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Design of Saturated Boundary Control for Hyperbolic Systems

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Abstract: This paper deals with the stabilization of 1-D linear hyperbolic systems with saturated feedback boundary control. By following a Lyapunov approach, sufficient conditions for global exponential stability in the $\mathcal{L}^2$ norm are given in the form of matrix inequalities. Numerical examples are presented to illustrate the theoretical results.

Keywords: Nonlinear systems, infinite dimensional systems, Lyapunov methods, saturation functions, hyperbolic systems

1. INTRODUCTION

Partial differential equations are mathematical expressions which are found to be of great importance in the modeling of many physical systems that are described simultaneously via spatial and temporal variables. Light propagation in optic fibers, blood flow in the vessels, plasma in laser, liquid metals in cooling systems, road traffic, acoustic waves, and electromagnetic waves are all examples of systems modeled via PDEs that can be seen in civil, nuclear, mechanical, quantum, and chemical engineering (see Bastin, Georges and J.-M Coron (2016) and Krstic and Smyshlyaev (2008) for more examples). That is why the study of control of PDEs, while challenging, is inevitable. The presence of actuator saturation in the system immediately threatens the controller by the risks of poor performance and instability (Hu and Lin, 2001). That is why control engineers almost always take into account this saturation in the system modeling and controller design to protect the actuators and maintain stability. Researchers have been studying several methods to tackle saturation problems in closed-loop systems as we can see in (Tarbouriech et al., 2011), (Zaccarian and Teel, 2011) and (Gomes da Silva Jr and Tarbouriech, 2005). However, stability analysis of PDEs in the presence of saturation is a relatively new topic and is still an open research area.
The aim of this paper is to focus control design of hyperbolic systems in the presence of saturation using boundary control (see more on global stabilization with bounded controls in Teel (1992) and Sontag (1984)). Several approaches exist for stability analysis of infinite dimensional systems such as the Lyapunov approach or the frequency domain approach (Jayawardhana et al., 2011). We use the Lyapunov stability approach (Jacob et al., 2019) by defining a suitable Lyapunov function to determine the regions of stability and constraints that the system exhibits. This is done using the help of LMIs to see whether eigenvalues lie in the right-hand side as in (Ferrante and Prieur, 2018). In addition, we will use a sector condition given by (Tarbouriech et al., 2011) to encapsulate the saturation into a conic sector.

This paper is organized as follows. In Section 2, the problem statement is defined by presenting the class of one-dimensional hyperbolic models on an appropriate domain and the class of nonlinear controllers. In Section 3, the wellposedness of the system and the conditions of global exponential stability are presented along with their proofs. In Section 4, our theoretical results are verified using a different numerical examples of a given hyperbolic system. Section 5 contains the conclusion and future perspective.

Notation:

The set $\mathbb{R}_{>0}$ represents the set of positive real scalars, $\mathcal{D}_p^n$ denotes the set of real diagonal positive definite matrices of dimension n, and X_t and X_z represents the partial derivatives of the function X with respect to time t and space z, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A. In partitioned symmetric matrices, the symbol * represents symmetric blocks. The letter I denotes the identity matrix. Let $U \subset \mathbb{R}$, $V \subset \mathbb{R}^n$, and $f : U \to V$, we denote by $||f||_{L^2} = (\int_U |f(x)|^2 dx)^{\frac{1}{2}}$ the L^2 norm of f. In particular, we say that $f \in L_2(U, V)$ if $||f||_{L_2}$ is finite.

2. PROBLEM STATEMENT

We consider the following linear hyperbolic system with a boundary input:

$$\begin{array}{ll} X_t(t,z) + \Lambda X_z(t,z) = 0 & \forall (t,z) \in \mathbb{R}_{>0} \times (0,1) \\ X(t,0) = HX(t,1) + Bu(t) & \forall t \in \mathbb{R}_{\geq 0} \\ X(0,z) = X_0(z) & \forall z \in [0,1] \end{array} \quad (1)$$

