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Nonlinear Observer design for Systems with Sampled Measurements: An LPV Approach

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Abstract: The aim of this work is to propose a design methodology of observers for a class of Lipschitz nonlinear dynamical systems with sampled measurements by using the differential mean value theorem (DMVT) which allows us to transform the nonlinear part of the estimation error dynamics into a linear parameter varying (LPV) system. The designed observer must ensure the stability of the estimation error subject to a sampled measurements. An LMI-based minimization problem is provided to ensure the stability and the existence of the observer using Lyapunov theory. Thus, the measurements sampling period is included in the LMI as a decision parameter. Indeed, this allows to widen the sampling period as much as possible, which helps optimization of energy consumption while guaranteeing the convergence of the observer. Finally, to illustrate the performance of the proposed methodology, a numerical example is presented.

Keywords: Observers design, Lipschitz nonlinarity, Sampled measurements, LMI Approach, LPV system .

1. INTRODUCTION

In real applications, most systems are naturally linear time-varying or nonlinear. Controlling this kind of system is not an easy task, because it requires the development of efficient control boxes with sufficient computation frequency to satisfy the process specifications. Generally, the high number of operation performed, implies a significant energy consumption. This presents a disadvantage in the case of embedded systems in some applications like autonomous mobile systems.

Nonlinear observer design has been a topic of great research in the last decades Krener and Respondek (1985), Simon (2006), Gauthier et al. (1992), Rajamani (1998), Khalil (2002), Arcak and Kokotovic (2001), Kravaris et al. (2004), those ones are required especially when we don't have access to all the system states. The estimated states obtained by the observer are used for different purposes such as the design of control laws for guarantee some desired performance. For example, in the field of autonomous vehicles, measurement of some variables, such as longitudinal distances, velocities and accelerations of other nearby vehicles, requires significant expense in terms of energy. Some of sensors, such as slip angle and roll angle, can be extremely expensive to measure, requiring sensors that cost thousands of dollars Rajamani (2012), Rajamani et al. (2000). In addition, several important tasks cannot be performed due to purely and simply unavailability of sensors.

Designing sampled-data observers has attracted much attention in recent years due to the growth of the number of applications where the measurements are sampled, see Bouraoui et al. (2015) and references therein. One of the major difficulties in the design of sampled-data observers is how to enlarge the sampling intervals and ensure the convergence of the estimations. Some results on the output feedback sampled-data control of nonlinear systems using high-gain observers are proposed in Dabroom and Khalil (2001). In Ahmed-Ali et al. (2013), the problem of global exponential sampled-data observers design for nonlinear systems with delayed measurements is studied. The exact and Euler approximate models are used by the authors of Abbaszadeh and Marquez (2008) to design Robust H_∞ observer for sampled-data Lipschitz nonlinear systems.

The aim of this note is to analyze and to design observers for a class of nonlinear systems with sampled measurements by using the LPV approach. Firstly, the nonlinear part of the considered system is analyzed. This is done by using the differential mean value theorem (DMVT) which allows us to transform the nonlinear part of the estimation error dynamics into a linear parameter varying (LPV) system (see Zemouche and Boutayeb (2013)). Moreover, the existence and stability conditions of such observer are given as an LMI-based minimization problem taking into account the sampling period of the measure. In fact, maximize the sampling time enable to reduce the access frequency to the energy source which quantifies the energy consumption. Finally, a simulation example is given to illustrate the effectiveness of the proposed approach.

Notation: In the sequel $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ will denote the n dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. A^T denotes the transpose of matrix A. A is symmetric positive definite matrix if and only if $A^T = A$ and $A > 0$. Sym$\{X\}$ is used to denote $X^T + X$.

