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Simple Utility Design for Welfare Games under Global Information ⋆

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Abstract: Welfare game is a game-theoretic model for resource allocation problem which is to find an allocation to maximize the welfare function. In order to determine it in a distributed way, each agent is assigned to an admissible utility function such that the resulting game possesses desirable properties, for example, scalability, existence and efficiency of pure Nash equilibria, and budget balance. In this paper, supposing that each agent can access the global information, marginal contribution based utility design is proposed. It is shown that utility functions based on the above design have scalability and existence of pure Nash equilibria. Furthermore, efficiency is the same as that of the conventional utility design via Shapley value.

Keywords: Game theory, resource allocation, distributed control.

1. INTRODUCTION

Multi-agent framework is useful to control a large scale system in a decentralized fashion. Many researches have been done, for example, consensus control, formation control, distributed Kalman filter, distributed optimization, and so on (Mesbahi and Egerstedt, 2010; Nedić et al., 2010). The main focus of these researches is that what type of global behavior is generated by a given local interaction rule. In this framework, it is important to consider how to determine players' utility functions. If the problem is to minimize sum of convex objective functions, we can select it as each convex objective function for each player (Nedić and Ozdaglar, 2009; Nedić et al., 2010; Masubuchi et al., 2019). However, this problem is difficult for general problems. To tackle this issue, game theoretic approach (Bauso, 2016) is studied in a context of multi-agent framework (Arslan et al., 2007; Marden et al., 2009; Li and Marden, 2013; Marden et al., 2013; Zhu and Martínez, 2013; Hatanaka et al., 2016; Fele et al., 2017; Jensen and Marden, 2018).

In this paper, we focus on welfare games which is a game-theoretic model for a resource allocation problem. The problem is to find an allocation to maximize the welfare function (Marden and Wierman, 2013b). The game-theoretic formulation is able to solve it in a distributed way. This is because each player can determine her action based on her own utility function. An application of the problem is a coverage problem which is to allocate sensors in regions. The goal of the problem is to maximize the probability of coverage of regions. Another application is a graph coloring problem which is to find a color allocation in a network such that each neighbouring node pair of the network has different colors. When a large scale problem is solved in a multi-agent framework, appropriate objective functions for each agent are needed. This problem is a fundamental problem for solving this issue. Notice that it is important to introduce utility functions of the players whose equilibrium is close to the optimal solution of the system-level objective function. Shapley value which is a solution concept of cooperative game theory (Peleg and Sudhölter, 2007) and the marginal contribution which is called as wonderful life are employed for determination of utility functions (Marden and Wierman, 2013b). A marginal contribution of the i-th player is difference between a welfare function of a subset of players which contains the player i and that of a subset of players except for it. Furthermore, we have to investigate how close between the equilibrium and the optimum. For these utility functions, it is shown that 1) existence of a pure-strategy Nash equilibrium, 2) the price of anarchy, 3) the price of stability, and 4) budget balancedness. Existence of a pure-strategy Nash equilibrium is the most fundamental property. If there is no equilibrium, game theoretic model cannot provide any solution. Then, there may exist multiple pure-strategy Nash equilibrium in a game. The price of anarchy/stability are worst/best case analysis of achievable performance. Budget balancedness mean that balancedness between obtained welfare and sum of utility functions. If utility functions correspond to payment to each player, this property is needed. From conventional researches in Marden and Wierman (2013b); Gopalakrishnan et al. (2014), it is shown that utility functions which satisfies the above four properties under distributed environment are weighted Shapley value which is a generalized version of Shapley value. However, computation of Shapley value requires us heavy computational task.

In this paper, we propose a simple utility design method via a marginal contribution. Our key idea is to relax the distributed problem setting slightly, that is, we assume that each agent can access the global information. If we consider a graph coloring problem with utility functions

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based on Shapley value, how many players uses a color is needed for computation of utility function of each player. From the information, each player decides their action. That is, there is a problem which is needed to access the global information when game theoretic approach is employed to solve it. Then, to evaluate our utility design method, we show that the above four properties are similar with conventional utility function designs such as Shapley value and wonderful life. Furthermore, if we consider a large scale system, scalability, that is, computational complexity for computing utility function, is an important property. Notice that the proposed utility function design is easy to compute its value. In fact, the number of terms in a utility function is proportional to the number of players.

