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Stability and phase transitions of dynamical flow networks with finite capacities

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Abstract: We study deterministic continuous-time lossy dynamical flow networks with constant exogenous demands, fixed routing, and finite flow and buffer capacities. In the considered model, when the total net flow in a cell —consisting of the difference between the total flow directed towards it minus the outflow from it— exceeds a certain capacity constraint, then the exceeding part of it leaks out of the system. The ensuing network flow dynamics is a linear saturated system with compact state space that we analyse using tools from monotone systems and contraction theory. Specifically, we prove that there exists a set of equilibrium points that is globally asymptotically stable. Such set of equilibrium points reduces to a single globally asymptotically stable equilibrium point for generic exogenous demand vectors. Moreover, we show that the critical exogenous demand vectors giving rise to non-unique equilibrium points correspond to phase transitions in the asymptotic behavior of the dynamical flow network.

Keywords: Dynamical flow networks, nonlinear systems, compartmental systems, network flows, robust control.

1. INTRODUCTION

The study of dynamical flows in infrastructure networks has attracted a considerable amount of attention in recent years. In particular, there is a growing body of literature in the control systems field dealing with issues of stability, optimality, robustness, and resilience in dynamical flow networks. See, e.g., Paganini (2002); Low et al. (2002); Fan et al. (2004); Como et al. (2013); Bauso et al. (2013); Como et al. (2015); Coogan and Arcak (2015); Como (2017); Nilsson and Como (2020) and references therein.

In this paper, we study deterministic continuous-time models of dynamical flow networks. We consider a finite number of cells exchanging some indistinguishable commodity among themselves and with the external environment. Cells possibly receive a constant exogenous inflow from outside the network and a constant flow is possibly drained out of them directly towards the external environment. We assume that the outflow from a cell is split among its immediately downstream cells in fixed proportions and that each cell has a finite flow and buffer capacity. When the total net flow in a cell —consisting of the difference between the total flow directed towards it minus the outflow from it— exceeds the cell's capacity, then the exceeding part of such net flow leaks out of the system. Also, when the difference between the total exogenous demand on a cell and the total inflow in it exceeds the cell's capacity, then the outflow towards the external environment is reduced by an amount equal to the exceeding part of this difference. The ensuing network flow dynamics tuns out to be a linear saturated system with compact state space that we analyse using tools from monotone systems and contraction theory.

The rest of the paper is organized as follows. The remainder of this section is devoted to the introduction of some notational conventions to be used throughout the paper. In Section 2 we present the class of dynamical flow network models to be studied. Section 3 presents the main results concerning the characterization of the set of equilibrium points and its global asymptotic stability, as well as the dependence of such equilibrium points on the exogenous demand vector. Finally, Section 4 and 5 contain the proofs needed to demonstrate such results.

We shall consider the standard partial order on $\mathbb{R}^n$ whereby the inequality $a \leq b$ for two vectors $a, b \in \mathbb{R}^n$ is meant hold true entry-wise. A dynamical system with state space $\mathcal{X} \subseteq \mathbb{R}^n$ will be referred to as monotone if it preserves such partial order. For two vectors $a, b \in \mathbb{R}^n$ such that $a \leq b$, we shall denote by

$$\mathcal{L}_a^b = \{x \in \mathbb{R}^n : a \leq x \leq b\} = \times_{i=1}^n [a_i, b_i]$$

the complete lattice and let $S_a^b : \mathbb{R}^n \to \mathcal{L}_a^b$ be the vector saturation function defined by

$$\left(S_a^b(y)\right)_i = \max\{a_i, \min\{y_i, b_i\}\}\,, \qquad (1)$$

for $y \in \mathbb{R}^n$ and $i = 1, \dots, n$. For subsets of indices $\mathcal{A}, \mathcal{B} \subseteq \{1, \dots, n\}$, we shall denote the restriction of a vector $x \in \mathbb{R}^n$ by $x_{\mathcal{A}} = (x_i)_{i \in \mathcal{A}}$ and the restriction of a matrix $M \in \mathbb{R}^{n \times n}$ by $M_{\mathcal{AB}} = (M_{ij})_{i \in \mathcal{A}, j \in \mathcal{B}}$.

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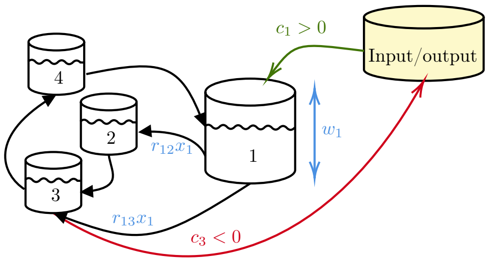

Fig. 1. A dynamical flow network with four cells.