where $z \mapsto X(.,z) \in \mathbb{R}^n$ is the state, $t \in \mathbb{R}_{\geq 0}$ and $z \in [0,1]$ are the two independent variables, respectively, "time" and "space", and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{D}_p^n$.
The system is controlled at the boundary $z = 0$ via the input $u \in \mathbb{R}^m$. Matrices $H \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given. Let $u = \sigma(KX(\cdot,1))$ where $K \in \mathbb{R}^{m\times n}$ is the control gain to be designed and the function $u \mapsto \sigma(u)$ is the symmetric decentralized saturation function with saturation levels $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_m \in \mathbb{R}_{>0}$, whose components for each $u \in \mathbb{R}^m$ are defined as

$$\sigma(u)_i = \sigma(u_i) := \min(|u_i|, \overline{u}_i)\mathrm{sign}(u_i) \quad i = 1, 2, \dots, m \tag{2}$$

Our goal is to design a static feedback control law under the effect of saturation, which stabilizes the system with improved convergence rate with respect to open-loop; i.e, when $u \equiv 0$. For convenience, we define the function $u \mapsto \phi(u)$ as the symmetric decentralized deadzone nonlinearity given by (see Tarbouriech et al. (2011), page 40):

$$\phi(u_i) := \sigma(u_i) - u_i \tag{3}$$

Hence, the boundary condition can be written as:

$$X(t, 0) = (H + BK)X(t, 1) + B\phi(KX(t, 1)) \tag{4}$$

which leads to the following representation of the closed-loop system:

$$\begin{aligned} &X_t(t, z) + \Lambda X_z(t, z) = 0 && \forall (t, z) \in \mathbb{R}_{>0} \times (0, 1) \\ &X(t, 0) = (H + BK)X(t, 1) + B\phi(KX(t, 1)) \\ & && \forall t \in \mathbb{R}_{\geq 0} \\ &X(0, z) = X_0(z) && \forall z \in [0, 1] \end{aligned} \tag{5}$$

From now on, we will refer to $\phi(KX(t, 1))$ simply as ϕ.

3. MAIN RESULTS

In this section, we prove that the closed-loop system (5) is wellposed using results in (Hastir et al., 2019) and (Tucsnak and Weiss, 2014). In addition, we propose sufficient conditions for the exponential stability of (5) in the form of matrix inequalities.

3.1 Wellposedness of the Cauchy Problem

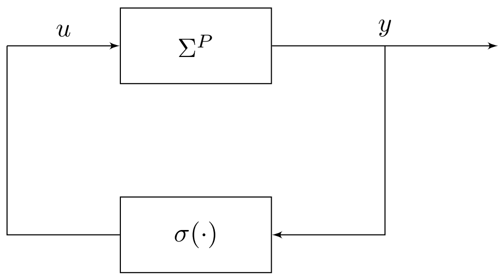

Fig. 1. Representation of Σ^σ

Proposition 1. *For every initial state $X_0 \in L^2((0, 1); \mathbb{R}^n)$, the closed-loop system (5) admits a unique solution $X \in \mathcal{C}([0, \infty); L^2((0, 1); \mathbb{R}^n))$.*

Proof. Consider the system:

$$\Sigma^P \begin{cases} X_t(t, z) + \Lambda X_z(t, z) = 0 & \forall (t, z) \in \mathbb{R}_{>0} \times (0, 1) \\ X(t, 0) = HX(t, 1) + Bu(t) & \forall t \in \mathbb{R}_{\geq 0} \\ X(0, z) = X_0(z) = 0 & \forall z \in [0, 1] \\ y(t) = KX(t, 1) & \forall t \in \mathbb{R}_{\geq 0} \end{cases} \tag{6}$$