The rest of the paper is organized as follows. In Section 2, a resource allocation problem is formulated. Welfare game as its game theoretic model is given in Section 3. Furthermore, performance criteria is introduced for evaluating utility functions. In Section 4, the utility design based on marginal contribution is provided and its performance is investigated. The utility design method is demonstrated through a numerical example in Section 5.

2. RESOURCE ALLOCATION PROBLEM

A resource allocation problem is to find an allocation $a^* = (a_1^*, a_2^*, \dots, a_n^*) \in \mathcal{A}$ which maximizes a (separable) welfare function

$$W(a) = \sum_{r \in \mathcal{R}} W^{(r)}\left(a^{(r)}\right),$$

where $\mathcal{N} = \{1, 2, \dots, n\}$ is a set of players, $\mathcal{R} = \{r_1, r_2, \dots, r_m\}$ is a finite set of resources, $a^{(r)} := \{i \in \mathcal{N} : r \in a_i\}$ is a set of players which utilises the resource r, $W^{(r)} : 2^{\mathcal{N}} \to \mathbb{R}^+$ is the welfare function for resource r, and and the set of action $\mathcal{A}$ is defined by $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$. Each player i selects an action $a_i \in \mathcal{A}_i \subseteq \mathcal{R}$. In this paper, we suppose that each welfare function $W^{(r)}$ is sub-modular, that is, $W^{(r)}$ satisfies

$$W^{(r)}(\mathcal{S}) + W^{(r)}(\mathcal{T}) \geq W^{(r)}(\mathcal{S} \cup \mathcal{T}) + W^{(r)}(\mathcal{S} \cap \mathcal{T})$$

for any subset $\mathcal{S} \subseteq \mathcal{N}$ and $\mathcal{T} \subseteq \mathcal{N}$. Note that a sub-modular function $W^{(r)}$ satisfies

$$W^{(r)}(\mathcal{S} \cup \{i\}) - W^{(r)}(\mathcal{S}) \geq W^{(r)}(\mathcal{T} \cup \{i\}) - W^{(r)}(\mathcal{T})$$

for any subset $\mathcal{S}$ and $\mathcal{T}$ such that $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{N}$. For example, the problem can apply to the following problem.

Example 1. (Coverage Problem). Now, we would like to cover given regions by using n sensors. Each region r has a relative value $v^{(r)} \geq 0$ and the probability that the i-th sensor successfully finds resource in the region $r \in a_i$ is $p_i^{(r)} \in [0, 1]$. Our goal is to maximize the sum of its probability, that is, maximize the welfare function which is given by

$$W(a) = \sum_{r \in \mathcal{R} : a^{(r)} \neq \emptyset} v^{(r)} \left(1 - \prod_{i \in a^{(r)}} [1 - p_i^{(r)}(a_i)]\right).$$

Note that $1 - \prod_{i \in a^{(r)}} [1 - p_i^{(r)}(a_i)]$ means that the probability that the allocation a successfully covers the region r. (Marden and Wierman, 2013a)

Example 2. (Graph Coloring Problem). Let us consider a finite set $\mathcal{C}$ of colors and graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is a set of nodes and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of edges. Each node $i \in \mathcal{N}$ corresponds to the player. Graph coloring problem is to find a color assignment $a = (a_1, a_2, \dots, a_n) \subseteq \mathcal{C}^n$ such that $a_i \neq a_j$ for any $(i, j) \in \mathcal{E}$.

When we introduce welfare function for color $c \in \mathcal{C}$ as

$$W(a) = \sum_{c \in \mathcal{C}} W^{(c)}(a^{(c)})$$

$$W^{(c)}(\mathcal{S}) = \begin{cases} 0 & \mathcal{S} = \emptyset \\ -1 & \mathcal{S} \neq = \emptyset, \end{cases}$$

we can rewrite the problem as a resource allocation problem. (Marden and Wierman, 2013a)

3. WELFARE GAMES

To solve the resource allocation problem in a distributed way, we introduce utility functions for the problem. Then, let us consider a welfare game $\mathcal{G} = \{\mathcal{N}, \mathcal{R}, \mathcal{A}, \{W_r\}_{r \in \mathcal{R}}, \{u_i\}_{i \in \mathcal{N}}\}$. Players determine their own action based on their utility functions.