2. A DYNAMICAL FLOW NETWORK MODEL WITH FINITE CAPACITY

We consider dynamical flow networks consisting of finitely many cells $i = 1, 2, \dots, n$, exchanging an indistinguishable commodity both among themselves and with the external environment as described below. (See also Figure 1)

Let $x_i(t)$ be the quantity of commodity contained in cell $i = 1, 2, \dots, n$ at time $t \geq 0$ and let $w_i > 0$ be its capacity. The state of the system is described by the vector $x(t) = (x_i(t))_{1 \leq i \leq n}$ and evolves in continuous time according to the following ordinary differential equation

$$\dot{x} = f(x)\,, \tag{2}$$

where $f(x) = (f_i(x))_{1 \leq i \leq n}$ is the vector of instantaneous net flows (inflows minus outflows) in the cells that will be assumed to satisfy the constraints

$$-x_i \leq f_i(x) \leq w_i - x_i\,, \qquad i = 0, \dots, n\,, \tag{3}$$

throughout the evolution of the system.

The leftmost inequality in (3) states that the outflow from cell i can never exceed the current inflow plus the total quantity of commodity in the cell, in particular implying the physically meaningful fact that the net flow $f_i(x)$ is nonnegative when the cell is empty (i.e., when $x_i = 0$) so that $x_i(t)$ can never become negative. On the other hand, the rightmost inequality in (3) guarantees that the sum of the current total mass and the inflow in a cell i and can never exceed the difference between its capacity w_i and the current outflow, so that in particular, when the mass $x(t) = w_i$ has reached the capacity, the net flow $f_i(x)$ is nonpositive, thus implying that the total mass will never exceed the capacity w_i if started below that. Notice that the complete lattice $\mathcal{L}_0^w$ is invariant for any dynamical flow network (2) satisfying (3).

Now, let each cell i possibly receive a constant exogenous inflow $\lambda_i \geq 0$ from outside the network and let a constant flow $\mu_i \geq 0$ possibly be drained directly from cell i towards the external environment, and let $c_i = \lambda_i - \mu_i$ be the exogenous net demand on cell i. Also, assume that constant fraction $R_{ij} \geq 0$ of the quantity of commodity x_i flows directly towards another cell $j \neq i$ in the network (fixed routing), while the remaining part $(1 - \sum_j R_{ij})x_i$ leaves the network directly. Notice that the routing matrix $R = (R_{ij}) \in \mathbb{R}^{n \times n}$ is necessarily sub-stochastic, i.e., with nonnegative entries and such that its rows all have sum less than or equal to 1.

Conservation of mass and the constraint (3) imply that the netflow in each cell $i = 1, \dots, n$ is given by

$$\begin{aligned} f_i(x) &= S_{-x_i}^{w_i - x_i}\left(\lambda_i - \mu_i + \textstyle\sum_j R_{ji}x_j - x_i\right) \\ &= S_0^{w_i}\left(\textstyle\sum_j R_{ji}x_j + c_i\right) - x_i\,. \end{aligned} \tag{4}$$

We may then rewrite the dynamical flow network (2)–(4) compactly as

$$\dot{x} = S_0^w\left(R^T x + c\right) - x\,, \tag{5}$$

where $w \in \mathbb{R}^n$ is the vector of the cells' capacities. Observe that the function $f(x)$ as defined in (4) is Lipschitz continuous on the whole $\mathbb{R}^n$, so that existence and uniqueness of a solution to the dynamical flow network (5) is ensured for every initial state $x(0) \in \mathcal{L}_0^w$.

In the dynamical network flow (5) it is understood that when the difference between the total flow $\lambda_i + \sum_j R_{ji}x_j$ directed towards a cell and the outflow $\mu_i + x_i$ from it exceeds the capacity w_i, then the exceeding part of it leaks out of the system. Moreover, the dynamical network flow (5) also assumes that, when the difference between the total exogenous demand μ_i on a cell i and the total inflow $\lambda_i + \sum_j R_{ji}x_j$ exceeds the cell's capacity w_i, then the outflow towards the external environment is reduced by an amount equal to the exceeding part of this difference.

3. MAIN RESULTS

In this section, we state the main results of this paper. These are concerned on the one hand with global asymptotic stability of the dynamical flow network (5) and on the other hand with the dependance (in particular, continuity and the lack thereof) of the equilibrium points of (5) on the exogenous demand vector $c \in \mathbb{R}^n$.

Before proceeding, let us gather some terminology that is used in our statements. The routing matrix R will be referred to as out-connected is for every $i = 1, \dots, n$ there exists $j \in \{1, \dots, n\}$ such that $\sum_k R_{jk} < 1$ and $(R^l)_{ij} > 0$ for some $l \geq 0$. It will be referred to as stochastic if all its rows sum up to 1 and irreducible if, for every nonempty proper subset $\mathcal{S} \subsetneq \{1, \dots, n\}$, there exists at least one $i \in \mathcal{S}$ such that $\sum_{j \in \mathcal{S}} R_{ij} < 1$. It is a standard fact that, if the routing matrix R is stochastic irreducible, then it admits a unique invariant probability vector $\pi = R^T \pi$ and such vector is strictly positive entry-wise. Moreover, as shown in Massai et al. (2019), for every zero-sum vector $v \in \mathbb{R}^n$, the vector series

$$Hv := \frac{1}{2}\sum_{k \geq 0}\left(\frac{I + R^T}{2}\right)^k v \tag{6}$$

is convergent and its limit satisfies

$$Hv = R^T Hv + v\,. \tag{7}$$

We start with the following stability results.

