Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Reduced-Complexity Affine Representation for Takagi-Sugeno Fuzzy Systems

Amine Dehak *, **Anh-Tu Nguyen**‡ *, **Antoine Dequidt** *, **Laurent Vermeiren** *, **Michel Dambrine** *

* *Laboratory LAMIH UMR CNRS 8201, Université Polytechnique Hauts-de-France, Valenciennes, France*
‡ *Corresponding author. Email: tnguyen@uphf.fr*

Abstract: This paper presents a systematic approach to reduce the complexity of sector nonlinearity TS fuzzy models using existing linear dependencies between local linear submodels. The proposed approach results in a decrease of the fuzzy model rules from 2^p to $p+1$ rules while maintaining equivalence to the TS fuzzy model. An LMI formulation is presented to obtain conditions for stability analysis and stabilizing controllers design with some examples to offer a comparison between the two models. The main purpose of reduced-complexity models is to keep the design and the structure of the nonlinear control and observer schemes as simple as possible for real-time implementation, especially when dealing with highly nonlinear systems with a very large number of premise variables. Two real-world robotics examples are provided to highlight the interests and the curent limitations of the proposed approach.

1. INTRODUCTION

Nonlinear control presents numerous challenges due to the complexity involved in the identification, modeling and control design for real physical systems endowed with nonlinear proprieties. Among the proposed control approaches available in literature stands the model-based fuzzy control methodology as a systematic approach for the control of nonlinear systems. It relies on the use of Takagi-Sugeno (TS) fuzzy models first introduced in Takagi and Sugeno (1985). In the last three decades, TS fuzzy model-based approaches have attracted a lot of attention from both academic and engineering researchers Lendek et al. (2011); Nguyen et al. (2014); Nguyen et al. (2019b).

A TS fuzzy model can be represented by a set of If-Then fuzzy rules with consequent parts being local linear representations. A state-space form is obtained via a convex sum of several local linear submodels with weights called membership functions. Proven to constitute a class of universal approximators Tanaka and Wang (2004), the fuzzy model approach provides a powerful solution for development of function approximation, system identification as well as systematic techniques to stability analysis and controller design. In fact, for sufficiently smooth nonlinear systems, an equivalent representation can be obtained semiglobally using the TS fuzzy modeling approach.

Recent contributions in the field focus on introducing new LMI conditions to improve feasibility and reduce the conservatism by introducing nonquadratic Lyapunov functions Lee et al. (2012); Guerra and Vermeiren (2004); Mozelli et al. (2009); Nguyen et al. (2019a), multiple-sum relaxation approaches Sala and Ariño (2007); Coutinho et al. (2020). However, an underlying problem with the TS fuzzy modeling approach is the number of local linear submodels used to build the state model as it exponentially increases in function of the number of the premise variables. For example, if the TS fuzzy model is based on p nonlinear terms then the number of fuzzy rules is 2^p. The complexity introduced by this increase can render the application of TS fuzzy approach limited to systems with a small number of nonlinearities. In this regard, we propose a reduced complexity model with $p+1$ fuzzy rules obtained directly from an initial TS fuzzy model using existing linear dependencies between the local submodels of the sector nonlinearity approach.

The paper is organized as follows. In Section 2, from the sector nonlinearity approach, we show the linear dependencies existing between the obtained local linear submodels. These dependencies allow for a substitution of a number of local linear submodels as an affine combination of the remaining submodels. Section 3 presents a systematic approach to obtain the reduced complexity model as an affine combination of a chosen $p+1$ local linear submodels of the sector nonlinearity model. We then present the expressions of the new weighting functions for the affine representation. Section 4, the LMI-based conditions for stability analysis and stabilization are given for the reduced-complexity model. A transformation from affine combination to conical sum representation is adopted to derive LMI conditions. The numerical experiments are presented in Section 5 to show the interests and the limitations of the proposed approach.

2. LINEAR DEPENDENCIES IN THE CLASSICAL SECTOR NONLINEARITY APPROACH

This section recalls the sector nonlinearity fuzzy approach to exhibit the existing nonlinear dependencies between the local linear submodels. Let us consider the class of nonlinear systems described by a state-space representation of the form

$$\dot{x} = M(x)x, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and M a mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}$. Let $z_1(x), \dots, z_p(x)$ be p scalar functions such that all entries of the matrix $M(x)$ can be rewritten as an affine function of these functions, i.e

$$M(x) = A(z(x))$$

where $A: \mathbb{R}^p \to \mathbb{R}^{n\times n}$ is an affine function of $z_i(x)$, for $\forall i \in 1, \dots, p$. We assume that the functions z_i are bounded on a compact set of the state space

$$\underline{z}_k \le z_k(x) \le \bar{z}_k, \quad \forall k \in \{1,\ 2, \dots, p\}.$$

The sector-nonlinearity approach Tanaka and Wang (2004) associates with each premise variable z_k two membership functions ω_k^0 and ω_k^1 given by

$$\omega_k^0(z_k) = \frac{\bar{z}_k - z_k}{\bar{z}_k - \underline{z}_k}, \quad \omega_k^1(z_k) = \frac{z_k - \underline{z}_k}{\bar{z}_k - \underline{z}_k}$$