2. PROBLEM FORMULATION AND PRELIMINARY RESULTS

2.1 Preliminaries and useful Lemmas

In this section we introduce some definitions and preliminaries which will be of crucial use in the proposed approach

Definition 1. (Zemouche and Boutayeb (2013)) Consider two vectors

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 0, ..., n$, we define an auxiliary vector $X^{Z_i} \in \mathbb{R}^n$ corresponding to X and Z as follows:

$$\begin{cases} X^{Z_i} = \begin{pmatrix} z_1 \\ \vdots \\ z_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} \quad \text{for } i = 1, ..., n \\ X^{Z_0} = X \end{cases} \tag{1}$$

Lemma 1. (Zemouche and Boutayeb (2013)). Consider a continuous function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then, for all

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$$

there exist functions $\psi_j : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, ..., n$ such that

$$\Psi(X) - \Psi(Z) = \sum_{j=1}^{j=n} \psi_j\Big(X^{Z_{j-1}}, X^{Z_j}\Big)\mathrm{e}_n^\top(j)\Big(X - Z\Big) \tag{2}$$

where $\mathrm{e}_n(j)$ is the j^{th} vector of the canonical basis of $\mathbb{R}^n$.

Lemma 2. (Zemouche and Boutayeb (2013)). Consider a function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then, the two following items are equivalent:

- Ψ is γ_Ψ-Lipschitz with respect to its argument, i.e.:
$$\Big\|\Psi(X) - \Psi(Z)\Big\| \le \gamma_\Psi \Big\|X - Y\Big\|, \quad \forall\ X, Z \in \mathbb{R}^n \tag{3}$$
- for all $i, j = 1, ..., n$, there exist functions
$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
and constants $\underline{\gamma}_{\psi_{ij}} \le 0$, $\bar{\gamma}_{\psi_{ij}} \ge 0$, such that $\forall\ X, Z \in \mathbb{R}^n$,
$$\Psi(X) - \Psi(Z) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \psi_{ij} H_{ij}\Big(X - Z\Big) \tag{4}$$
and
$$-\gamma_\Psi \le \underline{\gamma}_{\psi_{ij}} \le \psi_{ij} \le \bar{\gamma}_{\psi_{ij}} \le \gamma_\Psi \tag{5}$$
where
$$\psi_{ij} \triangleq \psi_{ij}\Big(X^{Z_{j-1}}, X^{Z_j}\Big) \quad \text{and} \quad H_{ij} = \mathrm{e}_n(i)\mathrm{e}_n^\top(j)$$

Lemma 3. Let X and Y two given matrices of appropriate dimensions. the following inequality is true:

$$X^TY + Y^TX \le \mu X^TX + \frac{1}{\mu}Y^TY. \tag{6}$$

Lemma 4. (Gu (2000)). For any constant symmetric positive matrix $M \in \mathbb{R}^{m\times m}$, scalars t_1, t_2 and vector function $v : [t_1, t_2] \rightarrow \mathbb{R}^m$ then the following inequality holds (Jensen Inequality):

$$\Big(\int_{t_1}^{t_2} v^T(\beta)d\beta\Big)M\Big(\int_{t_1}^{t_2} v(\beta)d\beta\Big) \le (t_2 - t_1)\int_{t_1}^{t_2} v^T(\beta)Mv(\beta)d\beta \tag{7}$$

Remark 1. Lemma 4 can be viewed as a direct application of a variant of Halanay's inequality Hien et al. (2015) or the well-known Razumikhin's Theorem.

Finally, we introduce the following main lemma, which will be used to guarantee exponential convergence of the estimation error.

Lemma 5. If there exist a Lyapunov function $V(t, x(t))$ and positive scalars $\alpha > 0$, $\beta > 0$, $\beta < \alpha$ and $\varphi = (\alpha - \beta)e^{-\alpha\tau_{max}}$ such that

$$\frac{d}{dt}V(x(t)) \le -\alpha V(x(t)) + \beta V(x(t_k)) \tag{8}$$
$$\forall t \in [t_k, t_{k+1}[, \quad k \ge 0 \tag{9}$$

Then, the function $V(x(t))$ converges exponentially to zero. i.e.

$$V(x(t)) \le e^{-\varphi(t-t_k)}V(x(0)) \tag{10}$$

Proof. By multiplying the inequality (8) by $e^{\alpha t}$ and integrating from t_k to t, leads to

$$\int_{t_k}^{t} e^{\alpha t}\frac{d}{dt}V(x(t))dt \le \int_{t_k}^{t} -\alpha e^{\alpha t}V(x(t))dt + \int_{t_k}^{t} \beta e^{\alpha t}V(x(t_k))dt \tag{11}$$
$$\forall t \in [t_k, t_{k+1}[, \quad k \ge 0$$