Theorem 1. Let $w \in \mathbb{R}^n$ be a positive vector and $R \in \mathbb{R}^{n \times n}$ a sub-stochastic matrix. Then,

(i) if R is sub-stochastic and out-connected, then, for every exogenous demand vector $c \in \mathbb{R}^n$ the dynamical flow network (5) admits a globally asymptotically stable equilibrium point $x^* \in \mathcal{L}_0^w$.

On the other hand, if R is stochastic and irreducible, then

(ii) for every exogenous demand vector $c \in \mathbb{R}^n$ the set of equilibrium points $\mathcal{X}(c)$ of the dynamical flow network (5) is a nonempty line segment joining two points $\underline{x} \leq \overline{x}$ on the boundary of the lattice $\mathcal{L}_0^w$;

(iii) for every initial state $x(0) \in \mathcal{L}_0^w$, the solution of (5) converges to the set of equilibrium points $\mathcal{X}(c)$ as t grows large;

(iv) the set of equilibrium points $\mathcal{X}(c)$ has positive length if and only if

$$\min_i \left\{ \frac{(Hc)_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - (Hc)_i}{\pi_i} \right\} > 0 \qquad (8)$$

Theorem 1 characterizes the set of equilibrium points $\mathcal{X}(c)$. In a given network, it is particularly relevant to study the dependance of $\mathcal{X}(c)$ on the exogenous net flow vector c as this might be subject to shocks that might affect the whole flow on the network. The resilience of the system with respect to such shocks heavily depends on the way equilibrium points depend on c. We will show existence of a set of critical vectors c such that the equilibrium points of (5) undergo a jump discontinuity, thus determining a phase transition in the asymptotic behavior of the system, and we will describe this critical set.

Let us introduce some further notation. Let

$$\mathcal{U} = \{c \in \mathbb{R}^n : |\mathcal{X}(c)| = 1\}, \quad \mathcal{M} = \mathbb{R}^n \setminus \mathcal{U}, \qquad (9)$$

be the subsets of exogenous flow vectors for which there is a unique equilibrium point and, respectively, there are multiple equilibrium points. Moreover, let $\underline{x}(c)$ and $\overline{x}(c)$ be the lowest and, respectively, the highest equilibrium points for a given vector c. For exogenous flow vectors $c \in \mathcal{U}$, we shall also use the notation

$$x^*(c) = \underline{x}(c) = \overline{x}(c)$$

for the unique equilibrium point.

We can now state the following result.

Theorem 2. Let $w \in \mathbb{R}_+^n$ be a nonnegative vector. Let $\mathcal{U}$ and $\mathcal{M}$ be defined as in (9). Then,

(i) if R is sub-stochastic and out-connected, then, for every exogenous demand vector $c \in \mathbb{R}^n$ the map $c \mapsto x^*(c)$ is continuous.

On the other hand, if R is stochastic and irreducible, then

(ii) $\mathcal{M}$ is linear sub-manifold of co-dimension 1;

(iii) the map $c \mapsto x^*(c)$ is continuous on the set $\mathcal{U}$;

(iv) for every $c^* \in \mathcal{M}$,

$$\liminf_{\substack{c \in \mathcal{U} \\ c \to c^*}} x^*(c) = \underline{x}(c^*), \qquad \limsup_{\substack{c \in \mathcal{U} \\ c \to c^*}} x^*(c) = \overline{x}(c^*).$$

Theorem 2 implies that the equilibrium points of (5) undergo a jump discontinuity when the vector c crosses the set $\mathcal{M}$ for which the uniqueness condition for equilibrium points fails to hold. This in turn implies that even a slight change in the exogenous flow may trigger a phase transition in the system and a huge impact on the quantity of commodities exchanged at equilibrium in the network. We show this phenomenon in the following example.

Example 1 Let us consider a flow model with an irreducible routing matrix R, in particular, we consider (5) with:

$$R = \begin{bmatrix} 0 & 0.75 & 0.25 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}, \; w = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The corresponding flow network is shown in Fig. 2.

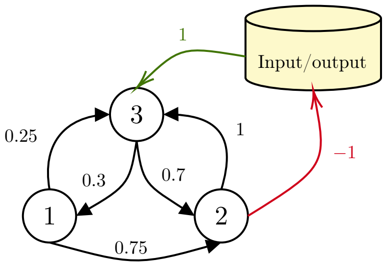

Fig. 2. Flow network with three cells.

Since $\mathbb{1}^T c = 0$ and $\min_i \left\{ \frac{(Hc)_i}{\pi_i} \right\} + \min_i \left\{ \frac{w_i - (Hc)_i}{\pi_i} \right\} \approx 9.62 > 0$ then (5) admits multiple equilibrium points because of Theorem 1(iv). Indeed one can compute $\overline{x}(c) \approx [1.62, 4, 5.41]^T$ and $\underline{x}(c) \approx [0.32, 0, 1.08]^T$. We highlight the big jump that occurs for this particular vector c; notice how in the largest solution $\overline{x}$, cell 2 can deliver its total outflow capacity 4 while in the smallest solution $\underline{x}$ it outputs 0. A slight change of the exogenous flow around c could then have a huge impact on the network. In Fig. 3 we show some trajectories (in red) for different initial conditions in the phase space; we also plot the two lattices $\mathcal{L}_0^w$ and $\mathcal{L}_{\underline{x}}^{\overline{x}}$ (in green and light blue respectively); finally, the segment of equilibrium points $\mathcal{X}$ is plot in orange.

