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Linear Direct Transcription for Nonlinear Constrained Optimal Control Problems

Zhong Wang, Yan Li

Department of Navigation, Guidance, and Control, Northwestern Polytechnical University, 710129, P.R. China (e-mail: zwang.nwpu@gmail.com, liyan@nwpu.edu.cn).

Abstract: Nonlinear optimal control problems are frequently transformed into nonlinear programming problems for which the solving process is generally time-consuming. In this paper, a linear direct transcription method is proposed for nonlinear optimal control problems. Taking advantage of the state-dependent coefficient parameterization method and the spectral discretization method, the nonlinear optimal control problem is successively linearized and turned into a sequence of efficiently solvable mixed linear complementarity problems. The proposed direct transcription method is linear in two ways: the nonlinear system is linearized using state-dependent coefficient parameterization; the resulting quadratic programming problem is converted into a mixed linear complementarity problem. Simulations are implemented, and numerical results verify the effectiveness and efficiency of the proposed method.

Keywords: nonlinear optimal control, quadratic programming, mixed linear complementarity problem.

1. INTRODUCTION

Nonlinear optimal control has a wide range of applications in areas like hypersonic gliding reentry vehicle trajectory optimization, process control and spacecraft maneuvering. However, in most cases, analytic solutions only exist for linear optimal control problems. Generally speaking, one should resort to numerical methods to implement optimal feedback for nonlinear systems.

Numerical methods for nonlinear optimal control problems have been widely studied, and numerous approaches have been proposed since 1960s. In Betts (1998), theses methods are classified into direct and indirect methods. In direct methods, the optimal control problem is discretized using methods like spectral collocation, and then it is usually turned into a nonlinear programming problem (NLP). Whereas the optimality condition is derived in the indirect methods, and numerical schemes are frequently employed to solve the resultant two-point boundary value problem (TPBVP) (see Peng (2013) and the references therein).

A state-dependent Riccati equation (SDRE) method is proposed for nonlinear infinite-horizon optimal control problems in Mracek (1998). In the SDRE method, the nonlinear system is expressed in a linear-like form using state-dependent coefficient (SDC) parameterization, which is also known as extended linearization. Then the sub-optimal control law is obtained by solving a linear quadratic regulator (LQR) problem in each step (Huang (2017)). Similarly, the SDRE method is also extended to finite-horizon problems, where state-dependent differential Riccati equations (SDDREs) should be solved in each step (see Heydari (2013)).

The SDRE/SDDRE method has been widely investigated because of its superior performance in design flexibility and numerical efficiency. Despite the fact that it is designed for optimal control problems, in most cases, the SDRE/SDDRE method fails to meet the optimality conditions. The reason for this is that the SDRE/SDDRE method only considers the linerized dynamics at current time step $t = t_0$. It implies that, in the SDRE/SDDRE method, the controller is just computed based on the dynamics at $t = t_0$, i.e., $\dot{\boldsymbol{x}} = A(\boldsymbol{x}(t_0))\,\boldsymbol{x} + B(\boldsymbol{x}(t_0))\,\boldsymbol{u}$, which further leads to the fact that the nonlinearity and evolution of the dynamics is not considered as a whole (Wang (2019)).

Based on the quasilinearization technique, in Li (2016), the optimal controller is calculated by linearizing the system using Taylor expansions. The Hamiltonian canonical equations and optimality conditions are then discretized using spectral collocation methods. Thus it belongs to the category of indirect methods (Betts (1998)). In Gomroki (2017), the successive state-dependent coefficient parameterization technique is combined with the spectral discretization method, and it transforms the optimal control problem into a sequence of quadratic programming problem. The convergence of this successive linearization technique is also proved for unconstrained problems in Banks (2000) and Çimen (2004). However, the resultant quadratic programming problem is inherently a nonlinear programming problem. Compared with the SDRE method, though a more accurate solution can be obtained, the employment of the quadratic programming techniques leads to a significant decrease in computational efficiency.

While nonlinear programming techniques are frequently used in previously proposed direct methods, a linear direct transcription method is proposed in this paper for a class of constrained nonlinear optimal control problems. Taking advantage of the successive state-dependent coefficient parameterization and direct transcription techniques, the nonlinear optimal problem is transformed into a sequence of efficiently solvable mixed linear complementarity problems. The proposed direct transcription method is *linear* in two aspects: the nonlinear system is *linearized* using state-dependent coefficient parameterization; the resulting optimization problem is turned into an efficiently solvable mixed *linear* complementarity problem (MLCP). Simulations also demonstrate the significant improvement in numerical efficiency.