Applying the integration by parts formula to the left hand side of the inequality (11), we get the following inequality

$$\Big(e^{\alpha t}V(x(t)) - e^{\alpha t_k}V(x(t_k))\Big) - \int_{t_k}^{t} \alpha e^{\alpha t}V(x(t))dt \le \int_{t_k}^{t} -\alpha e^{\alpha t}V(x(t))dt + \int_{t_k}^{t} \beta e^{\alpha t}V(x(t_k))dt \tag{12}$$
$$\forall t \in [t_k, t_{k+1}[, \quad k \ge 0$$

which can be simplified as

$$e^{\alpha t}V(x(t)) \le e^{\alpha t_k}V(x(t_k)) + \Big(\frac{\beta}{\alpha}e^{\alpha t}V(x(t_k)) - \frac{\beta}{\alpha}e^{\alpha t_k}V(x(t_k))\Big) \tag{13}$$
$$\forall t \in [t_k, t_{k+1}[, \quad k \ge 0$$

or

$$V(x(t)) \le \Big(e^{-\alpha(t-t_k)} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}e^{-\alpha(t-t_k)}\Big)V(x(t_k))$$
$$\forall t \in [t_k, t_{k+1}[, \quad k \ge 0. \tag{14}$$

Let $\varphi = (\alpha - \beta)e^{-\alpha\tau_{max}}$ since $\beta < \alpha$, then it can be easily shown that

$$e^{-\alpha(t-t_k)} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}e^{-\alpha(t-t_k)} \leq e^{-\varphi(t-t_k)} \tag{15}$$

from (14) and (15) and the fact that $V(x(t)) \in R^+$ is continuous we obtain the inequality (10). This ends the proof of Lemma 5.

2.2 System Description

For simplicity of the presentation and to explain well what we propose in this note, we consider the following triangular form of nonlinear systems as in Gauthier et al. (1992):

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \\ f(x) \end{bmatrix} \\ y = x_1(t_k) \end{cases} \tag{16}$$

with $f \ : \ \mathbb{R}^n \to \mathbb{R}$ satisfies the Lipschitz property formulated under the flowing form:

$$\Big| f(x_1 + \Delta_1, \ldots, x_n + \Delta_n) - f(x_1, \ldots, x_n) \Big| \leq \gamma_f \sum_{j=1}^{n} |\Delta_j| \ . \tag{17}$$

and $x \in \mathbb{R}^n$ denotes the state vector, $y \in \mathbb{R}$ the measurement output vector is sampled at instants t_k satisfy $0 \leq t_0 \leq \cdots \leq t_k \leq t_{k+1}$, the sampling interval $\tau_k = t_{k+1} - t_k$ is a positive constant satisfy $\tau_k < \tau_{max} \quad \forall k \geq 0$, where τ_{max} is the largest value of the sampling period τ_k.

System (16) can be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x) \\ y(t_k) = Cx(t_k), \end{cases} \tag{18}$$

with

$$B = [0 \ \ldots \ 0 \ 1]^T \ , \quad C = [1 \ 0 \ \ldots \ 0]$$

where the matrix A is defined as

$$(A)_{i,j} = \begin{cases} 1 \text{ if } j = i+1 \\ 0 \text{ if } j \neq i+1 \end{cases}.$$

To estimate the state of the system (18) subject to sampled measurements $y(t_k)$, we consider the following Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bf(\hat{x}) + L\Big(y(t_k) - C\hat{x}(t_k)\Big) \tag{19}$$

The dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ is then given by:

$$\dot{e}(t) = \big(A - LC\big)e(t) + B\big[f(x) - f(\hat{x})\big] + LC\Big(e(t) - e(t_k)\Big) \tag{20}$$

The aim consists in finding a matrix L that ensures exponential convergence of the estimation error $e(t)$ despite the availability of measures only at instants t_k.

