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Generalized $\mathcal{H}_2$ Control with Transients for Linear Hybrid Systems ⋆

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Abstract: This paper considers linear time-varying hybrid systems and introduces a notion of the finite-horizon generalized $\mathcal{H}_2$ norm with transients. It is defined as the worst-case peak value of the output in response to uncertain initial states and external disturbances. Such a measure represents the induced operator norm from L_2 to L_∞ because the peak value of the vector signal is considered as its generalized L_∞ norm. This approach allows to characterize the generalized $\mathcal{H}_2$ norm in terms of both the difference Lyapunov equation and difference linear matrix inequalities (DLMIs). By using the derived characterization the optimal control and Pareto optimal controls are synthesized minimizing the finite-horizon generalized $\mathcal{H}_2$ norm with transients. Finally, an example is given to illustrate the proposed technique.

Keywords: Optimal control of hybrid systems, Pareto optimal control, finite horizon, generalized $\mathcal{H}_2$-norm, LMIs

1. INTRODUCTION

For the first time, the generalized $\mathcal{H}_2$ norm of a continuous-time LTI system with zero initial conditions was introduced in Wilson (1989) as the worst-case peak value of the output (in terms of the L_∞ norm) in response to the external disturbance with the finite energy (in terms of the L_2 norm). Thus, it represents nothing but the induced norm from L_2 to L_∞. By using an approach based on linear operator theory the generalized $\mathcal{H}_2$ norm was characterized in terms of a solution to the Lyapunov equation.

The generalized $\mathcal{H}_2$ norm of sampled-data systems as the induced norm from L_2 to L_∞ was analytically formulated first in Bamieh et al. (1991) by employing the idea of the lifting technique, but no explicit computation method for the norm was provided. An explicit computation method for the induced norm was developed in Zhu and Skelton (1995). A discretization approach to compute the generalized $\mathcal{H}_2$ norm of LTI sampled-data systems is developed in Kim and Hagiwara (2017a). It is based on the lifting technique together with a gridding approximation method. Upper and lower bounds of the generalized $\mathcal{H}_2$ norm in sampled-data systems were obtained in Kim and Hagiwara (2017b).

The notion of the generalized $\mathcal{H}_2$ norm is close to the concept of finite-time stability of linear time-varying systems with a bounded initial state which was considered in Amato et al. (2006); Garcia et al. (2009). Similar results were received in Amato et al. (2011, 2013) for hybrid systems and impulse linear systems (systems with jumps). Moreover, in Nešíc et al. (2013) was received the characterization of finite-gain L_p stability of hybrid systems.

The optimal generalized $\mathcal{H}_2$ controllers for LTI systems over the infinite horizon were characterized in terms of solutions to Riccati equations, see Rotea (1993) and Wilson et al. (1998) for continuous-time and discrete-time systems, respectively. The characterization in terms of LMIs was derived in Balandin and Kogan (2017); Balandin et al. (2019a). For impulse linear systems the optimal generalized $\mathcal{H}_2$ controllers were obtained in Khargonekar and Sivashankar (1991) and expressed in terms of Riccati differential equations with jumps. However, the synthesis problem of minimizing the induced norm from L_2 to L_∞ of linear time-varying hybrid systems over a finite horizon has not been resolved until today.

In this paper, the concept of finite-horizon generalized $\mathcal{H}_2$ norm with transients for linear time-varying hybrid systems under external disturbance and initial states over finite-horizon is introduced. It is understood as the induced norm from L_2 to L_∞. The finite-horizon generalized $\mathcal{H}_2$ norm with transients is characterized in terms of both solutions to difference Lyapunov equations as well as solutions to difference LMIs. The DLMI characterization makes possible to synthesize optimal controllers including multi-objective ones minimizing generalized $\mathcal{H}_2$ norms of several outputs.

The paper is organized as follows. Section 2 provides some mathematical notations. Section 3 presents the definition of the finite-horizon generalized $\mathcal{H}_2$ norm with transients and its characterization in terms of solutions to difference Lyapunov equations (Theorem 1) and solutions to

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difference LMIs (Theorem 2). Optimal control including multi-objective one is synthesized in Section 4. An illustrative example concerning an optimal vibration isolation problem for the one-degree-of-freedom mechanical system is considered in Section 5. The conclusion is proposed in Section 6.

2. MATHEMATICAL NOTATIONS

This section gives the mathematical notations used in the paper. Let vector $x \in \mathbb{R}^n$ be partitioned as $x = \mathrm{column}(x_1, \ldots, x_m)$, where $x_k \in \mathbb{R}^{n_k}$ and $n_1 + \ldots + n_m = n$. Denote a $(1,2)$-norm and $(\infty,2)$-norm of the vector x through

$$|x|_{(1,2)} \triangleq \sum_{k=1}^{m} |x_k|_2, \qquad |x|_{(\infty,2)} \triangleq \max_{k=1,\ldots,m} |x_k|_2,$$

where $|x_k|_2$ is the Euclidean norm of the vector x_k. In particular, if $m = 1$, then $|x|_{(1,2)} = |x|_{(\infty,2)} = |x|_2$, and if $m = n$, then $|x|_{(1,2)} = |x|_1$ and $|x|_{(\infty,2)} = |x|_\infty$. Here $|x|_1$ and $|x|_\infty$ are standard 1-norm and ∞-norm of the vector x, respectively. The space of n-dimensional real vectors equipped with vector $(p,2)$-norm is denoted by $\mathbb{R}^n_{(p,2)}$ $(p = 1, \infty)$. It is clear that $(\mathbb{R}^n_{(1,2)})^* \cong \mathbb{R}^n_{(\infty,2)}$. The notation $\mathbb{R}^n_2$ denotes the space of n-dimensional real vectors equipped with the Euclidean norm $|\cdot|_2$.