To define utility functions, we introduce a distribution rule $f^{(r)} : \mathcal{N} \times 2^{\mathcal{N}} \to \mathbb{R}$ for each resource $r \in \mathcal{R}$. A distribution rule $\{f^{(r)}\}$ implies how to allocate $W^{(r)}(a^{(r)})$ which is the obtained value of the resource r via the action a.

According to $f^{(r)}$, we give the utility function of the i-th player as

$$u_i(a) = \sum_{r \in \mathcal{R}} f^{(r)}(i, a^{(r)}),$$

where $f(i, \mathcal{S})$ is a *distribution rule* which is an allocation of a value $W^{(r)}(\mathcal{S}$ of the welfare function at $\mathcal{S} \subseteq \mathcal{N}$ to the set $\mathcal{N}$ of players. Notice that there are a little difference between the above and definition in Marden et al. (2013). In the above paper, the i-th utility function is given by

$$u_i(a) = \sum_{r \in a_i} f^{(r)}(i, a^{(r)}).$$

That is, each agent has information on resource which is garnered by herself. On the other hand, our utility functions depend on all of the resources. That is, we assume that each agent know how many players utilize any resource r.

Thus we would like to relate the solution of the above game and the optimal solution of the resource allocation problem.

3.1 Pure-strategy Nash Equilibrium and Potential Games

We first define a solution concept of the game.

Definition 3. An action a^{ne} is *the pure-strategy Nash equilibrium* when

$$u_i(a_i^{\mathrm{ne}}, a_{-i}^{\mathrm{ne}}) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}^{\mathrm{ne}})$$

holds for any player $i \in \mathcal{N}$, where $a_{-i} \in_{j \neq i} \mathcal{A}_j$ is the action of players except for that of the player i.

Since each player determines her action based on her own utility function, we assume that each player play a game with pure-strategy Nash equilibrium. Thus, we analyze its property under the given utility functions. On the other hand, the game $\mathcal{G}$ does not always have a pure-strategy Nash equilibrium. We would like to guarantee that the welfare game which is defined by our utility function design has a pure-strategy Nash equilibrium. There are a few approaches for existence of a pure-strategy Nash equilibrium, a potential game approach (Monderer and Shapley, 1996), a supermodular game approach (Topkis, 1998), and an approach via discrete fixed point theorem (Iimura et al., 2005). We employ a potential game approach.

Definition 4. (Monderer and Shapley (1996)). The game $\mathcal{G}$ is called as *potential game* if there exists a potential function $\phi : \mathcal{A} \to \mathbb{R}$ which satisfies

$$\phi(a_i, a_{-i}) - \phi(\tilde{a}_i, a_{-i}) = u_i(a_i, a_{-i}) - u_i((\tilde{a}_i, a_{-i})$$

for any $i \in \mathcal{N}$, $a_i \in \mathcal{A}_i$ and $\tilde{a}_i \in \mathcal{A}_i$.

If the set of actions is finite, potential game always has a pure-strategy Nash equilibrium. Furthermore, one of the equilibrium points is the maximum point of the potential function. That is, if a game is a potential game, the solution can be derived in a decentralized way. This is because we can find a local maximum if each player tries to maximize their own utlity function. As the system becomes larger, it becomes more difficult to find the optimal solution in a centralized way. Therefore, it is necessary to consider a distributed way, and the potential game is an important property.

3.2 Price of Anarchy

In this paper, we employ price of anarchy for worst case analysis of quality on pure-strategy Nash equilibrium.

Definition 5. For the optimal solution a^* of the resource allocation problem and a pure-strategy Nash equilibrium a^{ne}, *the price of anarchy (PoA)* is defined by

$$PoA(G) := \max_{a^{\text{ne}} \in \mathcal{A}^{\text{ne}}} \left\{ \frac{W(a^*)}{W(a^{ne})} \right\}.$$

Note that the price of anarchy is always greater than equal to 1 from the above definition. When the price of anarchy is equal to 1, any pure-strategy Nash equilibrium is the optimal solution of the resource allocation problem. In general, we do not know that action of players converges to which Nash equilibrium in advance. Thus, the price of anarchy tell us the worst case analysis of achievable performance based on welfare game and it is important to utility design method which leads to small price of anarchy.