Consider the interconnection Σ^σ (see Figure 1) of the system Σ^P and feedback σ by taking:

$$u(t) = \sigma(t) \tag{7}$$

According to (Bastin, Georges and J.-M Coron, 2016), for every input $u \in L^2_{loc}([0, \infty); \mathbb{R}^m)$ and for every initial state $X_0 \in L^2((0, 1); \mathbb{R}^n)$, system Σ^P admits a unique solution $X \in \mathcal{C}([0, \infty); \mathbb{R}^n)$ and $y \in L^2_{loc}([0, \infty]; Y)$. In particular, the solution can be written in the form of four families of bounded linear operators as:

$$X(t) = \mathbb{T}_t X_0 + \Phi_t u, \tag{8}$$

$$\mathbf{P}_t y = \Psi_t X_0 + \mathbb{F}_t u \tag{9}$$

for all $t \in [0, \infty)$. For an interval $[0, t]$, $\mathbf{P}_t$ denotes the operator of truncation of a vector-valued function y defined on a larger set than $[0, t]$, to $[0, t]$. Moreover, on any bounded time interval $[0, t]$, $0 < t < \infty$, $X(t)$ and $\mathbf{P}_t y$ depend continuously on X_0 and on $\mathbf{P}_t u$.
In particular, by the method of characteristics, we can write:

$$X_i(t, z) = \begin{cases} 0, & t \leq \dfrac{z}{\lambda_i} \\ \displaystyle\sum_{j=1}^{n} H_{ij} X_j(t - \frac{z}{\lambda_i}, z) + \sum_{k=1}^{m} B_{ij} u_k(t - \frac{z}{\lambda_i}), & t > \dfrac{z}{\lambda_i} \end{cases}$$

where $i = 1, 2, \dots, n$

$$y_i(t) = \begin{cases} 0 & t \leq \dfrac{1}{\lambda_i} \\ \displaystyle\sum_{j=1}^{n} K_{ij} H_{ij} X_j(t - \frac{1}{\lambda_i}, 1) + \sum_{k=1}^{m} K_{ij} B_{ij} u_k(t - \frac{1}{\lambda_i}), & t > \dfrac{1}{\lambda_i} \end{cases} \tag{10}$$

Let $\delta := \inf_{t>0} ||\mathbb{F}_t||$. From (10), it follows that for $t \in [0, \frac{1}{\lambda})$, $\mathbb{F}_t = 0$, so $\delta = 0$. Each component σ_i of σ is differentiable almost everywhere and $|\sigma_i'(u_i)| \leq 1$ for almost all $u_i \in \mathbb{R}$. Thus, the saturation function is Lipschitz continuous with Lipschitz constant $L = 1$. According to (Tucsnak and Weiss, 2014), since $\delta L = 0 < 1$, the closed-loop system Σ^σ is wellposed and thus admits unique solution $X \in \mathcal{C}([0, \infty); L^2((0, 1); \mathbb{R}^n))$. □

3.2 Sufficient Conditions for Stability Analysis

In this section, we make use of the following global sector condition to derive sufficient conditions for global exponential stability of (5).

Lemma 1. *[Tarbouriech et al. (2011), page 41] For all $\nu \in \mathbb{R}^m$, the nonlinearity $\phi(\nu)$ satisfies the following inequality:*

$$\phi(\nu)^\top T(\phi(\nu) + \nu) \leq 0 \tag{11}$$

for any matrix $T \in \mathcal{D}_p^m$.

Theorem 1. *System (5) is globally exponentially stable in the norm $L^2((0, 1); \mathbb{R}^n))$ if there exist $P \in \mathcal{D}_p^n$, $T \in \mathcal{D}_p^m$, $K \in \mathbb{R}^{m \times n}$, and $\mu \in \mathbb{R}_{>0}$ such that the following inequality holds:*

$$N := \begin{pmatrix} A^\top P \Lambda A - e^{-\mu} P \Lambda & A^\top P \Lambda B - K^\top T \\ * & B^\top P \Lambda B - 2T \end{pmatrix} \leq 0 \tag{12}$$

where $A := H + BK$.