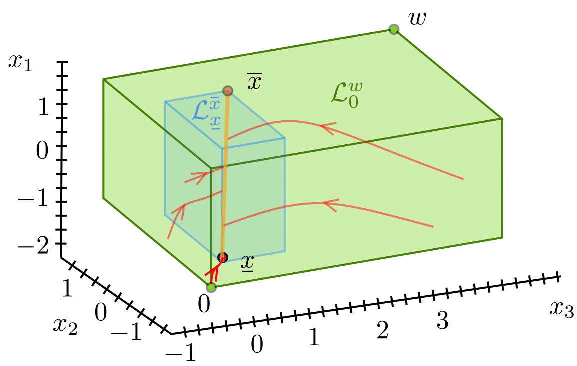

Fig. 3. Trajectories in the phase space in case of multiple equilibrium points.

We can notice how all trajectories (red curves) converge to the set of equilibrium points (orange segment).

Let us now change slightly the vector c by setting: $c = [\frac{\alpha}{3}, -1, \frac{2\alpha}{3}]^T$ with $\alpha \in [0, 9]$. Notice that we have multiple equilibrium points when $\alpha = 1 \implies c^* = [\frac{1}{3}, -1, \frac{2}{3}]^T$ as in that case one can check that condition of Theorem 1(iv) holds. In Fig. 4 we show the set of equilibrium points $\mathcal{X}(c)$ in the phase space as c varies as a function of α.

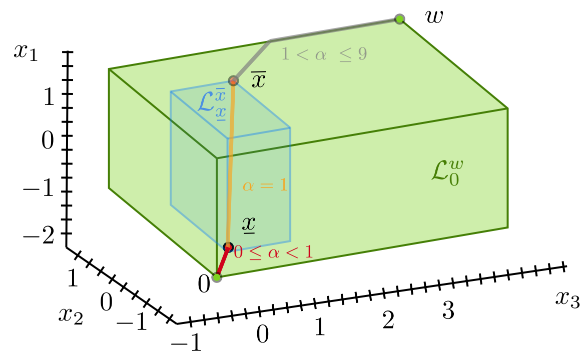

Fig. 4. Set of equilibrium points as α varies.

Notice that $x^*(c)$ is a piece-wise linear function. We can see that for $0 \le \alpha < 1$ the equilibrium points (red segment) start from 0, they are unique and located on $\partial\mathcal{L}_0^w$, then when $\alpha = 1$ (and $c = c^*$) we have multiple equilibrium points (orange segment) and finally when $\alpha > 1$ the unique equilibrium points (gray segment) are located on $\partial\mathcal{L}_0^w$ until they eventually reach w, which means that all cells output their maximal flow.

We appreciate a phase transition as the parameter α crosses the value $\alpha = 1$ and the equilibrium points undergo a jump discontinuity going from $\underline{x}(c^*)$ to $\overline{x}(c^*)$.

4. PROOF OF THE STABILITY RESULTS

This section is devoted to prove Theorem 1. We will first present some technical results concerning properties of the system (5) that we will need to prove the main statement.

We start with the following technical results, whose proofs are presented in Appendix A, Appendix B and Appendix C respectively.

Lemma 1. The dynamical system (5) is monotone and non-expansive in l_1-distance on $\mathcal{L}_0^w$.

Lemma 2. The dynamical system (5) always admits a maximal equilibrium point $\overline{x} \in \mathcal{L}_0^w$ and a minimal equilibrium point $\underline{x} \in \mathcal{L}_0^w$. Moreover, the sets

$$\mathcal{X}_\alpha = \left\{ x \in \mathcal{L}_{\underline{x}}^{\overline{x}} : \sum\nolimits_i x_i = \alpha \sum\nolimits_i \underline{x}_i + (1-\alpha) \sum\nolimits_i \overline{x}_i \right\} \tag{10}$$

for $0 \le \alpha \le 1$ are invariant for (5) and, for $x(0) \in \mathcal{L}_0^w$, the solution of (5) is such that $x(t) \overset{t \to +\infty}{\longrightarrow} \mathcal{L}_{\underline{x}}^{\overline{x}}$.

Lemma 3. Let x^* be an equilibrium point of the dynamical flow network (5) belonging to the interior of the lattice $\mathcal{L}_0^w$. Then, there exists an $\varepsilon > 0$ such that, every solution of (5) with initial condition $x(0) \in \mathcal{L}_0^w$ such that $||x(0) - x^*|| < \varepsilon$, coincides with the solution of the linear dynamics

$$\dot{x} = (R^T - I)x + c\,. \tag{11}$$

We are now ready to prove a first result that characterizes the set of equilibrium points.

Proposition 1. There exists a nondecreasing curve of equilibrium points joining $\underline{x}$ and $\overline{x}$ with image $\mathcal{X}$. Moreover, if R is stochastic irreducible, then such curve is entry-wise strictly increasing, while if R is sub-stochastic out-connected such curve is constant so that $\underline{x} = \overline{x}$.