3.3 Price of Stability

In addition to price of anarchy, we introduce a concept of the best case performance on pure-strategy Nash equilibrium.

Definition 6. For the optimal solution a^* of the resource allocation problem and a pure-strategy Nash equilibrium a^{ne}, *the price of stability (PoS)* is defined by

$$PoS(G) := \min_{a^{\text{ne}} \in \mathcal{A}^{\text{ne}}} \left\{ \frac{W(a^*)}{W(a^{ne})} \right\}.$$

The price of stability is also greater than equal to 1. When the price of anarchy is equal to 1, a pure-strategy Nash equilibrium is the optimal solution of the resource allocation problem. The price of stability is the best case analysis of achievable performance based on welfare game and If we can find the best Nash equilibrium, the price of stability is important performance criteria (Young, 2010).

3.4 Budget Balancedness

A distribution rule $\{f_r\}_{r \in \mathcal{R}}$ is *budget balanced* if the distribution rule satisfies

$$\sum_{i \in \mathcal{N}} f^{(r)}(i, \mathcal{S}) = W^{(r)}(\mathcal{S}), \qquad \forall r \in \mathcal{R},\ \forall \mathcal{S} \subseteq \mathcal{N}.$$

There is relationship between budget balancedness and price of anarchy in a sense of a rule of thumb. In fact, it is known that a utility function which has the optimal price of anarchy is budget balanced in some games (Marden and Wierman, 2013b).

4. A DISTRIBUTION RULE VIA MARGINAL CONTRIBUTION

In this section, we propose a distribution rule based on marginal contribution which is the main result of the paper. When we consider distribution rules for the resource r among the subset $\mathcal{S}$ of the players, we consider each player can obtain his marginal contribution. In addition to it, we allocate the rest $W^{(r)}(S) - \sum (W^{(r)}(S) - W^{(r)}(S/\{i\}))$of their marginal contribution to each player. That is, we define a distribution rule as

$$\begin{aligned} &f^{(r)}(i, \mathcal{S}) \\ &= \frac{1}{n}\left(W^{(r)}(\mathcal{S}) - \sum_{j \in \mathcal{S}} \left(W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S} \setminus \{j\}) \right) \right) \\ &\quad + W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S} \setminus \{i\}). \end{aligned}$$

However, from a view point of cooperative game, both distribution rules have similar properties are not strategic equivalent. Thus, these rules also have different properties. For example, a distribution rule via marginal contribution is budget balanced. However, a distribution rule via wonderful life is not budget balanced.

We would like to emphasize that computation of the distribution rule via marginal contribution is not hard task. This is because the number of terms in a distribution rule is linear function with respect to $|\mathcal{S}|$. This is a significant difference from a distribution rule via Shapley value.

Now, we investigate properties on the above distribution rule. We first show that the game is potential game.

Theorem 7. A game based on a distribution rule via marginal contribution is potential game and its potential function is given by

$$\begin{aligned} &\phi(a) = \sum_{r \in \mathcal{R}} \phi^{(r)}(a^{(r)}) \\ &\phi^{(r)}(\mathcal{S}) \\ &= \frac{1}{n}\left(W^{(r)}(\mathcal{S})) - \sum_{j \in \mathcal{S}} (W^{(r)}(\mathcal{S}) - W(\mathcal{S} \setminus \{j\})) \right) \\ &\quad + W^{(r)}(\mathcal{S}). \end{aligned}$$

Proof. The definition of the potential function leads to

$$\phi^{(r)}(\mathcal{S}) - \phi^{(r)}(\mathcal{S} \setminus \{i\})$$

$$= \frac{1}{n}\left(W^{(r)}(\mathcal{S}) - \sum_{j\in\mathcal{S}}\left(W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{j\})\right)\right)$$
$$+ W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{i\})$$
$$- \frac{1}{n}\left(W^{(r)}(\mathcal{S}\setminus\{i\})\right.$$
$$\left. - \sum_{j\in\mathcal{S}\setminus\{i\}}\left(W^{(r)}(\mathcal{S}\setminus\{i\}) - W^{(r)}(\mathcal{S}\setminus\{i,j\})\right)\right)$$
$$- W^{(r)}(\mathcal{S}\setminus\{i\}) + W^{(r)}(\mathcal{S}\setminus\{i\}).$$