Proof. Let $P \in \mathcal{D}_p^n$ and $\mu \in \mathbb{R}_{>0}$. Consider the following Lyapunov functional candidate:

$$V(X) = \int_0^1 e^{-\mu z} X^\top P X dz \tag{13}$$

The formal computation of the time derivative of the Lyapunov function along the solutions to (5) yields:

$$\dot{V}(X(t,\cdot)) = \int_0^1 e^{-\mu z}\left(\frac{\partial X}{\partial t}^\top PX + X^\top P\frac{\partial X}{\partial t}\right)dz \qquad \forall t \geq 0$$

Since P and Λ are both diagonal matrices, and using (1), we obtain:

$$\dot{V}(X(t,\cdot)) = -\int_0^1 e^{-\mu z}\frac{\partial}{\partial z}(X^\top P\Lambda X)dz \qquad \forall t \geq 0$$

performing integration by parts:

$$\begin{aligned}\dot{V}(X(t,\cdot)) &= -\mu\int_0^1 e^{-\mu z}X^\top P\Lambda X dz \\ &+ X(t,0)^\top P\Lambda X(t,0) - e^{-\mu}X(t,1)^\top P\Lambda X(t,1) \\ &\forall t \geq 0\end{aligned}$$

and finally using the boundary condition in (5):

$$\begin{aligned}\dot{V}(X(t,\cdot)) &= -\mu\int_0^1 e^{-\mu z}X^\top P\Lambda X dz + \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix}^\top \\ &\begin{pmatrix} A^\top P\Lambda A - e^{-\mu}P\Lambda & A^\top P\Lambda B \\ * & B^\top P\Lambda B \end{pmatrix}\begin{pmatrix} X(t,1) \\ \phi \end{pmatrix} \\ &\forall t \geq 0\end{aligned}$$

where $A := H + BK$. Then, $\forall t \geq 0$

$$\begin{aligned}\dot{V}(X(t,\cdot)) &\leq -\mu\lambda_{\min}(\Lambda)V(X(t,\cdot)) \\ &+ \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix}^\top M \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix}\end{aligned} \tag{14}$$

where $\lambda_{\min}(\Lambda)$ is the smallest eigenvalue of the matrix Λ, and

$$M := \begin{pmatrix} A^\top P\Lambda A - e^{-\mu}P\Lambda & A^\top P\Lambda B \\ * & B^\top P\Lambda B \end{pmatrix} \tag{15}$$

Using (11), we have:

$$\begin{aligned}\dot{V}(X(t,\cdot)) \leq \dot{V}(X(t,\cdot)) &- 2\phi(KX(1,t))^\top T(\phi(KX(1,t) \\ &+KX(1,t)) \qquad \forall t \geq 0\end{aligned} \tag{16}$$

We then can write:

$$\begin{pmatrix} X(t,1) \\ \phi \end{pmatrix}^\top M \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix} \leq \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix}^\top N \begin{pmatrix} X(t,1) \\ \phi \end{pmatrix} \tag{17}$$
$$\forall t \geq 0$$

where N is defined in (12). Hence, (12), (14) and (16) yield that

$$\dot{V}(X(t,\cdot)) \leq -\mu\lambda_{\min}V(X(t,\cdot)) \qquad \forall t \geq 0$$

We conclude that system (5) is globally exponentially stable if $N \leq 0$, thus concluding the proof. □

Remark 1. The derivation of the Lyapunov function was technically done for $\mathcal{C}^1$-solutions. However, similarly as in (Bastin, Georges and J.-M Coron (2016), page 67), it is shown that the stability analysis is also valid for the $\mathcal{L}^2$-solutions. In other words, the above Lyapunov analysis can be extended to weak solutions. ◦

Remark 2. The presence of the term μ in equation (14) contributes to enhancing the convergence rate of the L_2 system state. ◦

The result given next provides the representation of (12) in the form of a linear matrix inequality (5).

Corollary 1. *Assume there exist $\mu > 0$, $Q \in \mathcal{D}_p^n$, $S \in \mathcal{D}_p^m$, and $W \in \mathbb{R}^{m\times n}$ such that*

$$\begin{pmatrix} -Q\Lambda^{-1} & HQ + BW & BS \\ * & -e^{-\mu}\Lambda Q & -W^\top \\ * & * & -2S \end{pmatrix} \leq 0 \tag{18}$$

if we set the control gain by $K = WQ^{-1}$, then system (5) is globally exponentially stable.