2. PROBLEM FORMULATION

In this paper, we consider the optimal control problem for the control-affine nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$ are the state and input, respectively. And $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuously differentiable functions defined on a compact set $\Omega \in \mathbb{R}^n$. The constraints on the state and input are given by

$$\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) \leq 0. \tag{2}$$

Specifically, box constraints on the state and control are considered in this paper, and they can also be expressed elementwise as

$$\alpha^l_{\min} \leq x^l \leq \alpha^l_{\max}, \quad l = 1, \cdots, n, \tag{3a}$$

$$\beta^l_{\min} \leq u^l \leq \beta^l_{\max}, \quad l = 1, \cdots, m. \tag{3b}$$

The performance index is defined as

$$J(\boldsymbol{x}_0) = \frac{1}{2}\int_{t_0}^{t_f} \left(||\boldsymbol{x}||^2_Q + ||\boldsymbol{u}||^2_R\right) dt \tag{4}$$

where Q and R are positive definite matrices.

The problem considered in this paper is to find the optimal control for the system (1) with constraints (3) and cost functions (4). Besides, the well-posedness of the optimal control problem is also assumed in this paper.

3. LINEAR DIRECT TRANSCRIPTION METHOD

3.1 Direct Transcription Using Spectral Discretization

Numerous methods have been proposed to solved the continuous-time optimal control problems, and pseudospectral method is one of the most frequently used algorithms (Rao (2010)). Pseudospectral method employs a direct transcription framework, in which different kinds of collocation points, such as the Legendre-Gauss (LG) points, Legendre-Gauss-Radau (LGR) points and Legendre-Gauss-Lobatto (LGL) points, have been used to discretize the considered system. In this paper, the LGR collocation points are used to discretize the optimal control problems (Wang (2019)).

The LGR points lie in the interval $[-1, 1)$. Firstly, the domain transformation function

$$t = \phi(\tau) = \frac{t_f - t_0}{2}\tau + \frac{t_0 + t_f}{2} \tag{5}$$

and its derivative

$$T(\tau) = \dot{\phi}(\tau) = \frac{t_f - t_0}{2} \tag{6}$$

are employed to change the considered time domain $t \in [t_0, t_f]$ into $\tau \in [-1, 1]$.

Let $-1 = \tau_1 < \cdots < \tau_N < +1$ denote the LGR points. The state and control are approximated by (Wang (2019),Wang (2020))

$$\boldsymbol{x}(\tau) \approx \sum_{j=1}^{N} L_j(\tau)\boldsymbol{x}(\tau_j) \tag{7a}$$

$$\boldsymbol{u}(\tau) \approx \sum_{j=1}^{N} L_j(\tau)\boldsymbol{u}(\tau_j) \tag{7b}$$

where $L_j(\tau)$ is the Lagrange interpolation basis function which is given by

$$L_j(\tau) = \prod_{i=1, i\neq j}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}, \ j = 1, \cdots, N. \tag{8}$$

The time derivative of the state is calculated as

$$\dot{\boldsymbol{x}}(\tau) \approx \sum_{j=1}^{N} \dot{L}_j(\tau)\boldsymbol{x}(\tau_j). \tag{9}$$

Evaluate the derivatives at the LGR collocation points, and one has (Wang (2019))

$$\dot{\boldsymbol{x}}(\tau_i) \approx \sum_{j=1}^{N} \dot{L}_j(\tau_i)\boldsymbol{x}(\tau_j), \quad i = 1, \cdots, N. \tag{10}$$

Taking advantage of the Gauss quadrature rule, the performance index is computed using

$$J \approx \frac{t_f - t_0}{4}\sum_{i=1}^{N} w_i \left(||\boldsymbol{x}(\tau_i)||^2_Q + ||\boldsymbol{u}(\tau_i)||^2_R\right) \tag{11}$$

where w_i is the corresponding quadrature weight.

