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Some insights on rightmost spectral values assignment for time delay systems

Souad Amrane * **Islam Boussaada** **,*** **Fazia Bedouhene** *
Silviu-Iulian Niculescu **

* *Laboratoire de Mathématiques Pures et Appliquées, Université Mouloud Mammeri, Tizi-Ouzou, Algeria, (e-mail: amranesouad@yahoo.fr, fazia.bedouhene@ummto.dz)*
** *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes (L2S), Inria Saclay, DISCO Team, 3 rue Joliot-Curie, F-91192 Gif-sur-Yvette cedex, France (e-mail: Islam.Boussaada@l2s.centralesupelec.fr, Silviu.Niculescu@l2s.centralesupelec.fr)*
*** *IPSA, 63 Boulevard de Brandebourg, 94200 Ivry-sur-Seine, France*

Abstract: This paper investigates the stability and stabilization of some generic dynamic second-order linear time-invariant system including a single delay in the mathematical representation. As a first result, an appropriate stability criterion based on the manifold defined by the coexistence of the maximal number of negative spectral values is derived. Second, such ideas are exploited in the context of delayed output feedback by an appropriate "partial" pole placement guaranteeing simultaneously the stability in closed-loop and an appropriate exponential decay rate of the corresponding solution for the closed-loop system. To perform such an analysis, the argument principle is explicitly used. An illustrative example completes the presentation.

Keywords: Time-delay systems, frequency-domain approaches, controller design, argument principle.

1. INTRODUCTION

Time-delay is encountered in many fields, such as in Physics, Biology and Economy. It is often use in modeling transportation and propagation phenomena, population dynamics or in approximating models described by partial differential equations. A strong and ever-growing interest on time-delay systems has been reported over the years, see, for instance, Niculescu [2001], Bellman and Cooke [1963], Hale and Lunel [1993], Stépán [1979] and references therein.

In the stability analysis of time-delay systems, many approaches have been developed in both time- and frequency-domain by generalizing methods and techniques used in the stability characterization of finite-dimensional systems. As in finite-dimension, a wide range of frequency-domain techniques have been derived, addressing the distribution of the roots of the corresponding characteristic equation and related properties, see, for instance, Olgac and Sipahi [2002], Michiels and Niculescu [2007], Walton and Marshall [1987], Sipahi et al. [2011], Boussaada et al. [2015], Cooke and van den Driessche [1986].

If the scalar case is completely understood and the existing links between the spectral abscissa, the maximal allowed multiplicity of the characteristic roots, and the system's parameters is completely characterized, the second-order system still needs a deeper analysis. This paper addresses such problems.

This work is motivated by recent studies Boussaada and Niculescu [2016b], Boussaada et al. [2018], Boussaada and Niculescu [2016b,a] where a property called *Multiplicity-Induced-Dominancy* (MID) is emphasized, which consists in characterizing the exponential decay rate of the trivial solution. Recently, in Boussaada and Niculescu [2016b], a result by Polya and Szegô is revisited and exploited allowing to explicitly derive a *bound* for the number of real roots of the corresponding characteristic function. Such a bound is nothing else that *degree of the quasipolynomial* Boussaada et al. [2018],. In Boussaada and Niculescu [2018], an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems is explicitly provided.

Quite recently, Amarne et al. [2018] showed that the *multiplicity of a real root* itself is not important as such but its *connection* with the *dominancy* of this root is a meaningful tool in constructing controllers. As a matter of fact, it is proven that, under appropriate conditions, the coexistence of the maximal number of negative distinct roots guarantees their dominancy. In this case, an adequate factorization is derived in the scalar and second-order delay differential equations with single delay allowing to write the quasipolynomial in an some appropriate integral operator form. Furthermore, if these (real) roots are negative, the asymptotic stability of the trivial solution follows straightforwardly.

It is worth mentioning that, in most of the cases, such a factorization is hard to be established, especially for the quasipolynomials with higher-degrees. The stability criteria initially proposed and developed in Stépán [1979] and generalized later in Hassard [1997] allow to compute analytically the number of characteristic roots in the right-half complex plane (i.e., unstable roots) for time-delay system of retarded type by avoiding the direct calculation of the corresponding improper integral. Our study is mainly inspired by Boussaada et al. [2020], where the property of MID is investigated for the generic second-order retarded differential equation. In other words, the present paper focuses on the application of the Stépàn-Hassard approach to show the dominancy of real spectral values which are not necessarily multiple. To illustrate our results, we consider the problem of stabilizing the Mach number of a transonic flow in a wind tunnel. The corresponding mathematical model is a second-order delay-differential equation where the propagation phenomena was represented by using an appropriate constant delay.

