following inequality

$$\mathcal{M}_\xi + \mathcal{N}_{L_\xi^i, F_\xi^i} - (E_\xi L_\xi^i)^T (E_\xi L_\xi^i) \succ 0. \quad (35)$$

Using similar procedures to the previous proof for (29)/(30) on (35)/(25), giving (31) and (32), respectively. □

To summarize, the problem of updating the RCI sets of the estimation and control errors simultaneously can be posed as the convex semidefinite program

$$\min_{P_{\tilde{x}}, b_{\tilde{x}}, L, (D_{\tilde{x}}^i)^{-1}, W_{\tilde{x}}^i, Q_{\tilde{x}}, \overline{D}_{\tilde{x}}^{-1}, P_\xi, b_\xi, K, (D_{\tilde{x}\xi}^i)^{-1}, (D_\xi^i)^{-1}, W_\xi^i, Q_\xi, \overline{D}_\xi^{-1}} \quad log\ detQ_{\tilde{x}}^{-1}$$
$$s.t. \quad (29), (30), (31), (32), Q_{\tilde{x}} = Q_\xi. \quad (36)$$

Remark 12. Since the identity $\mathcal{N}_{L_0,D_0} = \mathcal{L}_{L_0,D_0}^{L_0,D_0}$ in Lemma 10 ensures that the constraints (29)-(32) are also feasible by setting the corresponding optimized variables equal to their initial value, then problem (36) results in a no more conservative solution than the initial one, namely the volume of the RCI set defined by $Q_{\tilde{x}}$ would be smaller or at least equal to the initial set defined by $Q_{\tilde{x}}^0$.

Remark 13. Note that the constraints in the optimization problem (36) includes the equality constraint $Q_{\tilde{x}} = Q_\xi$. This means that only one ellipse is used to circumscribe the two polytopes simultaneously. This leads to some conservatism in the updating algorithm, the best approach is to consider two ellipses circumscribing two polytopes separately, and then to optimize the total volume of two ellipses; however, this will be a direction for future work.

Finally, the complete computation algorithm for the RCI sets of the estimation and control errors based on successive iterations is summarized as follows.

Algorithm 1

1) Initial data: *Given system (1) and disturbance set $\mathcal{W}$, choose an initial polytope $\mathcal{P}(P_r, b_r)$ and tolerance level tol.*

2) Initial solution: *Compute the initial RCI sets of the estimation and control errors by the optimizations in* (22) *and* (28) *separately.*

3) Update: *Update the two sets simultaneously by the optimization in* (36).

4) Stopping condition: *Stop if the absolute value of the difference between the current and previous values of $log\ detQ_{\tilde{x}}^{-1}$ is less than tol.*

6. NUMERICAL EXAMPLE

6.1 Example 1

We consider a scalar system:

$$x^+ = 1.1x + u + d,$$
$$y = x + v,$$

with additive disturbance $d \in \mathcal{W}^d := \{d \in \mathbb{R} \mid |d|_\infty \le 0.5\}$ and $v \in \mathcal{V} := \{v \in \mathbb{R} \mid |v|_\infty \le 1\}$. The invariant sets of the estimation and control errors obtained with the proposed Algorithm 1 and Liu (2017) are shown in Table 1 below.

Table 1. The comparison of calculated invariant set boundaries for Example 1

Methods	$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$	$\mathcal{P}(P_\xi, b_\xi)$	$\mathcal{X} - \overline{\mathcal{X}}$	$\mathcal{U} - \overline{\mathcal{U}}$
Liu (2017)	1.6000	2.8600	4.4600	3.1460
Algorithm 1	2.0490	2.0490	4.0980	2.2539

Note that $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$ denote the invariant sets of the estimation and control errors, respectively. $\mathcal{X} - \overline{\mathcal{X}} = \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \oplus \mathcal{P}(P_\xi, b_\xi)$ and $\mathcal{U} - \overline{\mathcal{U}} = K\mathcal{P}(P_\xi, b_\xi)$ represent the tightened invariant tube on state and tightened constraint on input, respectively. As shown in Table 1, Liu (2017) obtains a smaller invariant set of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ with the computed $K = -1.1$ and $L = 1.1$ while the proposed Algorithm 1 achieves less conservative results for total volumes of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \oplus \mathcal{P}(P_\xi, b_\xi)$ with $K = -1.1$ and $L = 0.6720$, this leads to less tightened constraints on the nominal system state by using our Algorithm 1. Note also that the tightened constraint on input obtained by Liu (2017) is $\mathcal{U} - \overline{\mathcal{U}} = [-3.1460,\ 3.1460]$ while we achieve a smaller interval of $[-2.2539,\ 2.2539]$.