That is, since

$$\phi(a_i, a_{-i}) - \phi((a_i^{\emptyset}, a_{-i})) = u_i(a_i, a_{-i}) - u_i(a_i^{\emptyset}, a_{-i}),$$

we see that

$$\begin{aligned}
&\phi(a_i, a_{-i}) - \phi((\tilde{a}_i, a_{-i}))\\
&= \phi(a_i, a_{-i}) - \phi((a_i^{\emptyset}, a_{-i})) - \phi(\tilde{a}_i, a_{-i}) + \phi((a_i^{\emptyset}, a_{-i}))\\
&= u_i(a_i, a_{-i}) - u_i(a_i^{\emptyset}, a_{-i}) - u_i(\tilde{a}_i, a_{-i}) + u_i(a_i^{\emptyset}, a_{-i})\\
&= u_i(a_i, a_{-i}) - u_i(\tilde{a}_i, a_{-i})
\end{aligned}$$

for any action $\tilde{a}_i \in \mathcal{A}_i$.

Notice that the potential games guarantee existence of a pure-strategy Nash equilibrium.

Theorem 8. A distribution rule via marginal contribution is budget balanced.

Proof. According to summation of $f^{(r)}(i,\mathcal{S})$ among all of the players, we have

$$\begin{aligned}
&\sum_{i\in\mathcal{N}} f^{(r)}(i,\mathcal{S})\\
&= W^{(r)}(\mathcal{S}) - \sum_{j\in\mathcal{S}}(W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{j\}))\\
&\quad + |\mathcal{S}|W^{(r)}(\mathcal{S}) - \sum_{i\in\mathcal{S}} W^{(r)}(\mathcal{S}\setminus\{i\})\\
&= W_r(\mathcal{S}).
\end{aligned}$$

That is, we immediately see that the distribution rule is budget balanced.

Theorem 9. If the welfare functions $W^{(r)}$ are submodular, the price of anarchy of a game based on a distribution rule via marginal contribution is less than or equal to 2.

Proof. To prove the theorem, we show that the game is a valid utility game. Then, we show that

$$W^{(r)}(\mathcal{S}) - \sum_{j\in\mathcal{S}}\left(W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{j\})\right) \geq 0 \qquad \forall \mathcal{S}\subseteq\mathcal{N} \qquad (1)$$

by employing the inductive method. Now, we define $\mathcal{S} = \{s_1, s_2, \ldots, s_k\}$.

Due to submodularity of $W^{(r)}$, we have

$$\sum_{i=1}^{k} W^{(r)}(\mathcal{S}\setminus\{s_i\}) \geq (k-1)\sum_{i=1}^{k} W^{(r)}(\mathcal{S}).$$

This is because, for any s_i and $\mathcal{S}_{-i} \subset \mathcal{S}$ such that $s_i \notin \mathcal{S}_{-i}$,

$$\begin{aligned}
&W^{(r)}(\mathcal{S}\setminus\{s_i\}) + W^{(r)}(\mathcal{S}\setminus\mathcal{S}_{-i})\\
&\geq W^{(r)}((\mathcal{S}_{k+1}\setminus\{s_1\}) \cup (\mathcal{S}_{k+1}\setminus\mathcal{S}_{-i}))\\
&\quad + W^{(r)}((\mathcal{S}_{k+1}\setminus\{s_1\}) \cap (\mathcal{S}_{k+1}\setminus\mathcal{S}_{-i}))\\
&= \sum_{i=1}^{k+1} W^{(r)}(\mathcal{S}_{k+1}) + \sum_{i=1}^{k+1} W^{(r)}(\mathcal{S}_{k+1}\setminus(s_i\cup\mathcal{S}_{-i}).
\end{aligned}$$

According to this relationship, we immediately see that

$$\begin{aligned}
&W^{(r)}(\mathcal{S}) - \sum_{j\in\mathcal{S}}\left(W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{j\})\right)\\
&\geq kW^{(r)}(\mathcal{S}) - \sum_{j\in\mathcal{S}} W^{(r)}(\mathcal{S}) = 0,
\end{aligned}$$

that is, eq. (1) holds. The definition of the distribution rule and the above relationship leads to

$$f^{(r)}(i,\mathcal{S}) \geq W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{i\}).$$

Thus,

$$\begin{aligned}
u_i(a) &= \sum_{r\in\mathcal{R}} f^{(r)}(i, a^{(r)})\\
&\geq \sum_{r\in\mathcal{R}} W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S}\setminus\{i\})\\
&= W(a) - W(a_i^{\emptyset}, a_{-i}).
\end{aligned}$$

We can conclude that the condition (2) in Definition 13 holds.