The remaining of the paper is organized as follows. In Section 2, we present the problem formulation and we recall some known results on the spectrum distribution of retarded delay systems useful in the forthcoming sections. Next, conditions on the system parameters guaranteeing the co-existence of the maximal number of real roots is carried out in Section 3. In the particular case of equidistributed real roots the argument principle is applied to prove the dominancy of such real spectral values. All illustrative example completes the presentation and some concluding remarks end the paper.

The notations are standard.

2. PRELIMINARIES

2.1 Problem Statement

Consider the following generic LTI system including one delay:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{1}$$

under appropriate initial conditions, where $x(t) \in \mathbb{R}^n$ and A and B are $n \times n$ real matrices. The corresponding spectral values are characterized by evaluating for s the following characteristic equation:

$$\Delta(s) = \det(sI - A - Be^{-\tau s}) = 0. \tag{2}$$

It is well known that the asymptotic stability of the trivial solution is guaranteed if all roots of (2), called characteristic roots, are located in the left-half plane. Notice also that real spectral values correspond to non oscillatory solutions of (1).

It is also well known that second-order linear systems capture the dynamic behavior of many natural phenomena and have found numerous applications in a variety of fields, such as vibration, structural analysis and human balancing. To the best of the authors' knowledge, the first studies have been done by Gorelik [1939] (discussing the effect of the electron transit time in a vacuum tube) and Minorsky [1942] (in connection with the self-excited oscillations in a ship-stabilization problem). Finally, the first discussion on the multiple characteristic roots of such systems seems to be proposed by Pinney [1958] more than 60 years ago.

In this paper, we focus on planar systems (1) (i.e. $n = 2$) and restrict our analysis to systems where the corresponding quasipolynomials writes under the form:

$$\Delta(s,\tau) = P(s) + Q(s)e^{-s\tau}$$

with $\deg(P) = 2$ and $\deg(Q) = 1$. As mentioned in the Introduction, we will explicitly *investigate the effect of the coexistence of real roots on the stability of the trivial solution.* In other words, we wish to give an answer to the following question: *Is the coexistence of sufficiently many negative roots guarantees the location of the remaining roots in the left-half plane?*

2.2 Prerequisites

In complex analysis, it is well-known that the *argument principle* is a consequence of Cauchy theorem, and it connects the winding number of a closed rectifiable curve (see, e.g., Conway [1984] for a deeper discussion on the topics) with the number of zeros and poles inside the curve. This result gives insights on the location of zeros and poles of a given meromorphic function.

Theorem 1. (Argument principle): let $V \in \mathbb{C}$ be a bounded domain with smooth boundary Γ positively oriented (counter-clockwise) and let f be a meromorphic function inside the contour Γ, then

$$\frac{1}{2i\pi}\oint_\Gamma \frac{f'(z)}{f(z)}dz = Z - P, \tag{3}$$

where P and Z denote the number of poles and zeros of f in V, counted with their multiplicities.
In particular, if $f(z)$ is analytic inside Γ, then the left-hand side of (3) gives the number of zeros of $f(z)$ inside Γ.

We need also to recall the statement of Hassard's Theorem:
Theorem 2. (Hassard's Theorem). Let $A_1, ..., A_m$ be real n by n matrices, and let $\tau_1, \cdots, \tau_m$ be nonnegative reals. Let

$$\Delta(s,\tau) = \det(sI - \sum_{j=1}^{m} e^{-s\tau_j}A_j). \tag{4}$$

Let $\rho_1, \cdots, \rho_J$ be the positive zeros of $R(y) = \Re(i^{-n}\Delta(iy))$, counted by multiplicity and ordered so that $\rho_1 \geq \cdots \geq \rho_j > 0$. For each $j = 1, \cdots, J$ such that $\Delta(i\rho_j) = 0$, assume that the multiplicity of $i\rho_j$ a zero of $\Delta(\lambda)$ is the same as the multiplicity of ρ_j as a zero of $R(y)$. Then, the number of roots of the characteristic equation$\Delta(\lambda) = 0$ which lie in $\Re(s) > 0$, counted by multiplicity, is given by the formula

$$\frac{n-K}{2} + \frac{1}{2}(-1)^J sgnS^{(\kappa)}(0) + \sum_{j=1}^{J}(-1)^{j-1}sgnS(\rho_j) \tag{5}$$

where K is the number of zeros of $\Delta(s,\tau)$ on $\Re(\lambda) = 0$, counted by multiplicity, k is the multiplicity of $s = 0$ as a root of $\Delta(s,\tau) = 0$, and $S(y) = Im(i^{-n}\Delta(iy))$. Furthermore, the count (5) is odd if $\Delta^{(k)}(0) < 0$ and is even if $\Delta^{(k)}(0) > 0$. If $R(y)$ has no positive zeros, set $r = 0$ and omit the summation term in (5). If $\lambda = 0$ is not a root of the characteristic equations, set $\kappa = 0$ and interpret $S(0)^{(0)}$ as $S(0)$ and $\Delta(0)^{(0)}$ as $\Delta(0)$.