Proof. Applying the Schur complement lemma (see Bernstein (2009)) to (12) yields:

$$\begin{pmatrix} -\Lambda^{-1}P & PH + PBK & PB \\ * & -e^{-\mu}P\Lambda & -K^\top T \\ * & * & -2T \end{pmatrix} \leq 0$$

which is equivalent to:

$$C^\top \begin{pmatrix} -\Lambda^{-1}P & PH + PBK & PB \\ * & -e^{-\mu}P\Lambda & -K^\top T \\ * & * & -2T \end{pmatrix} C \leq 0$$

where $C = C^\top = \begin{pmatrix} P^{-1} & 0 & 0 \\ * & P^{-1} & 0 \\ * & * & T^{-1} \end{pmatrix}$

The previous inequality is equivalent to:

$$\begin{pmatrix} -P^{-1}\Lambda^{-1} & HP^{-1} + BKP^{-1} & BT^{-1} \\ * & -e^{-\mu}\Lambda P^{-1} & -P^{-1}K^\top \\ * & * & -2T^{-1} \end{pmatrix} \leq 0$$

By setting $P^{-1} = Q$, $T^{-1} = S$ and $W = KP^{-1}$, the previous inequality turns into (18). So, inequality (18) is equivalent to (12).

$$\begin{pmatrix} -Q\Lambda^{-1} & HQ + BW & BS \\ * & -e^{-\mu}\Lambda Q & -W^\top \\ * & * & -2S \end{pmatrix} \leq 0 \tag{19}$$

where $P^{-1} = Q$, $T^{-1} = S$ and $W = KP^{-1}$. □

Remark 3. It is important to note that the condition given in (19) is nonlinear because of $e^{-\mu}Q$ but we deal with this issue by performing a line search on μ (see Ferrante and Prieur (2018)). ◦

3.3 Application on a Scalar System

In this section, we consider the following closed-loop scalar hyperbolic system:

$$\begin{aligned} &X_t(t,z) + \Lambda X_z(t,z) = 0 && \forall (t,z) \in \mathbb{R}_{>0}\times(0,1) \\ &X(t,0) = (H+K)X(t,1) + \phi(KX(t,1)) && \forall t \in \mathbb{R}_{\geq 0} \\ &X(0,z) = X_0(z) && \forall z \in [0,1] \end{aligned} \tag{20}$$

where $z \mapsto X(.,z) \in \mathbb{R}$, $t \in \mathbb{R}_{\geq 0}$ and $z \in [0,1]$. The terms H and $K \in \mathbb{R}$ and P, Λ, and $T \in \mathbb{R}_{>0}$. Let $R = P\Lambda$. Based on Section 3.2, system (20) is globally exponentially stable in the norm $L^2((0,1);\mathbb{R})$ if the following inequality holds:

$$N_s := \begin{pmatrix} (H+K)^2R - R & (H+K)R - KT \\ * & R - 2T \end{pmatrix} \leq 0 \quad (21)$$

where matrix $N_s \in \mathbb{R}^{2\times 2}$. In this case, $N_s \leq 0$ if and only if the determinant $\det(N_s) \geq 0$ and $\mathrm{trace}(N_s) \leq 0$. Following the procedure done in (Prieur et al., 2016), we compute the determinant and trace of this matrix, and provide conditions to ensure $N_s \leq 0$.
For simplicity of the calculation, consider $\mu = 0$. However, due to the continuity with respect to μ, we know that if inequality (12) holds for $\mu = 0$, then there exist ϵ such the that inequality (12) also holds for $\mu \in [0, \epsilon]$.

Discussion 1. *Inequality (21) admits a feasible solution if and only if* $|H| < 1$.

The determinant of matrix N_s is given by:

$$\begin{aligned}\det(N_s) = -2T(H^2R - R) - (R + KT)^2 \\ -2RKT(H+1)\end{aligned} \quad (22)$$

The trace of matrix N_s is given by:

$$\mathrm{trace}(N_s) = (H+K)^2R - 2T \quad (23)$$

Case 1: $|H| < 1$

thesAs stated earlier, for matrix (21) to be negative semi-definite, (22) must be nonnegative and (23) nonpositive, simultaneously. To simplify the problem, we take several values of H between -1 and 1 and fix $R = 1$. We allow K and T to be free and observe the regions under which the determinant and trace conditions have suitable signs simultaneously. Figures 2 and 3 represent the K and T points at which the two conditions are satisfied simultaneously for $H = 0.5$ and $H = -0.99$ respectively. Similar graphs are generated for all other values of H within this range. As noticed, there exist several T and K pairs to solve this problem which means that global exponential stability for system (20) using (21) as a sufficient condition is feasible when H is strictly between 1 and -1.