Proof. For $0 \le \alpha \le 1$ the convex compact set $\mathcal{X}_\alpha$ defined in (10) is invariant for (5) by Lemma 2. Then, since $f(x)$ is Lipschitz-continuous, Lemma 1 in Lajmanovich and Yorke (1976) implies that (5) has at least one equilibrium point in $\mathcal{X}_\alpha$. For $0 \le \alpha < \beta \le 1$, if $x^*(\alpha)$ is an equilibrium point of (5) in $\mathcal{X}_\alpha$, then the same argument can be applied to show existence of an equilibrium point $x^*(\beta) \in \mathcal{X}_\beta \cap \mathcal{L}_{x^*(\alpha)}^{\overline{x}}$. Moreover, clearly $\lim_{\beta \downarrow \alpha} x^*(\beta) = x^*(\alpha)$. Similarly, one can prove existence of an equilibrium point $x^*(\beta) \in \mathcal{X}_\beta \cap \mathcal{L}_{\underline{x}}^{x^*(\alpha)}$ for $0 \le \beta < \alpha$ and that $\lim_{\beta \uparrow \alpha} x^*(\beta) = x^*(\alpha)$. This shows that there exists a nondecreasing curve of equilibrium points $[0,1] \ni t \mapsto x^*(t)$ joining $x_0^* = \underline{x}$ to $x_1^* = \overline{x}$.

In order to prove the second part of the claim, fix $0 \le \alpha < \beta \le 1$ and let $\mathcal{S} \subseteq \{1,\dots,n\}$ be the set of those cells i such that $x_i^*(\alpha) < x_i^*(\beta)$. If R is stochastic irreducible and $\mathcal{S}$ is a strict subset of $\{1,\dots,n\}$, then

$$\begin{aligned}
\beta - \alpha &= \sum_{i \in \mathcal{S}} x_i(\beta) - x_i(\alpha) \\
&= \sum_{i \in \mathcal{S}} S_0^{w_i} \left(\sum\nolimits_j R_{ji} x_j(\beta) + c_i \right) \\
&\quad - \sum_{i \in \mathcal{S}} S_0^{w_i} \left(\sum\nolimits_j R_{ji} x_j(\alpha) + c_i \right) \\
&\le \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} R_{ji}(x_j(\beta) - x_j(\alpha)) \\
&< \sum_{i \in \mathcal{S}} (x_j(\beta) - x_j(\alpha)) \\
&= \beta - \alpha\,,
\end{aligned} \tag{12}$$

where the last inequality follows from the fact that $\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} R_{ji} z_j < \sum_{i \in \mathcal{S}} z_i$ for every positive z and every strict subset $\mathcal{S} \subsetneq \{1,\dots,n\}$. It then follows that necessarily $x_i(\beta) > x_i(\alpha)$ for every $i = 1,\dots,n$. Finally, notice that if R is sub-stochastic out-connected then (12) remains valid for every nonempty subset $\mathcal{S} \subseteq \{1,\dots,n\}$, thus implying that necessarily $\overline{x} = \underline{x}$ in thus case. ■

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1

(i) It follows from Lemma 2 and Proposition 1 that, when R is sub-stochastic out-connected $\underline{x} = \overline{x} = x^*$ is a global asymptotically stable equilibrium point.

(ii) From Proposition 1 we know that there exists a strictly increasing curve joining $\underline{x}$ and $\overline{x}$, which means that either the system has a unique equilibrium point or it has a continuum of them. In the latter case, since the curve is strictly increasing, all non extremal equilibrium points $x^* \in \mathcal{X}(c) \setminus \{\underline{x}, \overline{x}\}$ must belong to the interior of the lattice $\mathcal{L}_0^w$ and hence satisfy

$$x^* = R^T x^* + c\,. \tag{13}$$

Now observe that, since R is row-stochastic, we have $\mathbb{1}^T x = \mathbb{1}^T R^T x + \mathbb{1}^T x + \mathbb{1}' c$ so that $\mathbb{1}^T c = 0$, i.e., for the linear system (13) to admit solutions it is necessary that c is a zero-sum vector. In fact, since the stochastic matrix R is irreducible, we have that $I - R^T$ has rank $n-1$ and for every zero-sum vector c the set of solutions x^* of the linear system (13) coincides with the line

$$\mathcal{H} = \{x^* = Hc + \alpha\pi : \alpha \in \mathbb{R}\}\,. \tag{14}$$

Hence, we have that:

$$\mathcal{X}(c) = \mathcal{H} \cap \mathcal{L}_0^w = [\underline{x}(c), \bar{x}(c)] \quad (15)$$

is a line segment joining $\underline{x} \in \partial\mathcal{L}_0^w$ and $\bar{x} \in \partial\mathcal{L}_0^w$.