3. MAIN RESULTS

Consider now, the second-order system

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = u(t) \tag{6}$$

where u is the unknown control. Assume that the system (6) is unstable in the uncontrolled case ($u(t) = 0$). Our aim is to construct an appropriate delayed-state-feedback controller of the form:

$$u(t) = -\alpha_0 x(t-\tau) - \alpha_1 \dot{x}(t-\tau), \tag{7}$$

allowing to guarantee the stability of the system (6) in closed-loop. The characteristic quasi-polynomial function corresponding to the closed-loop system is described as follows:

$$\Delta(s,\tau) = P(s) + Q(s)e^{-s\tau} = 0, \tag{8}$$

where $P(s)$ and $Q(s)$ are polynomials in s with degree of $Q(s)$ is less then the degree of $P(s)$. In our case, we will have explicitly

$$P(s) = s^2 + a_1 s + a_0 s \quad \text{and} \quad Q(s) = \alpha_1 s + \alpha_0. \tag{9}$$

3.1 On qualitative properties of s_1 as a root of (8)

The following proposition gives conditions on the parameter coefficients that allows assigning a maximum number of spectral values of the second-order system (6)-(7).

Proposition 3. The following assertions hold:

- The quasipolynomial (8) admits four distinct real spectral values s_1, s_2, s_3 and s_4 with $s_4 < s_3 < s_2 < s_1$ if and only if the parameters a_1, a_2 , α_1 and α_0 satisfy

$$\begin{cases} a_1(\tau) = \dfrac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1^2 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2^2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3^2 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4^2 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ a_0(\tau) = \dfrac{1}{Q(\tau)} \begin{vmatrix} s_1^2 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ s_2^2 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ s_3^2 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ s_4^2 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ \alpha_1(\tau) = \dfrac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1 & s_1e^{-s_1\tau} & s_1^2 \\ 1 & s_2 & s_2e^{-s_2\tau} & s_2^2 \\ 1 & s_3 & s_3e^{-s_3\tau} & s_3^2 \\ 1 & s_4 & s_4e^{-s_4\tau} & s_4^2 \end{vmatrix}, \\ \alpha_0(\tau) = \dfrac{1}{Q(\tau)} \begin{vmatrix} 1 & s_1 & s_1^2 & s_1e^{-s_1\tau} \\ 1 & s_2 & s_2^2 & s_2e^{-s_2\tau} \\ 1 & s_3 & s_3^2 & s_3e^{-s_3\tau} \\ 1 & s_4 & s_4^2 & s_4e^{-s_4\tau} \end{vmatrix}, \end{cases} \tag{10}$$

  where

$$Q(\tau) = \begin{vmatrix} 1 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}. \tag{11}$$

- The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such that

$$a_1(\tau_0) + s_2 = 0. \tag{12}$$

Sketch of the Proof:

- According to the G. Pólya and G. Szegö Theorem Pólya and Szegő [1972], the number of real roots of (8) is four, hence we investigate the existence of four distinct real spectral values $s_1 > s_2 > s_3 > s_4$. The coefficients of the quasi-polynomial function (8) described by (10) and (11) are obtained by solving the following system

$$s_i^2 + a_1 s_i + a_0 + (\alpha_1 s_i + \alpha_0)\exp(-s_i\tau) = 0, \; i = 1 \ldots 4, \tag{13}$$

  that we can represents under the more suitable form:

$$\begin{bmatrix} 1 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -s_1^2 \\ -s_2^2 \\ -s_3^2 \\ -s_4^2 \end{bmatrix} \tag{14}$$

  System (13) admits a unique solution because the Vandermonde-type matrix in $Q(\tau)$ is invertible, for every $\tau > 0$. Indeed, its determinant is positive since $\tau \mapsto Q(\tau)$ is increasing from 0 to ∞, with $Q(0) = 0$. To get the formulas (10), then just apply the Cramer's rule.
- The negativity of the root s_1 is shown through the variation of the function

$$\tau \mapsto a_1(\tau) + s_2. \tag{15}$$

  So, for all values of $\tau \in \mathbb{R}^{+*}$ ($\tau > 0$), this later function is continuous and increasing from $-\infty$ to $-s_1$, from which we deduce that $\tau \mapsto a_1(\tau) + s_2$ takes a positive values if and only if $s_1 < 0$. See Amarne et al. [2018].