Case 2: $H \geq 1$ **or** $H \leq -1$

We repeat the procedure done for Case 1. We try several values for H including -1 and 1. Numerical analysis shows that there exist no pair for K and T to solve the problem given for values of H within the range of this case. Therefore, when $H \geq 1$ or when $H \leq -1$, the problem of finding a suitable K to render system (20) globally exponentially stable using (21) as a sufficient condition is unfeasible.

4. NUMERICAL EXAMPLES

This section provides numerical results based on different H selections. We use MATLAB to solve initial-boundary value problems for first order hyperbolic equations. See (Shampine, 2005) for more details. To do that, we solve (18) in MATLAB using the YALMIP package (Löfberg, 2004) combined with the solver SDPT3 (Toh et al., 1999). Consider the example given in (Ferrante and Prieur, 2018)

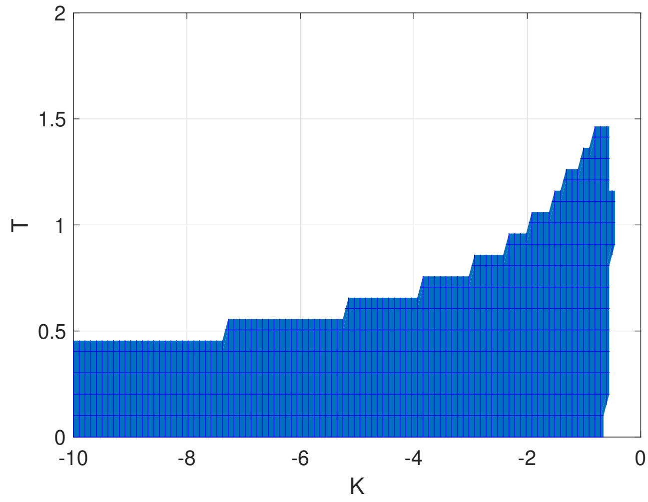

Fig. 2. Feasible region of the pair K, T where the conditions for global exponential stability are satisfied when $H = 0.5$

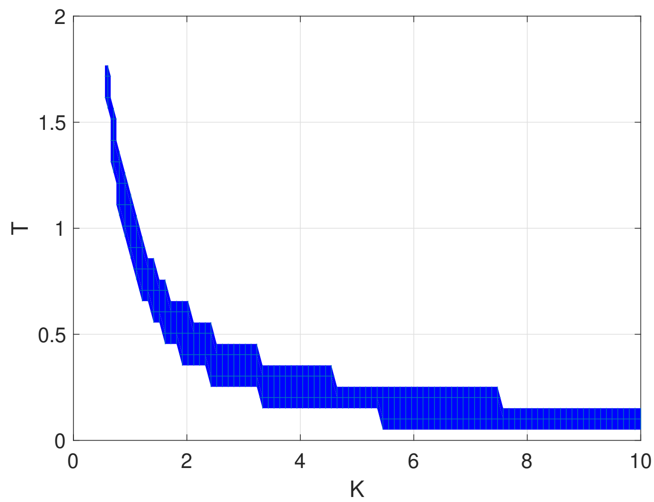

Fig. 3. Feasible region of the pair K, T where the conditions for global exponential stability are satisfied when $H = -0.99$

in which we have the following data:
$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, $B = I$ and the initial condition

$$z \in [0,1] \mapsto X_0(z) := 10 \begin{pmatrix} \cos(4\pi z) - 1 \\ \cos(2\pi z) - 1 \end{pmatrix} \quad (24)$$

Consider the saturation level $\overline{u} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

We consider three cases with different selections for the matrix H to assess the performance and stability of system (5) in different scenarios. Numerical experiments show that μ can be arbitrary large when H has eigenvalues strictly between -1 and 1. This means that the presence of the controller provides better convergence rate than in the case of an open-loop system.