3. MAIN RESULTS

3.1 LPV/LMI-Based Approach

In this section, the problem of designing an observer for a non-linear systems is considered, where the non-linear function satisfies the Lipschitz property defined as (17). Then, by using the Lemma 2, this allows us to transform the non-linear part in the dynamic of the estimation error as LPV term.

Since $f(.)$ is γ_f-Lipschitz, then following Lemma 2 there are functions

$$\psi_j \ : \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \ \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_j}$ and $\bar{\gamma}_{\psi_j}$, such that

$$f(x) - f(\hat{x}) = \Big[\sum_{j=1}^{j=n} \psi_j e_n^\top(j)\Big]e \tag{21}$$

and

$$\underline{\gamma}_{\psi_j} \leq \psi_j \leq \bar{\gamma}_{\psi_j} \tag{22}$$

where

$$\psi_j \triangleq \psi_j\Big(x_k^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big)$$

is defined as in Lemma 2. For the sake of brevity, we use only ψ_j instead of $\psi_j\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big)$.

Now, define the matrix function

$$\mathcal{A}\Big(\Psi\Big) = A + B\sum_{j=1}^{j=n} \psi_j e_n^\top(j), \ \forall \Psi \in \mathbb{R}^n \tag{23}$$

Consequently, the dynamics (20) can be rewritten as

$$\dot{e}(t) = \Big[\mathcal{A}\Big(\Psi\Big) - LC\Big]e(t) + LC\Big(e(t) - e(t_k)\Big) \tag{24}$$

According to (22), the vector parameter Ψ belongs to a bounded convex set $\mathcal{H}_n$ for which the set of vertices is defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \Big\{\Phi \in \mathbb{R}^n \ : \ \Phi_j \in \Big\{\underline{\gamma}_{\psi_j}, \bar{\gamma}_{\psi_j}\Big\}\Big\}. \tag{25}$$

In the following theorem, the problem of designing an observer of Lipschitz systems will be formulated as solving a LMI subject to some constraints.

Theorem 1. The observer (19) is asymptotically convergent if there exist a symmetric positive definite matrix $\mathcal{P}$ and a matrix $\mathcal{Y}$ of appropriate dimension such that the following LMI conditions hold:

$$\mathcal{A}(\Psi)^T\mathcal{P} - C^T\mathcal{Y}^T + \mathcal{P}\mathcal{A}(\Psi) - \mathcal{Y}C + \mathcal{Y}\mathcal{Y}^T + \mathcal{R} \leq -\mu I, \quad \forall \ \Phi \in \mathcal{V}_{\mathcal{H}_n} \tag{26}$$

where

$$\mathcal{R} = 2\tau_{\max}^2 \delta R^T R$$
$$1 < \delta, \quad R = [0, 1, 0, \ldots, 0]$$

Hence

$$(t - t_k) < \tau_{\max} \qquad \forall t \in [t_k, t_{k+1}[$$
$$0 < \beta = 2\tau_{\max}^2 \lambda_{max}(P)L_1^2$$
$$0 < \alpha_1 = \mu\lambda_{min}(P)$$
$$0 < \alpha_2 = 2\tau_{\max}(\delta - 1))$$
$$\beta < \min\{\alpha_1, \alpha_2\}$$

and, the observer gain is given by

$$L = \mathcal{P}^{-1}\mathcal{Y}.$$

Proof. To analyse the stability of the estimation error, we introduce the following candidate Lyapunov function

$$V(t) = V_1(t) + \delta V_2(t) \tag{27a}$$

where

$$V_1(t) = e^T(t)Pe(t) \tag{27b}$$

$$V_2(t) = \int_{t_k}^{t}\int_{\varsigma}^{t}\Big(e_2(s)\Big)^2 ds d\varsigma \tag{27c}$$

The derivative of $V_1(t)$ along the solution of (24) is given as follows

$$\frac{d}{dt}V_1(t) = e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\}\Big)e(t) + \underbrace{\text{Sym}\Big\{e^T(t)PLC\Big(e(t) - e(t_k)\Big)\Big\}}_{\Upsilon_1} \tag{28}$$