□

Due to the computation complexity of these time-delay dependent coefficients, we consider only the case of equidistance roots. This allows also to decrease the number of parameters to handle. So, we suppose that the roots of (8) are such that:

$$d = |s_1 - s_2| = |s_2 - s_3| = |s_3 - s_4|. \tag{16}$$

Substituting the formula (16) in (10) and (11), the above parameter coefficients can be rewritten as follows:

$$\begin{cases} a_1(\tau) = \dfrac{1}{e^{d\tau}-1}\left(2s_1 - 5d + de^{d\tau} - 2s_1e^{d\tau}\right) \\ a_0(\tau) = \dfrac{1}{(e^{d\tau}-1)^2}\Big(6d^2 - ds_1e^{2d\tau} + 6ds_1e^{d\tau} \\ \qquad -5ds_1 + s_1^2e^{2d\tau} - 2s_1^2e^{d\tau} + s_1^2\Big) \\ \alpha_1(\tau) = 2de^{-2d\tau}\dfrac{e^{\tau s_1}}{e^{-d\tau}-1} \\ \alpha_0(\tau) = \dfrac{-2de^{-2d\tau}e^{\tau s_1}}{(e^{-d\tau}-1)^2}\left(3d - s_1 + s_1e^{-d\tau}\right). \end{cases} \tag{17}$$

The following theorem gives conditions on the negativeness of s_1.

Theorem 4. The spectral values s_1 is negative if and only if one of the following equivalent conditions is satisfied:

(1) The spectral value s_1 is explicitly given by:

$$s_1 = -4\frac{d}{e^{d\tau}-1}.$$

(2) The delay τ takes the value

$$\tau^* = \frac{1}{d}\ln\frac{1}{s_1}\left(-4d + s_1\right).$$

(3) The distance d between the real roots satisfies

$$d = \left[\left\{ \frac{1}{4}s_1 - \frac{1}{\tau}\text{LambertW}\left(l, \frac{\tau s_1}{4}e^{\frac{\tau s_1}{4}}\right) \mid l \in \mathbb{Z} \right\} \setminus \left\{ \frac{2i\pi l}{\tau} \mid l \in \mathbb{Z} \right\} \right] \cap]0, +\infty[.$$

Sketch of the Proof: We use the characterization of the negativeness of s_1 given by equation (12). Observe that, as $\tau \to \infty$

$$a_1(\tau) = \frac{1}{e^{d\tau} - 1}\left(2s_1 - 5d + de^{d\tau} - 2s_1e^{d\tau}\right) \to d - 2s_1$$

with the property that $\tau \mapsto a_1(\tau)$ is bounded by $-s_1$. So if there exists τ^* such that $a_1(\tau^*) + s_1 - d = 0$, we obtain that $s_1 < 0$, and vis versa. Now, to get the value of s_1, τ^* and d respectively, we just have to solve the equation

$$4d - s_1 + s_1e^{d\tau} = 0. \tag{18}$$

Easy computation gives the value of s_1 and $\tau^\star$. For the value of d, it can be calculated using the Lambert W function since the unknown (d) appears both outside and inside the exponential function. The Lambert W function is defined as the multivalued function that satisfies

$$x = W(x)e^{W(x)},$$

for any complex number x. Equivalently, it may be defined as the inverse of the complex function $f(x) = xe^x$.
In order to solve equation (18) for d we rewrite it as follows:

$$\left(d - \frac{s_1}{4}\right)e^{-\tau d} = -\frac{s_1}{4},$$

this implies that

$$-\tau\left(d - \frac{s_1}{4}\right)e^{-\tau\left(d - \frac{s_1}{4}\right)} = \frac{s_1\tau}{4}e^{\frac{s_1\tau}{4}}.$$

From the later equation, we have

$$\text{LambertW}\left(\frac{\tau s_1}{4}e^{\frac{\tau s_1}{4}}\right) = -d\tau + \frac{\tau s_1}{4}. \tag{19}$$