Define the differentiation matrix (Wang (2019))

$$D_{i,j} = \dot{L}_j(\tau_i), \quad i = 1, \cdots, N; \ j = 1, \cdots, N. \tag{12}$$

By collocating at the LGR points, the nonlinear system is transformed into the equation

$$\sum_{j=1}^{N} D_{i,j}\boldsymbol{x}(\tau_j) = T(\tau_i)[\boldsymbol{f}(\boldsymbol{x}(\tau_i)) + \boldsymbol{g}\boldsymbol{u}(\tau_i)] \tag{13}$$

Then based on the direct transcription procedure, one obtains the nonlinear programming problem

$$\begin{aligned}\min_{\boldsymbol{x},\boldsymbol{u}} \quad & \frac{t_f - t_0}{4}\sum_{i=1}^{N} w_i \left(||\boldsymbol{x}(\tau_i)||^2_Q + ||\boldsymbol{u}(\tau_i)||^2_R\right) \\ s.t. \quad & \sum_{j=1}^{N} D_{i,j}\boldsymbol{x}(\tau_j) - T(\tau_i)[\boldsymbol{f}(\boldsymbol{x}(\tau_i)) + \boldsymbol{g}\boldsymbol{u}(\tau_i)] = 0 \\ & \boldsymbol{h}(\boldsymbol{x}(\tau_i), \boldsymbol{u}(\tau_i)) \leq 0\end{aligned}$$

To guarantee the existence of a solution to the discretized problem, the following relaxed problem

$$\begin{aligned}\min_{\boldsymbol{x},\boldsymbol{u}} \quad & \frac{t_f - t_0}{4}\sum_{i=1}^{N} w_i \left(||\boldsymbol{x}(\tau_i)||^2_Q + ||\boldsymbol{u}(\tau_i)||^2_R\right) \\ s.t. \quad & \left\|\sum_{j=1}^{N} D_{i,j}\boldsymbol{x}(\tau_j) - T(\tau_i)[\boldsymbol{f}(\boldsymbol{x}(\tau_i)) + \boldsymbol{g}\boldsymbol{u}(\tau_i)]\right\|_\infty \leq \varepsilon \\ & \boldsymbol{h}(\boldsymbol{x}(\tau_i), \boldsymbol{u}(\tau_i)) \leq \varepsilon\end{aligned}$$

is frequently introduced (Ross (2012)).

The direct transcription method shown above is often termed as the pseudospectral method, and it has been applied to many areas like zero-propellant maneuvers of the International Space Station (Ross (2012)). Though many off-the-shelf commercial solvers are available for the resultant nonlinear programming problem, the solving process for the pseudospectral method is still time consuming.

3.2 Successive Linearization Based on State-Dependent Coefficient Parameterization

State-dependent coefficient parameterization, i.e., extended linearization, is a method of factorizing the nonlinear system into a linear-like structure, see Çimen (2010),Wang (2019) and Wang (2020). Under the mild assumption that $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{f}(\cdot) \in C^1(\Omega)$, there should exist at least one parameterization structure such that the considered nonlinear system (1) can be expressed as

$$\dot{\boldsymbol{x}} = A(\boldsymbol{x})\boldsymbol{x} + B(\boldsymbol{x})\boldsymbol{u} \tag{14}$$

where

$$A(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) \tag{15a}$$
$$B(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}). \tag{15b}$$

Here $A : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ is a nonlinear state-dependent matrix-valued function.

In this paper, it is assumed that the considered optimal control problems are well-posed and the optimal controls also exist. Different from the SDRE method which takes the nonlinear dynamics as a linearized system, the proposed method takes the whole time horizon into account and calculates the optimal control in an iterative manner (Çimen (2004),Wang (2019),Wang (2020)). In the kth iteration, the linearized system is given by

$$\dot{\boldsymbol{x}}_k = A_k(t)\boldsymbol{x}_k + B_k(t)\boldsymbol{u}_k, \quad \boldsymbol{x}_k(t_0) = \boldsymbol{x}_0 \tag{16}$$

where $A_k(t) = A(\boldsymbol{x}_{k-1}(t))$ and $B_k(t) = B(\boldsymbol{x}_k(t))$, and the state and control in the kth iteration are defined by $\boldsymbol{x}_k = [x_k^1, x_k^2, \ldots, x_k^n]^T$ and $\boldsymbol{u}_k = [u_k^1, u_k^2, \ldots, u_k^m]^T$, respectively. The performance index in the kth iteration is given by

$$J_k(\boldsymbol{x}_0) = \frac{1}{2}\int_{t_0}^{t_f} ||\boldsymbol{x}_k||_Q^2 + ||\boldsymbol{u}_k||_R^2 dt. \tag{17}$$