In addition to it, we show that the condition (3):

$$\sum_{i\in\mathcal{N}} u_i(a) \leq W(a), \qquad \forall a\in\mathcal{A}$$

in Definition 13 holds based on budget balancedness of the distribution rule.

$$\begin{aligned}
\sum_{i\in\mathcal{N}} u_i(a) &= \sum_{i\in\mathcal{N}}\sum_{r\in\mathcal{R}} f^{(r)}(i, a^{(r)})\\
&= \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}} f^{(r)}(i, a^{(r)})\\
&= \sum_{r\in\mathcal{R}} W^{(r)}(a^{(r)})\\
&= W(a)
\end{aligned}$$

The game is a valid utility game and $W^{(r)}$ is submodular. From Lemma 14, we see that the price of anarchy is less than or equal to 2.

If we solve the problem in a centralized way, we will obtain an approximate solution whose price of anarchy is less than or equal to $e/(1-e) \approx 1.5819$ (Ageev and Sviridenko, 2004). On the contrary, the price of anarchy in the above theorem implies performance of distributed algorithms. It is natural that the performance of a distributed algorithm is worse than centralized algorithms. However, its performance degradation from 1.58 to 2 is acceptable.

Corollary 10. The maximizer $\bar{a} = \arg\max\phi(a)$ of the potential function is a pure-strategy Nash equilibrium of the game.

Proof. Due to definition of $\bar{a}$ and potential function, we see that

$$\begin{aligned}
&u_i(\bar{a}) - u_i(a_i, \bar{a}_{-i})\\
&= \phi(\bar{a}) - W(\bar{a}\setminus\{i\}) - \phi(a_i, \bar{a}_{-i}) + W((a_i, \bar{a}_{-i})\setminus\{i\})\\
&\geq -W(\bar{a}\setminus\{i\}) + W((a_i, \bar{a}_{-i})\setminus\{i\}) = 0.
\end{aligned}$$

If $\bar{a}$ is not a pure-strategy Nash equilibrium, there exists an action $a_i \in \mathcal{A}_i$ such that

$$u_i(\bar{a}) - u_i(a_i, \bar{a}_{-i}) < 0.$$

Table 1. Properties on utility functions (PG: Potential Game, BB: Budget Balance)

Method	PG	BB	PoA	PoS
Marginal contribution	Yes	Yes	2	1 ($n \to \infty$)
Shapley value	Yes	Yes	2	2
Wonderful life	Yes	No	2	1

However, this contradicts $u_i(\bar{a}) - u_i(a_i, \bar{a}_{-i}) \geq 0$. We therefore see that $\bar{a}$ is a pure-strategy Nash equilibrium of the game.

From the above statement, we can show that the price of stability of the game converges to 1 when the number of players goes to infinity.

Corollary 11. If the number of players goes to infinity, the price of stability converges to 1.

This statement means that, when the number of players is large enough, we can expect there exists a pure-strategy Nash equilibrium whose welfare function is close to its maximum value. We summaries these properties in Table 1.

Theorem 12. In a game based on a distribution rule via marginal contribution, the number of terms in the utility function is $\sum_{r\in R}(|a|^{(r)}+1)$, where $|a|^{(r)}$ is the number of players which utilize the resource r. Its maximum number of terms is $(n+1)m$.