Applying Lemma 3 on Υ_1, where $\mu = 1$, leads to the following inequality

$$\Upsilon_1 \leq \Big(e(t) - e(t_k)\Big)^T C^T C\Big(e(t) - e(t_k)\Big) + \Big(e^T(t)PLL^TP^Te(t)\Big) \tag{29}$$

From (28) and (29), we obtain the following inequality

$$\frac{d}{dt}V_1(t) \leq e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T\Big)e(t) + \underbrace{\Big(e(t) - e(t_k)\Big)^T C^T C\Big(e(t) - e(t_k)\Big)}_{\Upsilon_2} \tag{30}$$

Since

$$\begin{aligned}\Upsilon_2 &= \Big(e_1(t) - e_1(t_k)\Big)^T\Big(e_1(t) - e_1(t_k)\Big)\\ &= \Big\|\Big(e_1(t) - e_1(t_k)\Big)\Big\|^2 = \Big\|\int_{t_k}^{t}\dot{e}_1(s)ds\Big\|^2\\ &\leq (t - t_k)\int_{t_k}^{t}\Big\|\dot{e}_1(s)\Big\|^2 ds\\ &\leq (t - t_k)\int_{t_k}^{t}\Big\|e_2(s) - L_1e_1(t_k)\Big\|^2 ds\\ &\leq 2\tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 + \Big(L_1e_1(t_k)\Big)^2 ds\end{aligned} \tag{31}$$

this leads to

$$\frac{d}{dt}V_1(t) \leq e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T)e(t) + 2\tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 + \Big(L_1e_1(t_k)\Big)^2 ds \tag{32a}$$

which is equivalent to

$$\frac{d}{dt}V_1(t) \leq e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T)e(t) + 2\tau_{\max}^2\Big(L_1e_1(t_k)\Big)^2 + 2\tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds \tag{32b}$$

The dynamic of the Lyapunov function $V_2(t)$ along the solution trajectories of (24) is

$$\begin{aligned}\frac{d}{dt}V_2(t) &= \tau_{\max}\Big(e_2(t)\Big)^2 - \int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds\\ &\leq 2\tau_{\max}^2\Big(e_2(t)\Big)^2 - 2\tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds\end{aligned} \tag{33}$$

then, by substituting (28) and (33) in (27a) we obtain the derivative of $V(t)$ defined in (27a) along the solution trajectories of (24) is given as

$$\begin{aligned}\frac{d}{dt}V(t) \leq{}& e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T)e(t)\\ &+ 2\tau_{\max}^2\delta\Big(e_2(t)\Big)^2 + 2\tau_{\max}^2\Big(L_1e_1(t_k)\Big)^2\\ &+ 2\tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds - 2\tau_{\max}\delta\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds\end{aligned}$$

From the above inequality, we obtain

$$\begin{aligned}\frac{d}{dt}V(t) \leq{}& e^T(t)\Big(\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T\\ &+ 2\tau_{\max}^2\delta R^TR\Big)e(t) + 2\tau_{\max}^2\Big(L_1e_1(t_k)\Big)^2\\ &- 2\tau_{\max}(\delta - 1)\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds\end{aligned} \tag{34}$$

Let $1 < \delta$, from (27c), we have

$$V_2(t) = \int_{t_k}^{t}\int_{\varsigma}^{t}\Big(e_2(s)\Big)^2 ds d\varsigma \leq \tau_{\max}\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds \tag{35}$$

and,

$$-2\tau_{\max}(\delta - 1)\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds \leq -\frac{3}{2}(\delta - 1)V_2(t) \tag{36}$$