Taking advantage of the domain transformation method in Section 3.1, the linearized system is transformed into

$$\dot{\boldsymbol{x}}_k = T(\tau)\left[A_k(\tau)\boldsymbol{x}_k + B_k(\tau)\boldsymbol{u}_k\right]. \tag{18}$$

Note that the boundary condition for the new finite-horizon optimal control problem is

$$\boldsymbol{x}_k(-1) = \boldsymbol{x}_0. \tag{19}$$

And the performance index is given by

$$J_k(\boldsymbol{x}_0) = \frac{1}{2}T(\tau)\int_{-1}^{1} ||\boldsymbol{x}_k||_Q^2 + ||\boldsymbol{u}_k||_R^2 dt. \tag{20}$$

Similar to Section 3.1, the sequence of linear time-varying optimal control problems can also be discretized and solved based on spectral collocation methods.

As in Section 3.1, the differential equation (18) is discretized and transformed into the linear equations

$$\sum_{j=1}^{N} D_{i,j}\boldsymbol{x}_k(\tau_j) - T(\tau_i)[A_{k-1}(\tau_i)\boldsymbol{x}_k(\tau_i) - B_{k-1}(\tau_i)\boldsymbol{u}_k(\tau_i)] = 0, \quad 1 \le i \le N. \tag{21}$$

The box constraints are then given by

$$\alpha_{\min}^l \le x_k^l(\tau_i) \le \alpha_{\max}^l, \quad l = 1, \cdots, n, \tag{22a}$$
$$\beta_{\min}^l \le u_k^l(\tau_i) \le \beta_{\max}^l, \quad l = 1, \cdots, m. \tag{22b}$$

The direct transcription of the linearized system is as follows. Discretize the objective function (20) using Gaussian quadrature method and one has

$$J_k \approx \frac{t_f - t_0}{4}\sum_{i=1}^{N} T(\tau_i)w_i\left(||\boldsymbol{x}_k(\tau_i)||_Q^2 + ||\boldsymbol{u}_k(\tau_i)||_R^2\right). \tag{23}$$

Taking the states and controls at the LGR collocation points, i.e., the vector

$$\begin{aligned}\boldsymbol{\gamma}_k = &[x_k^1(\tau_1), \ldots, x_k^1(\tau_N), \ldots, x_k^n(\tau_1), \ldots, x_k^n(\tau_N),\\ &u_k^1(\tau_1), \ldots, u_k^m(\tau_N)]^T\end{aligned} \tag{24}$$

as the variables, the discretized optimal control problem, i.e., objective function (23) and constraints (19), (21) and (22), can be reformulated into the quadratic programming problem

$$\begin{aligned}\min_{\boldsymbol{\gamma}_k} \quad & \frac{1}{2}\boldsymbol{\gamma}_k^T H\boldsymbol{\gamma}_k\\ s.t. \quad & \mathcal{A}_k\boldsymbol{\gamma}_k = \boldsymbol{b}_k\\ & \mathcal{C}_k\boldsymbol{\gamma}_k \ge \boldsymbol{d}_k\end{aligned} \tag{25}$$

where $\mathcal{A}_k$, $\boldsymbol{b}_k$, $\mathcal{C}_k$ and $\boldsymbol{d}_k$ are the matrix and vector corresponding to the equality and inequality constraints. It is also straightforward to see that H is the matrix with scaled Q and R on its diagonal. It is assumed that the quadratic programming problem (24) obtained in each iteration has at least one feasible solution.

3.3 Mixed Linear Complementarity Problem

As shown in Gomroki (2017), successive extended linearization can improve the efficiency for solving nonlinear optimal control problems. However, the obtained quadratic programming problem is inherently a nonlinear programming problem. The computational efficiency can be significantly improved by transforming it into mixed linear complementarity problem. Based on the method of Lagrange multipliers, by incorporating the multipliers $\boldsymbol{\lambda}_k$ and $\boldsymbol{\mu}_k$, the Lagrangian for the optimization problem (24) is given by

$$\mathcal{L}(\boldsymbol{\gamma}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k) = \frac{1}{2}\boldsymbol{\gamma}_k^T H\boldsymbol{\gamma}_k - \boldsymbol{\lambda}_k^T(\mathcal{A}_k\boldsymbol{\gamma}_k - \boldsymbol{b}_k) - \boldsymbol{\mu}_k^T(\mathcal{C}_k\boldsymbol{\gamma}_k - \boldsymbol{d}_k). \tag{26}$$