Hence

$$d = \left[\left\{ \frac{1}{4}s_1 - \frac{1}{\tau}\text{LambertW}\left(l, \frac{\tau s_1}{4}e^{\frac{\tau s_1}{4}}\right) \mid l \in \mathbb{Z} \right\} \setminus \left\{ \frac{2i\pi l}{\tau} \mid l \in \mathbb{Z} \right\} \right] \cap]0, +\infty[.$$

□

4. ILLUSTRATIVE EXAMPLE

As an illustration of the presented result, we consider the problem of stabilizing the Mach number of a transonic flow in a wind tunnel. The analysis of transonic flows is a challenging problem in compressible fluid dynamics, since a full model of the flow would involve considering the Navier–Stokes equations in a three-dimensional domain and boundary controls for temperature and pressure regulation.

A simplified model was considered in Armstrong and Tripp [1981] in order to analyze the response of the Mach number of the flow to changes in the guide vane angle. Instead of using a PDE model, propagation phenomena are modeled through a time delay, leading to the time-delay system

$$\begin{cases} \kappa m'(t) + m(t) = k\vartheta(t - \tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases} \tag{20}$$

in which m, ϑ, and u represent perturbations of the Mach number of the flow, the guide vane angle, and the input of the guide vane actuator, respectively, with respect to steady-state values.

The parameters κ and k depend on the steady-state operating point and are assumed to be constant as long as m, ϑ, and u remain small, and satisfy $\kappa > 0$ and $k < 0$. Next, the parameters $\zeta \in (0, 1)$ and $\omega > 0$ come from the design of the guide vane angle actuator and are thus independent of the operating point. The time delay τ_0 is assumed to depend only on the temperature of the flow. Notice that, in the absence of control $(u(t) = 0)$, the open-loop system (20) is exponentially stable.

The design of the stabilizing feedback for (20) improving its stability properties has been considered in Manitius [1984], Boussaada et al. [2018]. The design we propose here is a delayed PD controller which can be written $u(t) = \beta_0 x(t - \tau_1) + \beta_1\dot{x}(t - \tau_1)$.

In closed-loop system, the corresponding characteristic equation writes as follows:

$$\Delta(s, \tau_1) = (s - a)\left((s\beta_1 + \beta_0 t)e^{-s\tau_1} + 2\,\omega\, s\zeta + \omega^2 + s^2\right)$$

Since the parameter a is fixed then, by focusing on the second factor only, one gets:

$$\Delta_4(s, \tau_1) = s^2 + 2\,\omega\, s\zeta + \omega^2 + (s\beta_1 + \beta_0)e^{-s\tau_1}. \tag{21}$$

To assign four equally distributed rightmost real spectral values $s_k = -k$ for $k \in [1, \ldots, 4]$, we may use directly (17).

For instance, if $\omega = 1.2$ and the damping factor $\zeta = 0.472$, then it is sufficient to set

$$\beta_0 = -\frac{3087}{6050},\ \beta_1 = -\frac{343}{3630},\ \tau_1 = \ln\left(\frac{22}{7}\right). \tag{22}$$

Figure 1 exhibits the spectrum distribution corresponding to (21).

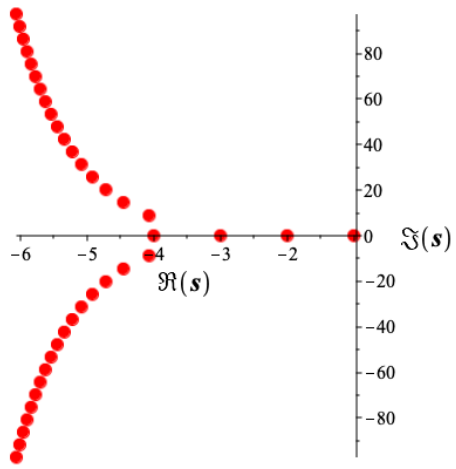

Fig. 1. Zeros distribution of quasipolynomial (21) with $\omega = 1.2$, $\zeta = 0.472$ and the controller gains given by .

5. CONCLUSION

A new dominancy result of the spectral values which are not necessarily multiple is established through this paper for a class of second-order time-delay system with a single delay. The stability analysis is based on the argument principle and the Stépàn-Hassard formula which allow counting the number of unstable roots of the corresponding characteristic equation. Further, to show the potential applicability of the result, an illustrative example is provided. The problem of stabilizing the Mach number of a transonic flow in a wind tunnel is discussed and a stabilizing delayed PD-controller is provided. This study is not achieved yet since we considered only the case where the real part of the quasipolynomial function has no real roots. The other case remains under investigation.

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