Note that $\omega_k^0, \omega_k^1 \in [0,1]$ and $\omega_k^0 + \omega_k^1 = 1$. The nonlinear system (1) is then represented by the fuzzy If-Then fuzzy rule as follows:

i-th rule:

$$\text{If } z_1 \text{ is } \Omega_1^{i_1},\ z_2 \text{ is } \Omega_2^{i_2},\ \dots,\ z_p \text{ is } \Omega_p^{i_p}. \text{ Then } \dot{x} = A_i x, \tag{2}$$

for $i = 1, 2, \dots, 2^p$, where $(i_1, i_2, \dots, i_p) \in \{0,\ 1\}^p$ are the digits of the binary representation of $i-1$, *i.e.*,

$$i - 1 = i_1 \times 2^{p-1} + \dots + i_{p-1} \times 2 + i_p,$$

and, for $k \in \{1, \dots, p\}$ and $i_k \in \{0,\ 1\}$, $\Omega_k^{i_k}$ is a fuzzy set admitting $\omega_k^{i_k}$ as membership function.

Using the inference scheme proposed by Takagi and Sugeno in Takagi and Sugeno (1985) leads to the representation of system (1) as

$$\dot{x}(t) = \sum_{i=1}^{2^p} h_i(z(t)) A_i x(t), \tag{3}$$

where $z(t)$ is the p-vector $(z_1(x(t)), \dots, z_p(x(t)))$, $h_i(.)$ are the normalized membership functions given by

$$h_i(z) = \prod_{k=1}^{p} \omega_k^{i_k}(z_k), \tag{4}$$

and

$$A_i = A(z^{[i]}) \quad \text{for } z^{[i]} \text{ such that} \tag{5}$$
$$\omega_1^{i_1}(z_1^{[i]}) = \omega_2^{i_2}(z_2^{[i]}) = \dots = \omega_p^{i_p}(z_p^{[i]}) = 1.$$

The normalized membership functions verify the convex-sum property $\sum_{i=1}^{r} h_i(.) = 1$, $h_i(.) \ge 0$.

The characterization given for the matrices A_i states that the components of the vector $z^{[i]}$ will take the values $\underline{z}_k$ or $\bar{z}_k$ according to the i^{th} rules of the fuzzy model so that

$$\begin{aligned}
A_1 &= A(\underline{z}_1, \underline{z}_2, \underline{z}_3, ..., \underline{z}_{p-2}, \underline{z}_{p-1}, \underline{z}_p)\\
A_2 &= A(\underline{z}_1, \underline{z}_2, \underline{z}_3, ..., \underline{z}_{p-2}, \underline{z}_{p-1}, \bar{z}_p)\\
A_3 &= A(\underline{z}_1, \underline{z}_2, \underline{z}_3, ..., \underline{z}_{p-2}, \bar{z}_{p-1}, \underline{z}_p)\\
&\vdots\\
A_{2^{p-1}} &= A(\underline{z}_1, \bar{z}_2, \bar{z}_3, ..., \bar{z}_{p-2}, \bar{z}_{p-1}, \bar{z}_p)\\
A_{2^{p-1}+1} &= A(\bar{z}_1, \underline{z}_2, \underline{z}_3, ..., \underline{z}_{p-2}, \underline{z}_{p-1}, \underline{z}_p)\\
&\vdots\\
A_{2^p} &= A(\bar{z}_1, \bar{z}_2, \bar{z}_3, ..., \bar{z}_{p-2}, \bar{z}_{p-1}, \bar{z}_p)
\end{aligned} \tag{6}$$

Let $\ell \in \mathbb{N}$ and $j \in \{1, \dots, p)$ such that $(\ell+1)2^j \le 2^p$, then the vertices $z^{[\ell 2^j+1]}$ and $z^{[(\ell+1)2^j]}$ share the same $p-i$ first components. The i last components all being fixed to the minimum values of the corresponding $\underline{z}_k$ for $z^{[\ell 2^j+1]}$ or to the corresponding maximal values for $z^{[(\ell+1)2^j]}$, which implies that the differences $A_{(\ell+1)2^j} - A_{\ell 2^j+1}$ are independent of ℓ:

$$\begin{aligned}
A_2 - A_1 &= A_4 - A_3 = \dots = A_{2^p} - A_{2^p-1}\\
A_4 - A_1 &= A_8 - A_5 = \dots = A_{2^p} - A_{2^p-3}\\
&\vdots\\
A_{2^{p-1}} - A_1 &= A_{2^p} - A_{2^{p-1}+1}
\end{aligned} \tag{7}$$

These linear dependencies allow to substitute a number of matrices A_i as linear combinations of the remaining matrices. The next section introduces a systematic substitution method to obtain a significant complexity-reduction for TS models.

3. REDUCED-COMPLEXITY REPRESENTATION OF TS FUZZY MODELS

Multiple choices are available to substitute a number of matrices A_i as a unitary linear combination of the remaining matrices using the relation (7). This section presents a specific choice for a systematic substitution resulting in a reduced complexity model with $r_a = p+1$ vertices compared to $r = 2^p$ of the TS fuzzy model (3). The modeling reduction procedure is summarized in the following theorem.