The KKT (Karush-Kuhn-Tucher) conditions for the optimization problem yield

$$\begin{aligned}H\boldsymbol{\gamma}_k - \mathcal{A}^T\boldsymbol{\lambda}_k - \mathcal{C}^T\boldsymbol{\mu}_k &= 0\\ \mathcal{A}_k\boldsymbol{\gamma}_k - \boldsymbol{b}_k &= 0\\ \mathcal{C}_k\boldsymbol{\gamma}_k - \boldsymbol{d}_k &\ge 0\\ \boldsymbol{\mu}_k^T(\mathcal{C}_k\boldsymbol{\gamma}_k - \boldsymbol{d}_k) &= 0\\ \boldsymbol{\mu}_k^T &\ge 0\end{aligned}$$

By introducing a slack variable

$$\boldsymbol{\nu}_k = \mathcal{C}_k\boldsymbol{\gamma}_k - \boldsymbol{d}_k \tag{27}$$

for inequality constraints, the KKT conditions can be rewritten into the mixed linear complementarity problem (MLCP)

$$\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{\nu}_k\end{bmatrix} = \begin{bmatrix}-\boldsymbol{b}_k\\ -\boldsymbol{d}_k\end{bmatrix} + \begin{bmatrix}\mathcal{A}_k\\ \mathcal{C}_k\end{bmatrix} H^{-1}\left[\mathcal{A}_k^T \; \mathcal{C}_k^T\right]\begin{bmatrix}\boldsymbol{\lambda}_k\\ \boldsymbol{\mu}_k\end{bmatrix} \tag{28}$$

$$\begin{aligned}\boldsymbol{\nu}_k &\geq 0\\ \boldsymbol{\mu}_k &\geq 0\\ \boldsymbol{\nu}_k^T \boldsymbol{\mu}_k &= 0\end{aligned}$$

The mixed linear complementarity problem is an important paradigm in mathematical programming. Due to its high efficiency, it has been widely used in many time critical applications like computer graphics, linear model predictive control (Bemporad (2009)) and so on.

The mixed linear complementarity problem is not a standard optimization problem, since it does not aim to minimize or maximize any objective function. However, the mixed linear complementarity problem can be efficiently solved using well-developed numerical methods, such as the celebrated Lemke's method (Cottle (2009),Li (2016)).

The Lemke's method solves the mixed linear complementarity problem in a direct way. The Lemke's method firstly incorporates an artificial variable into the considered mixed linear complementary problem, and then the introduced variable is driven to zero using a sequence of Gauss-Jordan pivoting. The Lemke's method is able to find a solution in a finite number of steps. The readers can refer to Cottle (2009) for more details on the mixed linear complementarity problem and Lemke's method.

After solving the mixed linear complementarity problem for $\boldsymbol{\lambda}_k$ and $\boldsymbol{\mu}_k$, the state and control for the optimal control problem are given by

$$\boldsymbol{\gamma}_k = H^{-1}\left(\mathcal{A}_k^T \boldsymbol{\lambda}_k + \mathcal{C}_k^T \boldsymbol{\mu}_k\right). \tag{29}$$

It should also be noted that when only equality constraints $\mathcal{A}_k \boldsymbol{\gamma}_k = \boldsymbol{b}_k$ are considered, the KKT conditions yield

$$\begin{bmatrix} H & \mathcal{A}_k^T \\ \mathcal{A}_k & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_k \\ \boldsymbol{\lambda}_k \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b}_k \end{bmatrix}. \tag{30}$$

The system of linear equations (30) can be efficiently solved using basic matrix operations.

Remark 1. The primal problem (24) and the dual problem (27) is equivalent. It is straightforward to verify that the matrix H is positive definite, and the equality and inequality constraint functions are both affine, then the quadratic programming problem (24) is a convex optimization problem. Since the Slater's condition holds, i.e., the feasible solutions are assumed to exist in each iteration, then the strong duality holds between the primal problem (24) and the dual problem (27) (Boyd (2004)).