- **Case 1:** $H = \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.6 \end{pmatrix}$, $\mu = 0.0001$ then,
$K = \begin{pmatrix} -0.415 & -0.42 \\ -0.37 & -0.48 \end{pmatrix}$
In this case we have a Schur matrix (eigenvalues

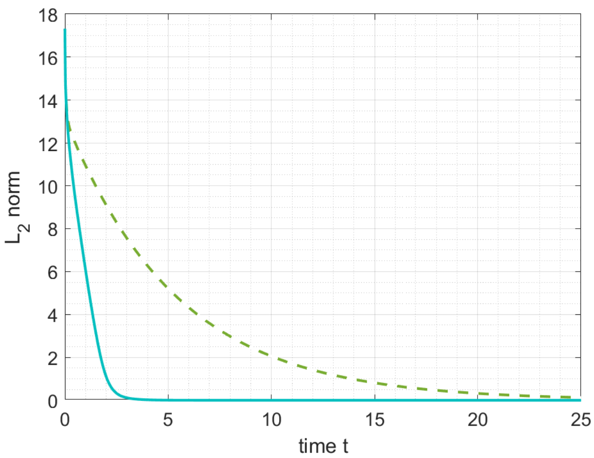

Fig. 4. The time evolution of the spatial norm $\mathcal{L}_2$ of $X(t,\cdot)$ in closed-loop (solid-line) and open-loop (dashed-line) for Case 1

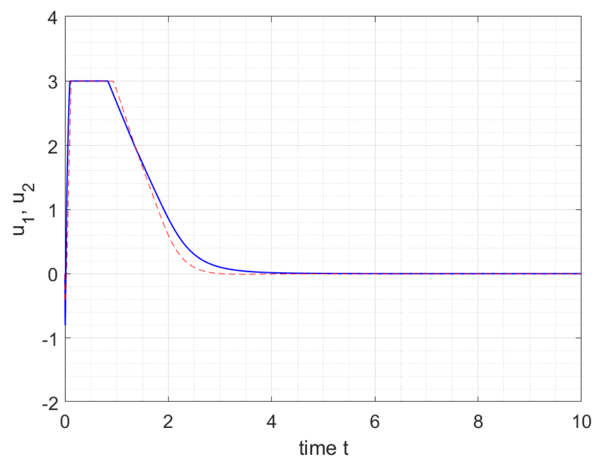

Fig. 5. Time evolution of $\sigma_1(K_1 X_1(t,1))$ (solid-line) and $\sigma_2(K_2 X_2(t,1))$ (dashed-line) with respect to time for Case 1

strictly between -1 and 1, and, as expected, we observe a stable response in Figure 4 where starting from an initial $\mathcal{L}_2$ norm of $X(t,\cdot)$ of value 17.7, the system converges with relatively high speed to zero. In the same figure, we observe the improved convergence rate of the closed-loop system compared to the open-loop one (about 27 time units faster). It is also important to note that the control values are exhibiting saturation levels in Case 1 as seen in Figure 5.

- **Case 2:** $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu = 0.001$ then, N is unstable and no feasible solution for K. When we consider a marginally stable matrix $H = I$, we obtain no feasible solution for inequality (12).
- **Case 3:** $H = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$, $\mu = 0.001$ then, N is unstable and no feasible solution for K.
When we consider H with relatively high values of eigenvalues (-2 and -1), we obtain no feasible solution for inequality (12).

5. CONCLUSION

In this paper, exponential stabilization of 1-D hyperbolic systems is studied using a saturated boundary controller. Lyapunov stability analysis and a global sector condition approach are used to derive a sufficient condition for exponential stability. An application on scalar system reveals the intervals of H on which a feasible controller can be designed. The results show that, using this saturated controller, open-loop exponential stability is necessarily needed to achieve closed-loop global exponential stability. Future work includes the extension to local exponential stability in order to relax the condition of open-loop stability condition.

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