Theorem 1. Consider a nonlinear system (1), represented by the $r = 2^p-$rule TS fuzzy model (3), then an equivalent model is given by

$$\dot{x}(t) = \sum_{j\in\mathcal{J}} \tilde{\omega}_j(z(x)) \mathcal{A}_j x(t), \tag{8}$$

where the set $\mathcal{J} = \{1, 2, 3, 7, \dots, 2^p - 1\}$ has $p+1$ elements, and the state-space matrices $\mathcal{A}_j$, for $\forall j \in \mathcal{J}$, are taken in the spanning set of all local linear matrices A_i, for $\forall i \in \{1, 2, \dots, 2^p\}$, of the TS fuzzy model (3). The scalar functions $\tilde{\omega}_j$, for $\forall j \in \mathcal{J}$, are constructed from the unitary linear combinations of the normalized membership functions (4), satisfying $-1 \le \tilde{\omega}_j(z(x)) \le 1$ and $\sum_{j\in\mathcal{J}} \tilde{\omega}_j = 1$.

Proof. It is sufficient to prove that every matrix A_i can be rewritten as a unitary linear combination of elements in $\mathcal{S} = \{A_i:\ i \in \mathcal{J}\}$. The general expression for the linear dependencies in (7) is given by

$$A_{2^k} - A_1 = A_{2^k\alpha} - A_{2^k(\alpha-1)+1}, \tag{9}$$

for $k \in \{1, 2, 3, \dots, p-1\}$ and $\alpha \in \{1, 2, 3, \dots, 2^{p-k}\}$. For $k = 1$, we get

$$A_2 - A_1 = A_{2^l} - A_{2^l-1}$$

for $l \in \{1, 2, \dots, p\}$, which yields

$$A_{2^l} = A_{2^l-1} + A_2 - A_1 \tag{10}$$

Thus, for all $k \in \{2, 3, \dots, p\}$, the matrices A_{2^k} can be substituted as a unitary linear combination of elements in $\mathcal{S}$ yielding a total of $p-1$ substitutions. One can easily verify that for $k, l, \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$

$$\begin{aligned}
A_{2^l-1} = A_{2^k\alpha+1} &\Rightarrow 2^l - 1 = 2^k\alpha + 1\\
&\Rightarrow 2^l = 2(2^{k-1}\alpha + 1)\\
&\Rightarrow 2^{l-1} = 2^{k-1}\alpha + 1\\
&\Rightarrow (l = 1, \alpha = 0) \text{ or } (k = 1, \alpha = 2^{l-1} - 1)
\end{aligned}$$

This means that among equations (9) only the first set (*i.e.*, for $k = 1$) contains elements of $\mathcal{S} \setminus \{A_1\}$.

The k-th set of equations (9) is written with 2^{p+1-k} matrices ($p - k + 1$ of them being of the form A_{2^l}) and involve all of the matrices appearing in the $k + 1$ set of equations. As a result, the number of matrices involved in the k-th set of equations which are not of the form A_{2^l} and not present in the following set of equations is equal to

$$2^{p+1-k} - (p - k + 1) - [2^{p-k} - (p - k)] = 2^{p-k} - 1.$$

Combining (10) with the general expression (9), for $k \in \{2, 3, \ldots, p - 1\}$, each set of equations allows the substitution of $2^{p-k} - 1$ terms as unitary linear combination of elements in $\mathcal{S}$ starting from $k = p-1$ with the substitution of

$$\begin{aligned} A_{2^{p-1}+1} &= A_{2^p} - A_{2^{p-1}} + A_1 \\ &= A_{2^p-1} + \cancel{A_2} - \cancel{A_1} - A_{2^{p-1}-1} - \cancel{A_2} + \cancel{A_1} + A_1 \\ &= A_{2^p-1} - A_{2^{p-1}-1} + A_1. \end{aligned}$$

Substituting successively to $k = 2$ leads to a total number of $\sum_{k=2}^{p-1} 2^{p-k} - 1 = 2^{p-1} - p$ substitutions. None of the substituted matrices are elements of $\mathcal{S}$ since the elements of $\mathcal{S} \setminus \{A_1\}$ are only involved in the first set of equations. The $p+1$ elements of $\mathcal{S}$ are not substituted, then for $k = 1$ we have $2^{p-1}-1-(p+1) = 2^{p-1}-p$ substitutions. Finally, the total number of substitutions amounts to

$$p - 1 + 2^{p-1} - p + 2^{p-1} - p = 2^p - p - 1,$$

adding to that the $p+1$ elements of $\mathcal{S}$ we obtain the number of rules $r = 2^p$ of the TS fuzzy model. Replacing each matrix $A_i \notin \mathcal{S}$ in (3) with its unitary linear combination, we obtain the result (8) where $\tilde{\omega}_j$ is a unitary linear combination of the normalized membership functions h_i of the TS fuzzy model (3). Since the h_i verifies the convex sum propriety (*i.e.*, $h_i \geq 0$, $\sum_{i=1}^{r} h_i = 1$), then $-1 \leq \tilde{\omega}_j \leq 1$, $\sum_{j\in\mathcal{J}} \tilde{\omega}_j = \sum_{i=1}^{r} h_i = 1$, for $\forall j \in \mathcal{J}$. This concludes the proof.