In each iteration, the mixed linear complementarity problem (27) is solved and $\boldsymbol{x}_k(\tau_i)$ and $\boldsymbol{u}_k(\tau_i)$ is calculated. Then the solution $\boldsymbol{x}_k(\tau_i)$ and $\boldsymbol{\lambda}_k(\tau_i)$ will be substituted into the next iteration. The solving process will be repeated until the defined relative error between the last two iterations is less than a specified threshold ε, i.e.,

$$\frac{|J_k - J_{k-1}|}{J_k} \leq \varepsilon. \tag{31}$$

4. NUMERICAL SIMULATIONS

The attitude stabilization problem for an axisymmetric spacecraft is considered in this section. The numerical simulations are conducted using MATLAB R2018b on a laptop equipped with 2.30 GHz CPU and 4.0 GB RAM. The dynamic equations of the spacecraft are taken from Tsiotras (1999) and it is given by

$$\begin{cases} \dot{x}_1 = amx_2 + u_1 \\ \dot{x}_2 = -amx_1 + u_2 \\ \dot{x}_3 = mx_4 + x_2x_3x_4 + (x_1/2)(1 + x_3^2 - x_4^2) \\ \dot{x}_4 = -mx_3 + x_1x_3x_4 + (x_2/2)(1 + x_4^2 - x_3^2) \end{cases} \tag{32}$$

where x_1 and x_2 represent angular velocities of the spacecraft, and x_3 and x_4 denote the states related to the Euler angles. As in Tsiotras (1999), the parameters are set to $a = 0.5$ and $m = -0.5$. Let $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$ and $\boldsymbol{u} = [u_1, u_2]^T$. The cost function is defined by

$$J = \int_{t_0}^{t_f} \boldsymbol{x}^T\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{u}\ \mathrm{d}t. \tag{33}$$

The time horizon is set to $t_f - t_0 = 5s$. In the state-dependent coefficient parameterization, the nonzero elements of $A(\boldsymbol{x})$ are given by (Wang (2019))

$$\begin{aligned}&A_{12} = -\tfrac{1}{4}, \quad A_{21} = \tfrac{1}{4}, \quad A_{31} = \tfrac{1}{2},\\ &A_{32} = x_3x_4, \quad A_{33} = \tfrac{1}{2}x_1x_3, \quad A_{34} = -\tfrac{1}{2} - \tfrac{1}{2}x_1x_4,\\ &A_{41} = x_3x_4, \quad A_{42} = \tfrac{1}{2}, \quad A_{43} = -\tfrac{1}{2}x_2x_3 + \tfrac{1}{2},\\ &A_{44} = \tfrac{1}{2}x_2x_4.\end{aligned}$$

To verify the effectiveness and efficiency of the proposed linear transcription method, two simulations are implemented. The performance is compared with the SDDRE method (Heydari (2013)) with integral control modification (Çimen (2010),Huang (2017)), pseudospectral method (Ross (2012),Rao (2010)) and quadratic programming based direct method (Gomroki (2017)).

4.1 Optimal Control Under Input Saturation

In this section, the nonlinear optimal control problem with only input saturation is considered. The initial state is $\boldsymbol{x}_0 = [0.35, -0.4, -0.35, 0.3]^T$ and the input constraints are $|u_1| \leq 0.15$ and $|u_2| \leq 0.15$. The pseudospectral method is known as an accurate method for constrained optimal control problems, and the SDDRE method can also conveniently deal with input saturation, so these two method are also implemented to evaluate the numerical performance of the proposed numerical method. In the linear direct transcription method, 40 collocation points are employed.

The state trajectories and the controls are shown in Fig. 1, 2 and 3. It can be seen that the input constraints are satisfied in all of these methods. More importantly, the comparisons also show that the linear direct transcription method and the pseudospectral method agree with each other very well, implying that the linear direct transcription method calculates a relatively accurate trajectory. The SDDRE method also successfully handles the input saturation. But as observed in Fig. 3, the SDDRE controller exhibits chattering phenomenon, which is undesirable for practical applications.