Proof. The distribution rule via marginal contribution can be rewritten as

$$f^{(r)}(i, a^{(r)}) =$$
$$\frac{n - |a|^{(r)} + 1}{n} W^{(r)}(a^{(r)}) +$$
$$\frac{1}{n} \sum_{j \in a^{(r)} \setminus i} W^{(r)}(a^{(r)} \setminus \{j\}) - \frac{n-1}{n} W^{(r)}(a^{(r)} \setminus \{i\}).$$

The number of terms in the above is $|a|^{(r)} + 1$. Since the utility function is the sum of distribution rule for all resources, the number of terms in utility function is $\sum_{r\in R}(|a|^{(r)}+1)$. When all players utilize all resources, that is, $|a|^{(r)} = n, \forall r \in R$, the number of terms is maximum and represented by $(n+1)m$.

The number of terms increases in proportion to the number of players in the system. From Corollary 16 in Appendix B, the number of terms in the utility function via Shapley value increases exponentially with the number of players. Fig. 1 shows the relation between the number of players and the number of terms in the utility function, with $m = 10$.

5. A NUMERICAL EXAMPLE

Let us consider a coverage problem in Example 1. In this example, there are 3 players and 2 regions, that is, $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{R} = \{r_1, r_2\}$. Supposing that $p_i \in (0, 1)$ does not depend on the region, the welfare function is given by

$$W^{(r)}(a^{(r)}) = v^{(r)} \left(1 - \prod_{i \in a^{(r)}} (1 - p_i) \right),$$

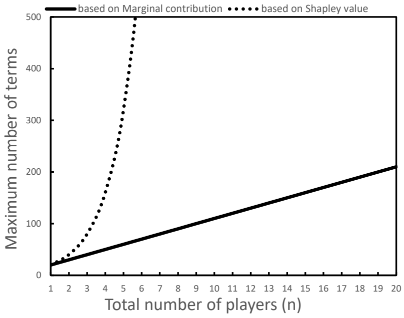

Fig. 1. Maximum number of terms in utility function

Table 2. Welfare function $W(a)$

3	r_1			r_2	
1 \ 2	r_1	r_2	1 \ 2	r_1	r_2
r_1	9.06	10.63	r_1	10.00	6.88
r_2	10.00	10.63	r_2	9.06	4.53

Table 3. Utility functions based on Marginal Contribution

3	r_1			r_2	
1 \ 2	r_1	r_2	1 \ 2	r_1	r_2
r_1	1.9792 2.6042 4.4792	1.2500 3.1250 6.2500	r_1	1.6667 4.1667 4.1667	3.1250 1.2500 2.5000
r_2	2.5000 2.5000 5.0000	0.8333 2.0833 7.7083	r_2	0.6250 5.3125 3.1250	0.9896 1.3021 2.2396

Table 4. Utility functions based on wonderful life

3	r_1			r_2	
1 \ 2	r_1	r_2	1 \ 2	r_1	r_2
r_1	0.3125 0.9375 2.8125	0.625 2.500 5.625	r_1	1.25 3.75 3.75	2.50 0.625 1.875
r_2	1.25 1.25 3.75	0.625 1.875 2.50	r_2	0.3125 2.8125 2.8125	0.1563 0.4688 1.4063

where $v^{(r)} \geq 0$ is the relative value of the region r. Each relative value and success probability are given by

$$p_1 = 0.25, \qquad p_2 = 0.5, \qquad p_3 = 0.75$$
$$v^{(r_1)} = 10, \qquad v^{(r_2)} = 5$$

Note that the above function is submodular. These parameters lead to the welfare function which is shown at Table 2. The optimal solutions are (r_1, r_2, r_1) and (r_2, r_2, r_1).

We have considered the distribution rules via marginal contribution (proposed method), Shapley value, and wonderful life. Each utility functions are shown in Tables 3, 4, and 5. Nash equilibriums are highlighted by under line. For any utility design, a set of pure-strategy Nash equilibriums includes the optimal solution of an resource allocation problem. That is, the price of stability for these games are 1. On the other hand, the price of anarchy of the proposed method is 1.063 and that of Shapley value and wonderful life are also 1.063. We therefore see that the

Table 5. Utility functions based on Shapley values

3	r_1			r_2	
1 \ 2	r_1	r_2	1 \ 2	r_1	r_2
r_1	0.42	$\underline{0.31}$	r_1	$\underline{0.63}$	2.50
	1.77	$\underline{2.50}$		$\underline{1.88}$	0.31
	2.50	$\underline{2.81}$		$\underline{3.75}$	0.93
r_2	1.25	$\underline{0.31}$	r_2	0.16	0.21
	0.63	$\underline{2.50}$		2.50	0.89
	1.88	$\underline{7.50}$		0.93	1.25

price of anarchy is less than or equal to 2 which consistent with the theorem.