We further provide expressions to systematically obtain the reduced-complexity representation (8) of the TS fuzzy model (3). For $p \geq 2$ premise variables, we have

$$\tilde{\omega}_1 = \sum_{l=1}^{2^{p-2}} h_{1+2^2(l-1)} - \sum_{l=1}^{2^{p-2}} h_{4+2^2(l-1)}, \tag{11}$$

$$\tilde{\omega}_2 = \sum_{i=1}^{2^{p-1}} h_{2i}, \quad \tilde{\omega}_{2^p-1} = \sum_{i=2^{p-1}+1}^{2^p} h_i. \tag{12}$$

Moreover, for $k \in \{2, 3, \ldots, p\}$, we have $\tilde{\omega}_{2^k-1} = \sum_{l=1}^{2^{p-k-1}} \Xi_l$ with

$$\Xi_l = \sum_{i=2^{k-1}+1}^{2^k} h_{i+(l-1)2^{k+1}} - \sum_{i=2^k+1}^{2^{k+1}-2^{k-1}} h_{i+(l-1)2^{k+1}}.$$

Remark 1. The expressions of $\tilde{\omega}_i$ are obtained using the linear dependency expression (9) to determine for each substitution of $A_i \notin \mathcal{S}$ the corresponding elements $\mathcal{A}_i \in \mathcal{S}$ of the unitary linear combination and their coefficients. Then, the inherent relationship between each occurrence of $\mathcal{A}_i$ in each substitution with the corresponding coefficient can be deduced.

Remark 2. While the proposed approach results in a reduced-complexity model, the equivalence to the nonlinear system is maintained $M(x) = \sum_{j\in\mathcal{J}} \tilde{\omega}_j(z(x))\mathcal{A}_j$.

4. QUADRATIC REDUCED-COMPLEXITY STABILITY ANALYSIS AND STABILIZATION

To obtain LMI-based conditions for the stability analysis of the reduced-complexity model, we propose a transformation to the representation (8) from an affine combination of elements in $\mathcal{S}$ to a conical combination while maintaining the equivalence to the nonlinear system.

From (11), we have $0 \leq \tilde{\omega}_2 \leq 1$ and $0 \leq \tilde{\omega}_p \leq 1$, and $-1 \leq \tilde{\omega}_k \leq 1$, for $k \in \mathcal{J} \setminus \{2, 2^p - 1\}$. To obtain a conical combination we suggest a normalization of $\tilde{\omega}_j$ such that

$$\sum_{j\in\mathcal{J}} \tilde{\omega}_j(z(x))\mathcal{A}_j = \mathcal{A}_0 + \sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))\mathcal{A}_j \tag{13}$$

with $\tilde{h}_2 = \tilde{\omega}_2$, $\tilde{h}_p = \tilde{\omega}_p$, $\tilde{h}_k = \tilde{\omega}_k+1$, for $\forall k \in \mathcal{J}\setminus\{2, 2^p-1\}$, and $\mathcal{A}_0 = -\sum_{j\in\mathcal{J}\setminus\{2,2^p-1\}} \mathcal{A}_j$. Note that

$$\tilde{h}_j \geq 0, \quad \sum_{j\in\mathcal{J}} \tilde{h}_j = p, \quad \forall j \in \mathcal{J}.$$

The reduced-complexity conical sum representation of the TS fuzzy system (3) is given by

$$\dot{x}(t) = \mathcal{A}_0 x(t) + \sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))\mathcal{A}_j x(t) \tag{14}$$

Theorem 2. Consider a TS fuzzy system (1) and its reduced-complexity conical sum representation (14). If there exist a definite positive matrix $P > 0$ and a matrix Q such that

$$\begin{aligned} &\mathcal{A}_0^T P + P\mathcal{A}_0 - Q < 0 \\ &\mathcal{A}_j^T P + P\mathcal{A}_j + \frac{1}{p} Q < 0, \quad \forall j \in \mathcal{J}. \end{aligned} \tag{15}$$

Then, the zero-solution of system (1) is asymptotically stable in the sense of Lyapunov.

Proof. Consider a Lyapunov function $V(x) = x^T P x$. Then, the time-derivative of this function is given by

$$\dot{V}(x(t)) = x^T(\mathcal{A}_0^T P + P\mathcal{A}_0 + \sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))(\mathcal{A}_j^T P + P\mathcal{A}_j))x.$$

If condition (15) is verified, then we have

$$\begin{aligned} &\mathcal{A}_0^T P + P\mathcal{A}_0 - Q < 0 \\ &\sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))(\mathcal{A}_j^T P + P\mathcal{A}_j) + \frac{1}{p}\sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))Q < 0. \end{aligned}$$

Since $\sum_{j\in\mathcal{J}} \tilde{h}_j(z(x)) = p$, it follows that

$$\sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))(\mathcal{A}_j^T P + P\mathcal{A}_j) + \cancel{Q} + \mathcal{A}_0^T P + P\mathcal{A}_0 - \cancel{Q} < 0,$$

which yields $\dot{V}(x(t)) < 0$. This concludes the proof.