4.2 Optimal Control Under Input and State Constraints

In this section, both input and state constraints are taken into consideration. The initial state is $\boldsymbol{x}_0 = [0.5, 0, -0.5, -0.5]^T$ and the constraints are $|x_2| \leq 0.2$ and

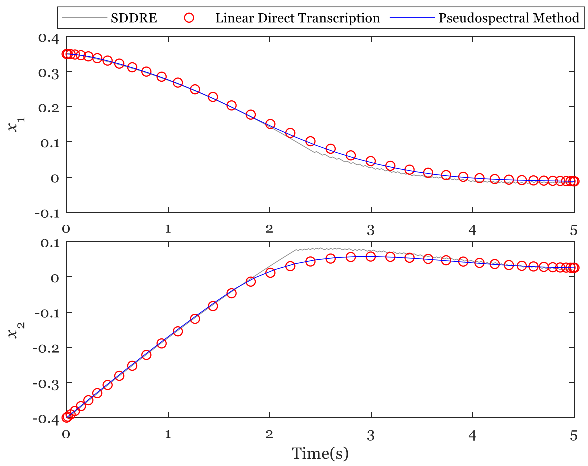

Fig. 1. Trajectories of x_1 and x_2 under input constraints

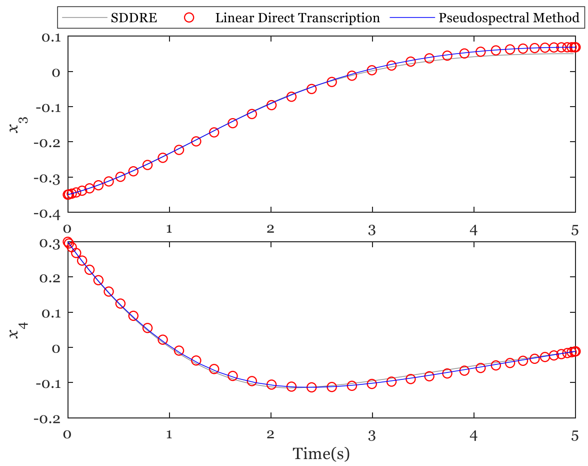

Fig. 2. Trajectories of x_3 and x_4 under input constraints

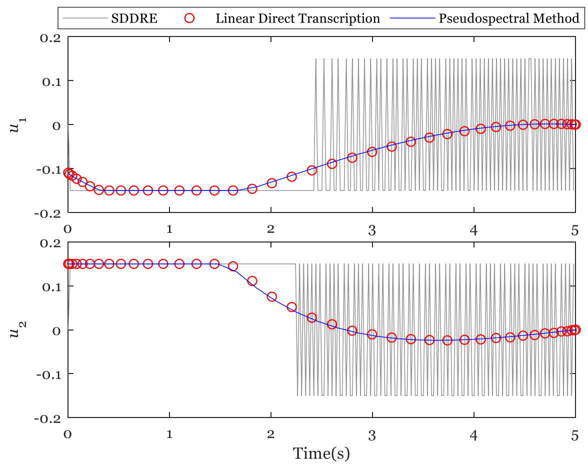

Fig. 3. Time history of u_1 and u_2 under input constraints

$|u_1| \leq 0.25$. To evaluate the efficiency of the proposed method, the quadratic programming based method in Gomroki (2017) is also implemented in this section. Specifically, the MATLAB syntax `quadprog` is used to solve the resultant quadratic programming problem. Besides, 60 collocation points are employed in the simulations. For these two methods, 10 iterations are carried out, and the numerical results for each iteration are shown in Table 1.

The algorithms are also run for 100 times to obtain the average computation time. The average times for these two methods are shown in Table 2.