Now, let

$$\text{Sym}\Big\{\Big(\mathcal{A}(\Psi) - LC\Big)^T P\Big\} + PLL^TP^T + 2\tau_{\max}^2\delta R^TR \leq -\mu I \tag{37}$$

then,

$$\frac{d}{dt}V(t) \leq -\mu e^T(t)e(t) - 2\tau_{\max}(\delta - 1)\int_{t_k}^{t}\Big(e_2(s)\Big)^2 ds + 2\tau_{\max}^2\Big(L_1e_1(t_k)\Big)^2$$

$$\frac{d}{dt}V(t) \leq -\alpha_1V_1(t) - \alpha_2V_2(t) + \beta V(t_k)$$

where

$$\begin{aligned}&0 < \beta = 2\tau_{\max}^2\lambda_{max}(P)L_1^2\\ &0 < \alpha_1 = \mu\lambda_{min}(P)\\ &0 < \alpha_2 = 2\tau_{\max}(\delta - 1)\end{aligned}$$

$$\frac{d}{dt}V(t) \leq -\mu V(t) + \beta V(t_k)$$

where $\alpha = \max\{-\alpha_1, -\alpha_2\}$

Then, according to Lemma 5, we have

$$V(x(t)) \leq e^{-\varphi(t-t_k)}V(x(t_k)) \qquad \forall t \in [t_k, t_{k+1}[, \quad k \geq 0 \tag{38}$$

where $\varphi = (\alpha - \beta)e^{-\alpha T}$ and $\beta < \alpha$, then $V(t)$ converge exponentially toward zero. This completes the theorem proof.

Remark 2. Let us distinguish between 2 cases. In the first case where $\alpha = \alpha_1$, we deduce from the fact $\beta < \alpha = \alpha_1$ that

$$\beta < \alpha_1 \Rightarrow \tau_{max} < T_1 = \sqrt{\frac{2\mu\lambda_{min}(P)}{3\lambda_{max}(P)L_1^2}} \tag{39}$$

In the second case where $\alpha = \alpha_2$, from the fact $\beta < \alpha = \alpha_2$ we deduce that

$$\beta < \alpha_2 \Rightarrow \tau_{max} < T_2 = \sqrt{\frac{(\delta - 1)}{\lambda_{max}(P) L_1^2}} \tag{40}$$

The obtained conditions (39) and (40) are summarised as

$$\tau_{max} < \min\{T_1, T_2\} \tag{41}$$

4. NUMERICAL EXAMPLE

In this section, the performance of the proposed observer (19) is tested using a numerical example. We consider a two-dimensional nonlinear system under the triangular form (16) given as follows

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [1 \ 0], \tag{42a}$$

and the nonlinear function $f(x) = [0 \ 0.3 * sin(x_1)]^T$ satisfies the Lipschitz property (17).

However, applying the proposed approach, we obtain

$$A(\alpha^1) = \begin{bmatrix} 0 & 1 \\ -L_f & -L_f \end{bmatrix}, \quad A(\alpha^2) = \begin{bmatrix} 0 & 1 \\ L_f & -L_f \end{bmatrix}, \tag{43a}$$

$$A(\alpha^3) = \begin{bmatrix} 0 & 1 \\ -L_f & L_f \end{bmatrix}, \quad A(\alpha^4) = \begin{bmatrix} 0 & 1 \\ L_f & L_f \end{bmatrix}, \tag{43b}$$

Then, by solving the LMI-based minimization problem proposed in Theorem 1 leads to deduces the matrix gain of the observer (19) and the sampling time. This can be done by using any toolbox dedicated for solving this kind of optimization problem.

A feasible solution of the optimization problem defined as (26) is given by the following matrices

$$P = \begin{bmatrix} 0.42729 & -0.45686 \\ -0.45686 & 0.5924 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.50133 \\ 0.19737 \end{bmatrix}, \tag{44a}$$

$$L = \begin{bmatrix} 8.7188 \\ 7.0571 \end{bmatrix}, \tag{44b}$$

for $\tau_{max} = 0.1s$ which ensures the stability of the estimation error.

To illustrate the effectiveness of the proposed method in this work, the evolution of the system states ant their estimates are plotted in figures 1 and 2. In addition, the sampled measurement and sampled output of the observer are also plotted in (3).

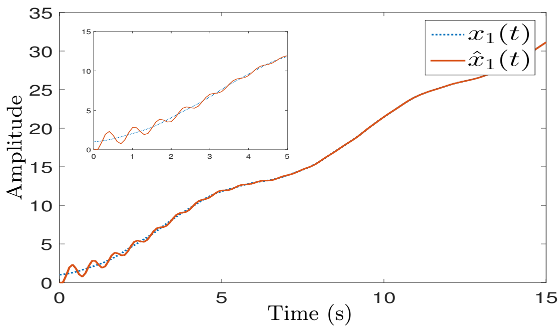

Fig. 1. The evolution of $x_1(t)$ and its estimate $\hat{x}_1(t)$

Due to scale effect, the figures 1, 2 and 3 are zoomed in order to show the convergence of the estimated states to the real states.