We consider now a nonlinear system with p nonlinear terms and the following TS fuzzy representation:

$$\dot{x}(t) = \sum_{i=1}^{2^p} h_i(z(x))(A_i x(t) + B_i u(t)). \tag{16}$$

Using the same fuzzy rules as in (2), the same linear dependencies expressed in (7) can be found using the representation (6) with $\bar{A}_i = [A(z)\ B(z)]$. The resulting reduced-complexity model is given by

$$\dot{x}(t) = \sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))\ (\mathcal{A}_j x(t) + \mathcal{B}_j u(t)) + \Sigma_0(t) \qquad (17)$$

where $\Sigma_0(t) = \mathcal{A}_0 x(t) + \mathcal{B}_0 u(t)$ with

$$\mathcal{A}_0 = -\sum_{j\in\mathcal{J}\setminus\{2,2^p-1\}} \mathcal{A}_j,\quad \mathcal{B}_0 = -\sum_{j\in\mathcal{J}\setminus\{2,2^p-1\}} \mathcal{B}_j.$$

The following theorem provide LMI-based stabilization conditions for reduced-complexity system (17).

Theorem 3. Consider a TS fuzzy model (16) and its reduced-complexity conical sum representation (17). If there exist positive definite matrix $P > 0$, matrices M_j and Q_j, for $\forall j \in \mathcal{J}$, such that

$$\begin{aligned} & A_0^T P + PA_0 + \mathcal{B}_0 M_j + M_j^T \mathcal{B}_0^T - Q_j < 0 \\ & X_{ii} < 0, \quad \forall i \in \mathcal{J} \\ & \frac{1}{p} X_{ii} + \frac{1}{2}(X_{ij} + Xji) < 0, \quad \forall i, j \in \mathcal{J},\ i \neq j \end{aligned} \qquad (18)$$

with $X_{ij} = \mathcal{A}_i^T P + P\mathcal{A}_i + \mathcal{B}_i M_j + M_j^T \mathcal{B}_i^T + \frac{1}{p^2} Q_j$. Then, the state-feedback control law $u(t) = \frac{1}{p}\sum_{j\in\mathcal{J}} \tilde{h}_j(z(x))K_j x(t)$, with control gains $K_j = M_j P^{-1}$, is a stabilizing with respect to system (17).

Proof. Using the same argument as in Theorem 2 and recalling that $\sum_{j\in\mathcal{J}} \tilde{h}_j = p$, the proof steams directly from the Lyapunov stability condition with the quadratic Lyapunov function $V(x) = x^T P^{-1} x$, and the relaxation result in Tuan et al. (2001).

Remark 3. It is important to note that the system form (17) is *strictly* equivalent to the TS fuzzy representation (16). Note also that although the convex sum propriety of functions $\tilde{h}_j$, for $j \in \mathcal{J}$, in (17) is lost, the conical sum $\sum_{j\in\mathcal{J}} \tilde{h}_j = p$ still allows deriving LMI-based design conditions as indicated in Theorem 3.

5. NUMERICAL EXPERIMENTS

Two real-world examples are given in this section to show the interest of the proposed approach. Both examples are concerned with the stability study of $n-$DoF robot manipulators, whose dynamics can be expressed as Spong and Vidyasagar (2008):

$$M(q)\ddot{q} + N(q,\dot{q})\dot{q} + G(q) = u. \qquad (19)$$

where $q \in \mathbb{R}^n$ is the vector of generalized coordinates in joint space, $u \in \mathbb{R}^n$ is the vector of generalized control forces, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $N(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis/centripetal matrix plus the viscous friction coefficients of the joints, and $G(q) \in \mathbb{R}^n$ represents the generalized gravity forces. Note that the dynamics (19) can account for many types of robot manipulators. Since the vector field $G(q)$ is smooth, we can then parameterize $G(q) = P(q)q$.

Let us denote $x = \left[q^T\ \dot{q}^T\right]^T$. The manipulator dynamics (19) can be rewritten in the following descriptor form:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A(x) & -E(x) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \qquad (20)$$

where

$$A(x) = \begin{bmatrix} O & I \\ -P(q) & -N(q,\dot{q}) \end{bmatrix}, \quad E(x) = \begin{bmatrix} I & 0 \\ 0 & M(q) \end{bmatrix}.$$

For illustrations, we consider the following cases where $n = 2$ for Example 1 and $n = 3$ for Example 2. All optimization problems are performed with YALMIP toolbox and SDPT3 solver Löfberg (2004), on a computer equipped with Intel(R) Xeon(R) CPU X5670 2.93GHz and a 12288MB RAM.