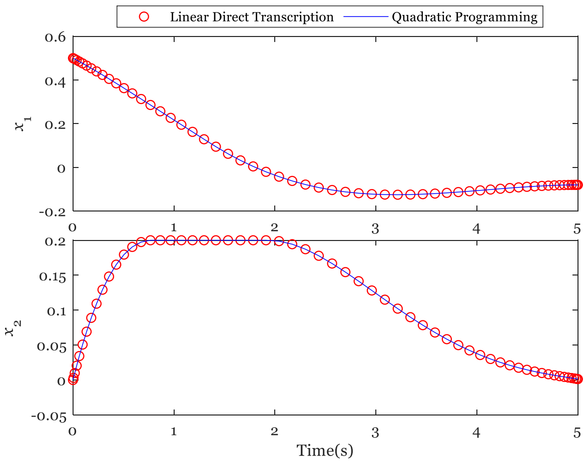

Fig. 4. Trajectories of x_1 and x_2 under input and state constraints

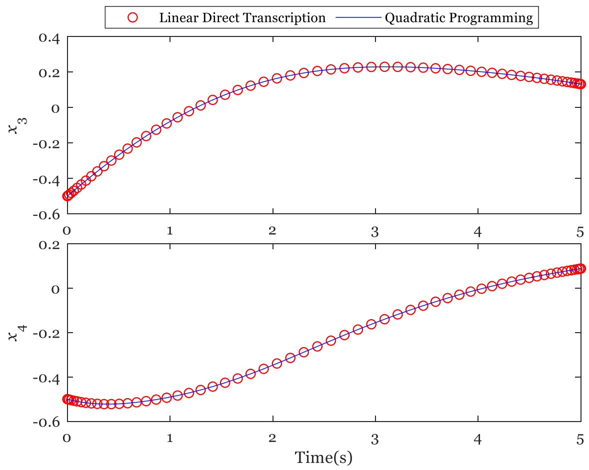

Fig. 5. Trajectories of x_3 and x_4 under input and state constraints

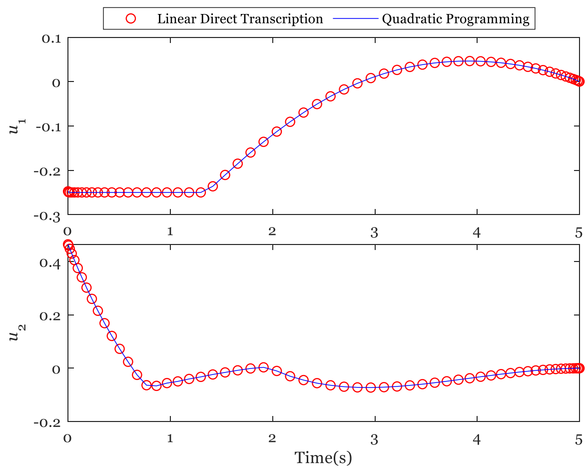

Fig. 6. Time history of u_1 and u_2 under input and state constraints

The state trajectories and the controls are shown in Fig. 4, 5 and 6. It can be observed form these figures that the

Table 1. Iteration Processes for Different Methods

	Linear Direct Transcription Method		Quadratic Programming Based Method	
Iteration	Relative Error	J	Relative Error	J
1	$--$	1.085934570731961	$--$	1.085934587527623
2	$8.5990788740e-2$	1.188100248174365	$8.5990897747e-2$	1.188100408246419
3	$1.4955705723e-2$	1.170593200742959	$1.4955837599e-2$	1.170593206357709
4	$1.7156063056e-3$	1.172604929153418	$1.7166921347e-3$	1.172606210215835
5	$1.2045290021e-4$	1.172463702500057	$1.2087377640e-4$	1.172464490005228
6	$6.4011121257e-6$	1.172471207619721	$6.4986547784e-6$	1.172472109496705
7	$2.6205844764e-7$	1.172470900363817	$2.7547991448e-7$	1.172471786504277
8	$7.8267281106e-9$	1.172470909540428	$9.0421256801e-9$	1.172471797105914
9	$1.0597879069e-10$	1.172470909416171	$1.8666891484e-10$	1.172471796887050
10	$6.0191204931e-12$	1.172470909409114	$2.0487303340e-12$	1.172471796884648

Table 2. Computation Time on Average

Method	Linear Direct Transcription	Quadratic Programming
Computational Time (s)	**0.35934**	0.75515

trajectories all satisfy the input and state constraints, and the performances of these two methods are very close to each other, which again verifies the effectiveness of the proposed method. It can also be seen from Table 1 that both methods have a fast convergence speed, and they are able to generate an accurate solution in just several iterations.

It should be noted that the proposed linear direct transcription method only takes 0.35934 seconds on average to complete the iterations, whereas the quadratic programming-based method needs 0.75515 seconds. It can then be concluded that, compared with the quadratic programming based method, the proposed method is able to achieve a relatively accurate solution with much less computational cost.

5. CONCLUSIONS

In this paper, a linear direct transcription method is proposed for nonlinear optimal control problems. Taking advantage of successive extended linearization and direct transcription, the nonlinear optimal problem is transformed into a sequence of efficiently solvable mixed linear complementarity problems. Numerical comparisons with the SDRE, pseudospectral and quadratic programming based methods are conducted, and simulation results verify the accuracy and efficiency of the proposed method.

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