6. CONCLUDING REMARKS

Utility design for welfare game has been considered. We assume that each agent can access the global information. Then we have proposed marginal contribution based simple utility design. It has been shown that the resulting game of the above utility functions possesses desirable properties. That is, scalability, existence of pure-strategy Nash equilibrium, and an upper bound of price of anarchy. Furthermore, these properties were shown in a numerical example. As we see in Fig. 1, proposed method allows us to determine utility functions even in larger systems.

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Appendix A. VALID UTILITY GAMES

In this section, we summarize definition of valid utility games and its property on the price of anarchy (Roughgarden, 2015).

Definition 13. The game is called *valid utility game* if it satisfies

(1) All of the welfare functions $W^{(r)}$, $r \in \mathcal{R}$ are submodular.

(2) For any player $i \in \mathcal{N}$,
$$u_i(a) \geq W(a) - W(a_i^{\emptyset}, a_{-i}), \qquad \forall a \in \mathcal{A}$$
holds, where $a_i^{\emptyset} = \emptyset$, that is, the player i does not employ any resource.

(3) For any action $a \in \mathcal{A}$, the following inequality holds.
$$\sum_{i \in \mathcal{N}} u_i(a) \leq W(a).$$

An upper bound of price of anarchy in valid utility games is given in Roughgarden (2015).

Lemma 14. If the game is valid utility game, the price of anarchy is less than or equal to 2.

Appendix B. THE DISTRIBUTION RULE BASED ON SHAPLEY VALUE

A distribution rule based on Shapley value is given by
$$f_{\mathrm{SV}}^{(r)}(i, \mathcal{S}) = \sum_{T \subseteq S \setminus \{i\}} \frac{|\mathcal{T}|!(|\mathcal{S}| - |\mathcal{T}| - 1)!}{|\mathcal{S}|!}$$

$$\cdot (W^{(r)}(T \cup \{i\}) - W^{(r)}(\mathcal{T}))$$

if the player $i \in \mathcal{N}$ utilizes the resource r and $f_{\mathrm{SV}}^{(r)}(i, \mathcal{S}) = 0$ otherwise.

Lemma 15. (Marden and Wierman (2013b)). Supposing submodularity of the welfare functions $W^{(r)}$, the distribution rule based on Shapley value leads to a potential game and satisfies budget balancedness. Then, its price of anarchy and price of stability is less than or equal to 2 and is also less than or equal to 2.

From lemma 15, we can easily evaluate the number of terms in a utility function.

Corollary 16. The number of terms in a utility function via Shapley value is $\sum_{r \in a_i} 2^{|a|^{(r)}}$. Its maximum number of terms is $2^n m$.

Shapley value is characterized by a potential function (Hart and Mas-Colell, 1989) in a research of cooperative game. Distribution rule based on Shapley value has several preferable properties. Furthermore, if we employ weighted Shapley value for a distribution rule, there exists a weight such that the price of stability is one. To find the weight, we need to execute a recursive algorithm (Marden and Wierman, 2013b). However, notice that computation of Shapley value is NP-hard (Deng and Papadimitriou, 1994). The above distribution rule requires us to compute Shapley value for any subset $\mathcal{S}$ of the set $\mathcal{N}$ of players.

Appendix C. THE DISTRIBUTION RULE BASED ON WONDERFUL LIFE

A distribution rule based on wonderful life is given by

$$f_{\mathrm{WL}}^{(r)}(i, \mathcal{S}) = W^{(r)}(\mathcal{S}) - W^{(r)}(\mathcal{S} \setminus \{i\})).$$

The marginal contribution of the player i is directly employed for distribution rule design.

Lemma 17. (Marden and Wierman (2013b)). Supposing submodularity of the welfare functions $W^{(r)}$, the distribution rule based on wonderful life leads to a potential game. Then, its price of anarchy is less than or equal to 2 and its price of stability is equal to 1.

Note that the distribution rule based on wonderful life does not satisfy budget balancedness in general.