The above results confirm our expectation, because Liu (2017) optimizes the two sets separately and can make sure that the invariant set of the estimation error is minimal only, but it might lead to a larger disturbance set for the control error $\mathcal{P}(P_\xi, b_\xi)$. Our algorithm uses a common set to optimize these two sets simultaneously, it is possible to achieve better trade-off between $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$ and therefore a smaller total volume.

6.2 Example 2

A double integrator system from Goulart and Kerrigan (2007) is considered in this example

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d,$$
$$y = [1\ \ 1]\, x + v,$$

with additive disturbances $d \in \mathcal{W}^d := \{d \in \mathbb{R}^2 \| d|_\infty \le 0.1\}$ and $v \in \mathcal{V} := \{v \in \mathbb{R} \mid |v|_\infty \le 0.1\}$. State and control input constraints are given by $\mathcal{X} := \{x \in \mathbb{R}^2 \mid -25 \le x_i \le 3\}$ and $\mathcal{U} := \{u \in \mathbb{R} \mid |u| \le 5\}$, respectively, where x_i denotes the ith element of x.

We set $m = 3$ and produce the same (randomly generated) initial polytope $\mathcal{P}(P_r, b_r)$ for $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$, where

$$P_r = \begin{bmatrix} -0.5817 & 0.9493 \\ -1.8301 & 0.7174 \\ -0.4491 & 2.2878 \end{bmatrix}, b_r = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Figure 1 shows the invariant tube $\mathcal{X} - \overline{\mathcal{X}}$ obtained by Algorithm 1 (yellow) and Liu (2017) (pink). We observe that our invariant tube is smaller, which could provide a larger admissible domain on the nominal system state. The state feedback and observer gains computed by the method in Liu (2017) are $L = [1\ \ 1]^T$ and $K = [-1\ -1.8]$, the constraint on nominal input is $\overline{\mathcal{U}} = [-1.5230,\ 1.5230]$. In contrast, the corresponding results obtained by our algorithm are $L = [1\ \ 0.3279]^T$ and $K = [-1\ -1.8]$,

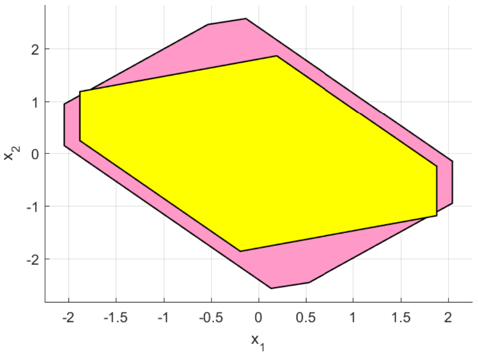

Fig. 1. Tube calculated using our Algorithm 1 (yellow) and Liu (2017) (pink)

and $\overline{\mathcal{U}} = [-2.6149,\ 2.6149]$. Note that our obtained $\overline{\mathcal{U}}$ is significantly larger compared to the method in Liu (2017).

The relation between the objective value and the number of iterations for the update of Algorithm 1 is shown as the following Figure 2. We note that the objective value are non-increasing with the number of iterations and it converge to its final value with an observed quadratic speed of convergence.

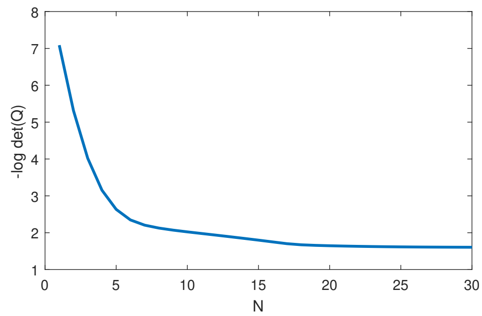

Fig. 2. The objective value (-log detQ) vs. iterations

7. CONCLUSION

We have presented a numerically efficient algorithm based on LMIs to compute invariant tubes for robust output MPC of DLTI systems with additive state and output disturbances. Instead of using pre–defined observer and control gains methods, or optimizing the invariant sets of the estimation and control errors separately as in Liu (2017), our algorithm optimizes the volumes of these two sets simultaneously. This enables us to take account of the effect of estimation error on the dynamics of the control error rather than treat them as decoupled problems. Two numerical examples are provided to illustrate that our algorithm can obtain less conservative tightened system state and input constraints for the nominal system.

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