(iii) If $\underline{x} = \bar{x}$, then the global convergence follows from Lemma 2. Hence, we focus on the case of infinitely many equilibrium points. Notice that, for $0 \le \alpha \le 1$, the set $\mathcal{X}_\alpha$ defined in (10) intersects the line segment $\mathcal{X}(c)$ in a single equilibrium point $x^*(\alpha) = \alpha\underline{x} + (1-\alpha)\bar{x}$. Moreover, as discussed in the proof of point (ii) above, $x^*(\alpha)$ belongs to the interior of the lattice $\mathcal{L}_0^w$ for $0 < \alpha < 1$ so that Lemma 3 implies that (5) reduces to the linear system (11) in a sufficiently small neighborhood of $x^*(\alpha)$. Observe that all solutions of (11) with initial condition $x(0) \in \mathcal{X}_\alpha$ converge to x^*_α as t grows large. It then follows that, for every $0 \le \alpha \le 1$, there exists some $\varepsilon > 0$ such that for every solution $x(t)$ of the dynamical flow network with initial condition $x(0) \in \mathcal{X}_\alpha$ such that $||x(0) - x^*(\alpha)|| < \varepsilon$ converges to $x^*(\alpha)$ as t grows large.

Let $\phi^t(x^\circ)$ be the solution of (5) started at $x(0) = x^\circ$. By Theorem 4.5 in Khalil (2002) our last finding implies that, for every $0 \le \alpha \le 1$ there exists a $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that $\|\phi^t(x) - x^*(\alpha)\| \le \beta(x - x^*(\alpha), t)$ for every $x \in \mathcal{X}_\alpha$ such that $\|x - x^*(\alpha)\|_1 \le \varepsilon$. To prove global convergence to the set $\mathring{\mathcal{X}}(c)$ we need to show that for any $x^\circ \in \mathcal{X}_\alpha$ such that $\|x^\circ - x^*\|_1 > \varepsilon$, there exists a finite time $T \ge 0$ such that $||\phi^T(x^\circ) - x^*(\alpha)||_1 \le \varepsilon$. For sake of notation, let us put $x^* = x^*(\alpha)$.

Now let $\hat{x} = x^* + \varepsilon\|x^\circ - x^*\|^{-1}(x^\circ - x^*)$, for which it is easily seen that $\|\hat{x} - x^*\|_1 = \varepsilon$, and

$$\|x^\circ - x^*\|_1 = \|x^\circ - \hat{x}\|_1 + \|\hat{x} - x^*\|_1 = \|x^\circ - \hat{x}\|_1 + \varepsilon$$

and consider the trajectories of the system starting from x° and $\hat{x}$. The l_1-non expansiveness ensured by Lemma 1 implies that $\frac{\mathrm{d}}{\mathrm{d}t}\|\phi^t(x^\circ) - \phi^t(\hat{x})\|_1 \le 0$, so that $\|\phi^t(x^\circ) - \phi^t(\hat{x})\|_1 \le \|x^\circ - \hat{x}\|_1$. By the triangle inequality,

$$\begin{aligned}\|\phi^t(x^\circ) - x^*\|_1 &\le \|\phi^t(x^\circ) - \phi^t(\hat{x})\|_1 + \|\phi^t(\hat{x}) - x^*\|_1 \\ &= \|x^\circ - \hat{x}\|_1 + \|\phi^t(\hat{x}) - x^*\|_1 \\ &= \|x^\circ - x^*\|_1 - \varepsilon + \|\phi^t(\hat{x}) - x^*\|_1 .\end{aligned}$$

Due to the properties of the $\mathcal{KL}$ functions, there exists $T_{\varepsilon/2} \ge 0$ such that $\beta(x-y,t) \le \varepsilon/2$ for all y such that $\|y - x^*\|_1 \le \varepsilon$ and for all $t \ge T_{\varepsilon/2}$. Thus, we have

$$\begin{aligned}\|\phi^t(x^\circ) - x^*\|_1 &\le \|x^\circ - x^*\|_1 - \varepsilon + \|\phi^t(\tilde{x}) - x^*\|_1 \\ &\le \|x^\circ - x^*\|_1 - \varepsilon/2\end{aligned}$$

for all $t \ge T_{\varepsilon/2}$. If $||\phi^{T_{\varepsilon/2}}(x^\circ) - x^*(\alpha)||_1 \le \varepsilon$ the proof is complete with $T^- = T_{\varepsilon/2}$. Otherwise, by the same argument, since each step the ℓ_1 distance between $\phi^t(x)$ and x^* decreases by at least $\varepsilon/2 > 0$ in no more than $\lceil 2\|x^\circ - x^*\|_1/\varepsilon\rceil$ steps, so that $\|\phi^T(x^\circ) - x^*\|_1 \le \varepsilon$ for $T \le \lceil 2\|x^\circ - x^*\|_1/\varepsilon\rceil T_{\varepsilon/2}$. ■

(iv) Because of what said in point (ii) of this proof, the set $\mathcal{X}(c)$ has positive length if and only if (15) defines a non-empty set, i.e. if and only if we can find values of $\alpha \in \mathbb{R}$ such that $0 < Hc + \alpha\pi < w$. As shown in Massai et al. (2019), this is the case if and only if

$$\min_i \left\{\frac{(Hc)_i}{\pi_i}\right\} + \min_i \left\{\frac{w_i - (Hc)_i}{\pi_i}\right\} > 0 , \quad (16)$$

thus completing the proof. ■

5. PROOF OF THE CONTINUITY RESULTS

This section is devoted to prove Theorem 2.

We need the following technical results.