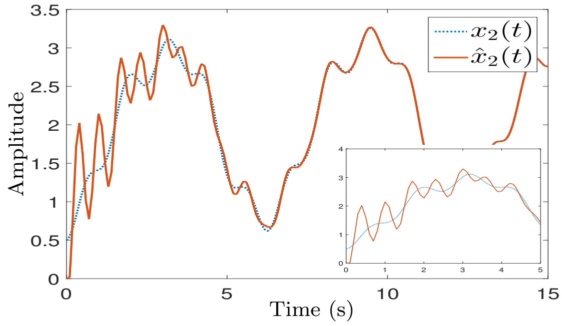

Fig. 2. evolution of $x_2(t)$ and its estimate $\hat{x}_2(t)$

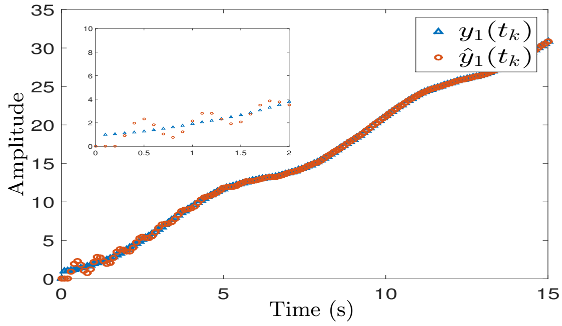

Fig. 3. The evolution of $y_1(t_k)$ and its estimate $\hat{y}_1(t_k)$

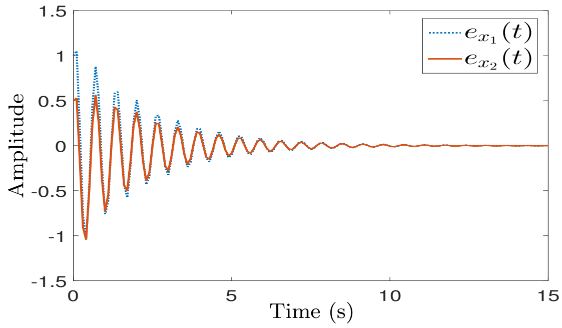

Fig. 4. The evolution of the estimation error

$$e_x(t) = x(t) - \hat{x}(t)$$

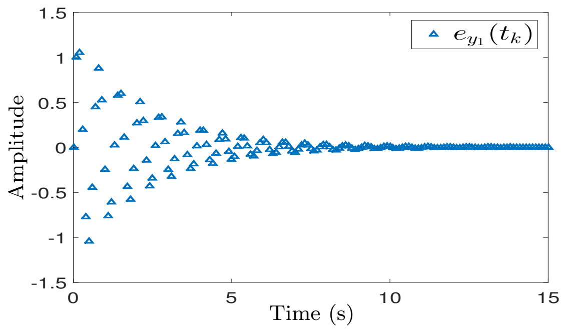

Fig. 5. The evolution of the estimation error

$$e_y(t_k) = y(t_k) - \hat{y}(t_k)$$

5. CONCLUSION

In this work, a design methodology of observers for a class of Lipschitz nonlinear dynamical systems with sampled measurements is investigated. Firstly, the nonlinear part of the estimation error dynamics is reformulated by using the differential mean value theorem (DMVT). Then, to ensure the stability of the estimation error subject to sampled measurements, an LMI-based optimization problem is addressed such that, the measurements sampling period is taken as a decision parameter. This allows to reduce the energy consumption. Finally, a simulation result is presented to illustrate the performance of the proposed methodology.

As future work, we aim to extend the results of this paper to other interesting estimation problems, namely for systems with noises and uncertainties. We also plan to extend the main ideas of this paper to systems with time-varying sampling period and event-triggered measurements.

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