5.1 Example 1: 2-DoF Robot Manipulator

The characteristics matrices of the robot manipulator in this case are given by

$$M(q) = \begin{bmatrix} c_1 + 2c_2\cos q_2 & c_3 + c_2\cos q_2 \\ c_3 + c_2\cos q_2 & c_3 \end{bmatrix},$$

$$N(q,\dot{q}) = \begin{bmatrix} 2c_2\dot{q}_2\sin q_2 & c_2\dot{q}_2\sin q_2 \\ -c_2\dot{q}_1\sin q_2 & 0 \end{bmatrix},$$

$$P(q) = \begin{bmatrix} -c_4\frac{\sin q_1}{q_1} - c_5\frac{\sin q_{12}}{q_{12}} & -c_5\frac{\sin q_{12}}{q_{12}} \\ -c_5\frac{\sin q_{12}}{q_{12}} & -c_5\frac{\sin q_{12}}{q_{12}} \end{bmatrix},$$

where $q_{12} = q_1 + q_2$, $c_1 = m_1 r_1^2 + I_1 + m_2 L_1^2 + m_2 r_2^2 + I_2$, $c_2 = m_2 L_1 r_2$, $c_3 = m_2 r_2^2 + I_2$, $c_4 = m_1 g r_1 + m_2 g L_1$, $c_5 = m_2 g r_2$. The details on the parameters and the TS fuzzy representation can be found in Nguyen et al. (2019). The five premise variables of this 2-DoF robot are defined as follows:

$$z_1 = \cos q_2,\ z_2 = -c_4\frac{\sin q_1}{q_1} - c_5\frac{\sin q_{12}}{q_{12}},$$

$$z_3 = -c_5\frac{\sin q_{12}}{q_{12}},\ z_4 = c_2\dot{q}_2\sin q_2,\ z_5 = -c_2\dot{q}_1\sin q_2.$$

From the 32-fuzzy rule TS model, the reduced-complexity model with 6 linear subsystems can be easily derived, whose details are omitted here for conciseness. To study the conservatism obtained with both modeling approaches, we consider both the open-loop stability analysis (with $u \equiv 0$) and the closed-loop control design.

For stability analysis, we solve LMI conditions in Theorem 2 for reduced-complexity model and the classical quadratic-based stability conditions Tanaka and Wang (2004) for the 32-fuzzy rule TS model. For illustrations, we defined $a = m_2$ with $1 \leq a \leq 7$, and $b = f_{v1} = 0.5f_{v2}$ with $0 \leq b \leq 50$. Fig. 1 shows the feasibility domains obtained with both cases. Observe that for stability analysis, without additional degrees of freedom from the LMI-based formulation, the reduced-complexity model may induce extra conservatism compared to the classical sector nonlinearity modeling approach.

For control design, we solve LMI conditions in Theorem 3 for reduced-complexity model and the classical quadratic-based stability conditions Tanaka and Wang (2004) for the 32-fuzzy rule TS model. As shown in Fig. 2, despite a significant complexity reduction, both control approaches lead to the same feasibility domain with the additional degree of freedom coming from the control gains.

Table 1 provides the number of scalar decision variables, the number of constraints as well as the SDP blocks of the LMI conditions for a single value of parameters a and b comparing the stability analysis and stabilization for both approaches on the 2-DoF serial robot.

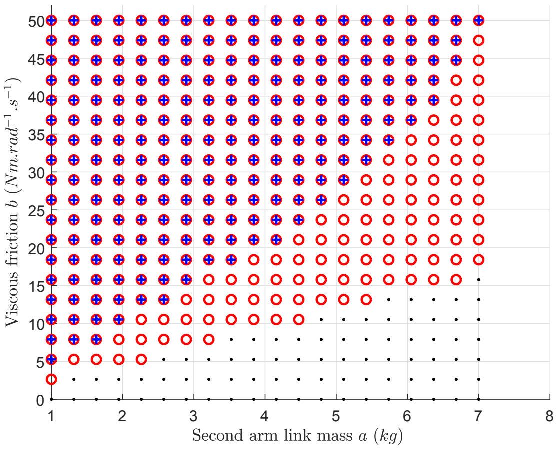

Fig. 1. Feasibility stability domain of 2-DoF robot: "$\circ$" classical TS fuzzy approach, "+" reduced-complexity conical sum approach.

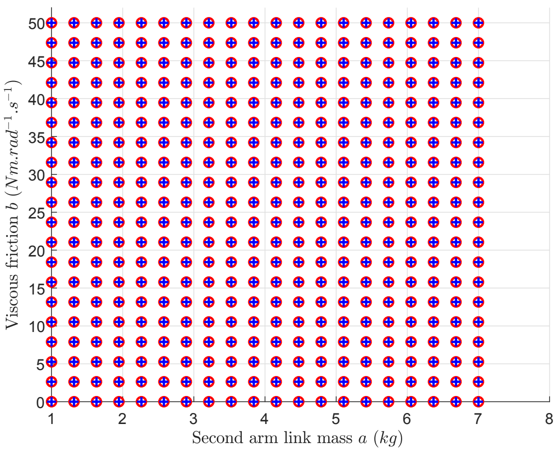

Fig. 2. Feasibility stabilisation domain of 2-DoF robot: "$\circ$" classical TS fuzzy approach, "+" reduced-complexity conical sum approach.