Lemma 4. Both $c \mapsto \underline{x}(c)$ and $c \mapsto \bar{x}(c)$ are monotone nondecreasing maps from $\mathbb{R}^n$ to $\mathcal{L}_0^w$.

Proof. Consider two vectors $c_1, c_2 \in \mathbb{R}^n$ such that $c_1 \le c_2$ and let $x_1(t)$ and $x_2(t)$ the two solutions of (5) with, respectively, $c = c_1$ and $c = c_2$, and with initial condition $x_1^\circ = x_2^\circ = 0$. Then we know that these two solutions must converge, respectively, to the minimal solutions $\underline{x}(c_1)$ and $\underline{x}(c_2)$. Since $S_0^w(\cdot)$ is monotone nondecreasing,

$$\begin{aligned}x_1(t) \le x_2(t) \Rightarrow \dot{x}_1(t) &= S_0^w\left(R^T x_1(t) + c_1\right) - x_1 \\ &\le S_0^w\left(R^T x_2(t) + c_2\right) - x_2 = \dot{x}_2(t) .\end{aligned}$$

Since $x_1^\circ = x_2^\circ = 0$, this implies $x_1(t) \le x_2(t)$ for all t. This yields $\underline{x}(c_1) \le \underline{x}(c_2)$. Thus $x(c)$ is nondecreasing. The same property for $\bar{x}(c)$ follows analogously. ■

Lemma 4 allows us to prove the following results that is the key for the proof of Theorem 2.

Lemma 5. Let $R \in \mathbb{R}_+^{n\times n}$ be a stochastic matrix and $w \in \mathbb{R}_+^n$ be a nonnegative vector. Then, for every $c^* \in \mathbb{R}^n$,

$$\limsup_{c\to c^*} \bar{x}(c) = \bar{x}(c^*) , \qquad \liminf_{c\to c^*} \underline{x}(c) = \underline{x}(c^*) .$$

Proof. Let (c_n) be a sequence in $\mathbb{R}^n$ such that $c_n \overset{n\to+\infty}{\longrightarrow} c^*$ and $\bar{x}(c_n) \overset{n\to+\infty}{\longrightarrow} z^*$. Let $d_n = \sup\{\max\{c_k, c^*\} : k \ge n\}$, for $n \ge 1$. Clearly, $d_n \overset{n\to+\infty}{\longrightarrow} c^*$, while $d_n \ge c_n$, $d_n \ge c^*$, and $d_{n+1} \le d_n$, for $n \ge 1$. Then, Lemma 4 implies that $\bar{x}(d_n) \ge \bar{x}(c_n)$, $\bar{x}(d_n) \ge \bar{x}(c^*)$, and $\bar{x}(d_{n+1}) \le \bar{x}(d_n)$, for $n \ge 1$. Thus, in particular, $\bar{x}(d_n)$ converges to some $z \in \mathcal{L}_0^w$ such that $z \ge z^*$ and $z \ge \bar{x}(c^*)$. On the other hand, $\bar{x}(d_n) \in \mathcal{X}(d_n)$ is an equilibrium point so that $\bar{x}(d_n) = S_0^w(R^T\bar{x}(d_n) + d_n)$ for $n \ge 1$. By taking the limit of both sides, continuity implies that $z^* = S_0^w(R^T z^* + c^*)$ so that $z \in \mathcal{X}(c^*)$ is such that $z \le \bar{x}(c^*)$. Then

$$\bar{x}(c^*) = z \ge z^* = \limsup_{c\to c^*} \bar{x}(c) = \bar{x}(c^*) .$$

The liminf part of can then be proven similarly. ■

Proof of Theorem 2

(i) Because of Theorem 1, in this case $\mathcal{U} = \mathbb{R}^n$ and hence by Lemma 5 it follows that, for $c^* \in \mathbb{R}^n$, we have

$$\begin{aligned}\limsup_{c\to c^*} \underline{x}(c) &\le \limsup_{c\to c^*} \bar{x}(c) = \bar{x}(c^*) = \underline{x}(c^*) \\ &= \liminf_{c\to c^*} \underline{x}(c) \le \liminf_{c\to c^*} \bar{x}(c) .\end{aligned}$$

Then, the inequalities above all hold as equalities.

(ii) By Theorem 1, $c^* \in \mathcal{M}$ satisfies (8). This determines a linear sub-manifold of co-dimension 1 in $\mathbb{R}^n$.

(iii) It follows from Lemma 5 that, for $c^* \in \mathcal{U}$, we have

$$\begin{aligned}\limsup_{c\to c^*} \underline{x}(c) &\le \limsup_{c\to c^*} \bar{x}(c) = \bar{x}(c^*) = \underline{x}(c^*) \\ &= \liminf_{c\to c^*} \underline{x}(c) \le \liminf_{c\to c^*} \bar{x}(c) .\end{aligned}$$

Then, the inequalities above all hold as equalities.