Table 1. Computational burden for 2-DoF serial robot.

Problem	Nb. variables	Nb. constraints	SDP blocks
Analysis-TS	260	42	33
Analysis-Reduced	60	78	8
Design-TS	260	298	33
Design-Reduced	60	142	8

5.2 Example 2: 3-DoF Robot Manipulator

The considered 3-DoF serial manipulator is composed of 3-arm links which are contained in a vertical plane. The characteristic matrices of this robot are given by

$$M(q) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ * & 2c_{11}z_{11} + c_{14} & c_{11}z_{11} + c_{15} \\ * & * & c_{15} \end{bmatrix}$$

$$N(q,\dot{q}) = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}$$

$$P(q) = \begin{bmatrix} c_1z_1 + c_2z_2 + c_3z_3 & c_2z_2 + c_3z_3 & c_3z_3 \\ * & c_2z_2 + c_3z_3 & c_3z_3 \\ * & * & c_3z_3 \end{bmatrix}$$

where $*$ denotes the symmetric matrix elements, and

$N_{11} = c_6z_6 + c_7z_7 + c_8z_8 + c_9z_9 + f_{v1}$,
$N_{12} = c_4z_4 + c_7z_7 + c_8z_8 + c_9z_9$,
$N_{13} = c_8z_8 + c_9z_9$,
$N_{21} = c_5z_5 + c_6z_6 + c_7z_7 + c_8z_8$,
$N_{22} = c_7z_7 + c_8z_8 + 2f_{v1}$, $\quad N_{23} = c_8z_8$,
$N_{31} = c_6z_6 + c_7z_7$, $\quad N_{32} = c_7z_7$,
$M_{11} = 2(c_{10}z_{10} + c_{11}z_{11} + c_{12}z_{12}) + c_{13}$,
$M_{12} = c_{10}z_{10} + 2c_{11}z_{11} + c_{12}z_{12} + c_{14}$,
$M_{13} = c_{11}z_{11} + c_{12}z_{12} + c_{15}$, $\quad N_{33} = 2f_{v1}$.

The constant parameters c_i, $i \in \{1, 2, \ldots, 10\}$ given in Table 4, while the viscous friction coefficients f_{v1} and f_{v2} are linked to the simulation varying parameter b. Similar to the Example 1, we define $a = m_3$ and $b = f_{v1} = 0.5f_{v2} = 0.5f_{v3}$, with $5 \leq a \leq 15$ and $0 \leq b \leq 100$. Note that there are $p = 12$ premise variables for this 3-DoF manipulator, which are defined in Table 2.

Table 2. Premise variables of 3-DoF serial robot.

Premise variable	Expression
z_1	$\sin(q_1)/q_1$
z_2	$\sin(q_1+q_2)/(q_1+q_2)$
z_3	$\sin(q_1+q_2+q_3)/(q_1+q_2+q_3)$
z_4	$2\dot{q}_1 + \dot{q}_2 \sin(q_2)$
z_5	$\dot{q}_1 \sin(q_2)$
z_6	$\dot{q}_1 \sin(q_2+q_3)$
z_7	$(\dot{q}_1+\dot{q}_2)\sin(q_3)$
z_8	$(\dot{q}_1+\dot{q}_2+\dot{q}_3)\sin(q_3)$
z_9	$\dot{q}_1+\dot{q}_2+\dot{q}_3)\sin(q_2+q_3)$
z_{10}	$\cos(q_2)$
z_{11}	$\cos(q_3)$
z_{12}	$\cos(q_2+q_3)$

Note that the classical TS fuzzy approach results in $2^{12} = 4096$ fuzzy rules, which is clearly inapplicable from practical viewpoints on both obtaining numerical solution and real-time implementation. Indeed, in this case we cannot find a stabilization solution for the 3-DoF robot manipulator with actual numerical solvers due to the excessive computational costs. Note that the proposed reduced-complexity approach leads to only $p + 1 = 13$ linear submodels. For comparison purposes, Table 3 represents the numerical complexity of LMI-based stabilization conditions related to two modeling approaches for the 3-DoF robot. In particular, the proposed approach can lead to a large stabilization feasibility domain as shown in Fig. 3.

6. CONCLUSIONS AND PERSPECTIVES

The proposed approach exploits the existing linear dependencies between the matrices of the fuzzy model to obtain

Table 3. Computational burden for 3-DoF serial robot.

Problem	Nb. variables	Nb. constraints	SDP blocks
Design-TS	49158	147549	4097
Design-Reduced	174	441	15

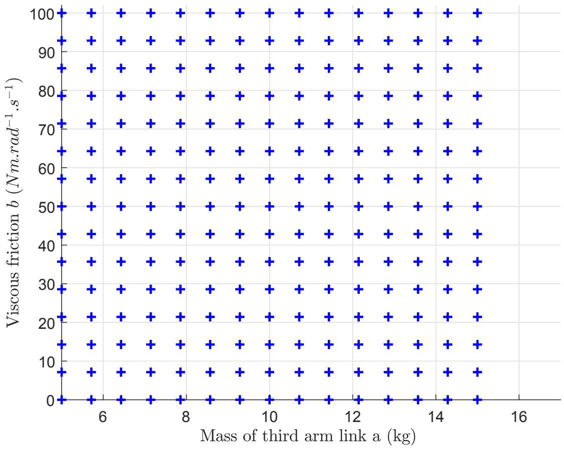

Fig. 3. Feasibility stabilization domain of 3-DoF serial robot obtained with the reduced-complexity conical sum approach.

Table 4. Mechanical constants of the 3-DoF serial robot.

Symbol	Expression	Value
c_1	$g(L_1m_2 + L_1m_3 + m_1r_1)$	121.64
c_2	$g(L_2m_3 + m_2r_2)$	67.69
c_3	gm_3r_3	23.54
c_4	$-L_1(L_2m_3 + m_2r_2)$	-6.9
c_5	$L_1L_2m_3 + L_1m_2r_2$	6.9
c_6	$L_1m_3r_3$	2.4
c_7	$L_2m_3r_3$	2.16
c_8	$-L_2m_3r_3$	-2.16
c_9	$-L_1m_3r_3$	-2.4
c_{10}	$L_1L_2m_3 + L_1m_2r_2$	6.9
c_{11}	$m_3r_3L_2$	2.16
c_{12}	$L_1m_3r_3$	2.4
c_{13}	$(m_2 + m_3)L_1^2 + L_2^2m_3 + \sum_{i=1}^{3} I_i$	19.75
c_{14}	$L_2^2m_3 + I_2 + I_3$	7.85
c_{15}	I_3	1.9

a reduced complexity model based on an affine sum of $p + 1$ matrices selected from the TS fuzzy model. The reduced complexity model obtained is also equivalent to the nonlinear system on the exact region of the state space where the TS fuzzy model was first defined. To obtain LMI-based conditions from the reduced-complexity affine model, a transformation into a conical sum representation was proposed. Within the quadratic-based framework, the obtained LMI-based conditions may be conservative in comparison with those obtained from the classical TS fuzzy approach for stability analysis. However, the proposed approach provides a clear interest in terms of numerical complexity reduction in case of stabilization, especially for complex systems with a very large number of premise variables. In these situations, the real-time applicability of the proposed approach has been clearly demonstrated compared to the classical TS model-based control approaches. Future works focus on reducing further the stability/satbilization conservatism and on real-time tracking control of high DoF robot manipulators.

REFERENCES

Coutinho, P., Araújo, R., Nguyen, A.T., and Palhares, R. (2020). A multiple-parameterization approach for local stabilization of constrained Takagi-Sugeno fuzzy systems with nonlinear consequents. *Inf. Sci.*, 506, 295–307.

Guerra, T.M. and Vermeiren, L. (2004). LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form. *Automatica*, 40(5), 823–829.

Lee, D., Park, J., and Joo, Y.H. (2012). A fuzzy lyapunov function approach to estimating the domain of attraction for continuous-time takagi-sugeno fuzzy systems. *Inf. Sci.*, 185, 230–248. doi:10.1016/j.ins.2011.06.008.

Lendek, Z., Guerra, T.M., Babuska, R., and De Schutter, B. (2011). *Stability analysis and nonlinear observer design using Takagi-Sugeno fuzzy models*. Springer.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in Matlab. In *IEEE Int. Symp. Comput. Aided Control Syst. Des.*, 284–289. Taipei.

Mozelli, L., Palhares, R., and Avellar, G. (2009). A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems. *Inf. Sci.*, 179(8), 1149–1162.

Nguyen, A.T., Dambrine, M., and Lauber, J. (2014). Lyapunov-based robust control design for a class of switching non-linear systems subject to input saturation: application to engine control. *IET Control Theory Appl.*, 8, 1789–1802.

Nguyen, A.T., Guerra, T., and Campos, V. (2019a). Simultaneous estimation of state and unknown input with l_∞ guarantee on error-bounds for fuzzy descriptor systems. *IEEE Control Systems Letters*, 3(4), 1020–1025.

Nguyen, A.T., Taniguchi, T., Eciolaza, L., Campos, V., Palhares, R., and Sugeno, M. (2019b). Fuzzy control systems: Past, present and future. *IEEE Comput. Intell. Mag.*, 14(1), 56–68.

Nguyen, V.A., Nguyen, A.T., Dequidt, A., Vermeiren, L., and Dambrine, M. (2019). Nonlinear tracking control with reduced complexity of serial robots: A robust fuzzy descriptor approach. *Int. J. Fuzzy Syst.*, 21(4).

Sala, A. and Ariño, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, 158(24), 2671–2686.

Spong, M.W. and Vidyasagar, M. (2008). *Robot Dynamics and Control*. John Wiley & Sons.

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. 15(1), 116–132.

Tanaka, K. and Wang, H.O. (2004). *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. John Wiley & Sons.

Tuan, H., Apkarian, P., Narikiyo, T., and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. 9(2), 324–332.