(iv) If $c^* \in \mathcal{M}$, then any $c \in \mathbb{R}^n$ such that $c > c^*$ or $c < c^*$ belongs to $\mathcal{U}$. This implies that the limit relations in Lemma 5 continue to hold true when c is restricted in $\mathcal{U}$ and the proof follows along the same lines. ■

6. CONCLUSIONS AND FUTURE WORK

We have introduced a nonlinear dynamical system modeling flow dynamics among cells with finite capacity. We have completely characterized the set of equilibrium points and proved the global convergence toward this set. Moreover, we have shown how the model exhibits critical phase transitions as the exogenous flow approaches a set of critical values. Future work includes more in-depth analysis of the discontinuities and their relationship to the network structure and extending the dynamical flow model to allow for nonlinearities in the dependence of the outflow from a cell on the mass of commodity in it.

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Appendix A. PROOF OF LEMMA 1

We first prove that $\mathcal{L}_0^w$ is invariant. This follows from the fact that, for every x in $\mathcal{L}_0^w$ such that $x_i = w_i$, we have

$$f_i(x) = S_0^{w_i}\left(\sum\nolimits_j R_{ji}x_j + c_i\right) - w_i \leq 0\,,$$

whereas for every x in $\mathcal{L}_0^w$ such that $x_i = 0$ we have

$$f_i(x) = S_0^{w_i}\left(\sum\nolimits_j R_{ji}x_j + c_i\right) \geq 0\,.$$

We now prove that (5) is a monotone system. For $i \neq k$,

$$\frac{\partial f_i}{\partial x_k}(x) = \begin{cases} 0 & \text{if } \sum_j R_{ji}x_j + c_i < 0 \\ R_{ki} & \text{if } 0 < \sum_j R_{ji}x_j + c_i < w_i \\ 0 & \text{if } \sum_j R_{ji}x_j + c_i > w_i \end{cases} \tag{A.1}$$

This shows that $\frac{\partial f_i}{\partial x_k}(x) \geq 0$ almost everywhere in $\mathcal{L}_0^w$, so (5) is a monotone system by (Kamke, 1929, Theorem 1.2).

Finally, we show that (5) is nonexpansive in l_1 distance on $\mathcal{L}_0^w$. By monotonicity and using the fact that

$$\sum_i \frac{\partial f_i}{\partial x_k}(x) \leq \sum_i R_{ki} - 1 \leq 0 \tag{A.2}$$

this follows from (Lovisari et al., 2015, Lemma 5). ■

Appendix B. PROOF OF LEMMA 2

From monotonicity and the fact that $\mathcal{L}_0^w$ is invariant, the two initial value problems

$$\begin{cases} \dot{x} = S_0^w\left(R^T x + c\right) - x \\ x_0 = 0 \end{cases} \qquad \begin{cases} \dot{x} = S_0^w\left(R^T x + c\right) - x \\ x_0 = w \end{cases} \tag{B.1}$$

admit unique solutions that converge to a lower equilibrium point $\underline{x}$ and largest equilibrium point $\bar{x} \geq \underline{x}$. Now, let $\underline{y} = \sum_i \underline{x}_i$ and $\overline{y} = \sum_i \overline{x}_i$. Consider an initial state $x(0) \in \mathcal{L}_{\underline{x}}^{\overline{x}}$. Since the system is non-expansive in l_1-distance, both $\|x(t) - \underline{x}\|_1$ and $\|x(t) - \overline{x}\|_1$ cannot increase in time, which implies that $\sum_i x_i(t)$ remains constant. Then, the sets $\mathcal{X}_\alpha = \{x \in \mathcal{L}_{\underline{x}}^{\overline{x}} : \sum_i x_i = \alpha\underline{y} + (1-\alpha)\overline{y}\}$ for $0 \leq \alpha \leq 1$ are all invariant. Finally, for any $x^\circ \in \mathcal{L}_0^w$, let $\phi^t(x^\circ)$ be the solution of (5) at $t \geq 0$. Since $\phi^t(0) \overset{t\to+\infty}{\longrightarrow} \underline{x}$ and $\phi^t(w) \overset{t\to+\infty}{\longrightarrow} \bar{x}$, then $\phi^t(x^\circ) \overset{t\to+\infty}{\longrightarrow} \mathcal{L}_{\underline{x}}^{\overline{x}}$ for $x^\circ \in \mathcal{L}_0^w$. ■

Appendix C. PROOF OF LEMMA 3

Observe that an equilibrium point x^* in the interior of $\mathcal{L}_0^w$ is such that $S_0^w(R^T x^* + c) = x^*$ belongs to the interior of $\mathcal{L}_0^w$ which in turn implies that

$$f(x^*) = S_0^w(R^T x^* + c) - x^* = (R^T - I)x^* + c\,.$$

Since the map $f(x)$ is continuous, there necessarily exists an $\varepsilon > 0$ such that $f(x) = (R^T - I)x + c$ for all $||x - x^*|| < \varepsilon$. Since R is stochastic, its spectrum is contained in the unitary disk centered in 0. Hence $R^T - I$ has all eigenvalues with nonpositive real part and x^* is stable (both for the linear dynamical system (11) and the nonlinear dynamical flow network (5), as they locally coincide), so that we can always find a number $\delta \leq \varepsilon$ such that if $\|x(0) - x^*\| < \delta$ then $\|x(t) - x^*\| < \varepsilon$ for all $t \geq 0$. This ensures that the trajectories of the system remain in the region where the dynamics is linear and hence the claim follows. ■