The norm of a matrix $A \in \mathbb{R}^{l\times n}$ as the induced norm of a mapping from $\mathbb{R}^n_{(1,2)}$ to $\mathbb{R}^l_2$ is defined as

$$\|A\|_{2/(1,2)} \triangleq \sup_{|x|_{(1,2)}=1} |Ax|_2.$$

It can be computed as follows. Let assume that the symmetric $(n\times n)$-matrix $P = A^\top A \geq 0$ be partitioned on blocks $P = (P_{i,j})$, $P_{i,j} \in \mathbb{R}^{n_i\times n_j}$, $i,j = 1,\ldots,m$, $n_1 + \ldots + n_m = n$, according to partition of vector $x \in \mathbb{R}^n_{(1,2)}$, then

$$\|A\|_{2/(1,2)} = \max_{i=1,\ldots,m} \lambda_{\max}(P_{i,i}),$$

where $\lambda_{\max}(P_{i,i})$ is the maximum eigenvalue of the matrix $P_{i,i}$. In particular, if $m = 1$, then $\|A\|_{2/(1,2)} = \lambda_{\max}(A^\top A)$, and if $m = n$, then $\|A\|_{2/(1,2)} = d_{\max}(A^\top A)$, where $d_{\max}(\cdot)$ is the maximum diagonal entry. For the sake of brevity, we denote by $\lambda_{\mathrm{gmax}}(P)$ the generalized maximum eigenvalue of P, i.e, the maximum of the maximum eigenvalue of the matrices $P_{i,i}$. Furthermore, by analogy with the notation of the matrix 2-norm, we can write the $2/(1,2)$-norm of matrix A as

$$\|A\|_{2/(1,2)} = \lambda_{\mathrm{gmax}}(A^\top A).$$

For a real vector-valued finite sequence $w = \{w_k\}$ the notation $\|w\|_{l_\infty}$ is used to denote the $l_\infty([M,N], \mathbb{R}^n_{(\infty,2)})$ norm under the spatial $(\infty,2)$-norm, i.e.,

$$\|w\|_{l_\infty} = \max_{k=M,\ldots,N} |w_k|_{\infty,2}.$$

Similarly, the notation $\|w\|_{l_1}$ means the $l_1([M,N], \mathbb{R}^n_{(1,2)})$ norm under the spatial $(1,2)$-norm, i.e.,

$$\|w\|_{l_1} = \sum_{k=M}^{N} |w_k|_{1,2}.$$

Finally, the notations $\|v\|_{L_2}$ and $\|w\|_{l_2}$ are used to mean the $L_2\big([t_0,t_N], \mathbb{R}^n_2\big)$ norm of a real vector-valued function $v(t)$ and the $l_2\big([M,N], \mathbb{R}^n_2\big)$ norm of a real vector-valued finite sequence w_k, respectively, i.e.,

$$\|v\|^2_{L_2} \triangleq \int_{t_0}^{t_N} |v(t)|_2^2 dt, \qquad \|w\|^2_{l_2} \triangleq \sum_{k=M}^{N} |w_k|_2^2.$$

3. FINITE-HORIZON GENERALIZED $\mathcal{H}_2$ NORM

Consider a linear time-varying hybrid system described by the set of differential and difference equations

$$\begin{aligned} \dot{x} &= A_c(t)x + \Delta_c(t)\xi_k + B_c(t)v, \\ \xi_{k+1} &= A_{d,k}\xi_k + \Delta_{d,k}x(t_k) + B_{d,k}w_k, \\ z_k &= C_{c,k}x(t_k) + C_{d,k}\xi_k, \end{aligned} \tag{1}$$

here $t \in [t_k, t_{k+1})$, $k = 0,\ldots,N-1$, $x \in \mathbb{R}^{n_x}_2$ and $v(t) \in L_2([t_0,t_N], \mathbb{R}^{n_v}_2)$ are the continuous-time state and disturbance, $\xi_k \in \mathbb{R}^{n_\xi}_2$ and $\{w_k\} \in l_2([0,N-1], \mathbb{R}^{n_w}_2)$ are the discrete-time state and disturbance, and $z_k \in \mathbb{R}^{n_z}_{(\infty,2)}$ is the target output. It is assumed that the system initial states $x(t_0) = x_0$ and ξ_0 are unknown. The matrix functions A_c, B_c, Δ_c are piecewise continuous and bounded.

The system (1) generates the linear operator $\mathcal{S}$ mapping the initial state (x_0, ξ_0) and the disturbances v, w to output $z \in l_\infty([0,N], \mathbb{R}^{n_z}_{(\infty,2)})$, i.e.,

$$\mathcal{S} : \big(x_0, \xi_0, v(t), \{w_k\}\big) \mapsto \{z_k\}. \tag{2}$$

Define the norm of the element (x_0, ξ_0, v, w) in the space $\mathbb{R}^{n_x}_2 \times \mathbb{R}^{n_\xi}_2 \times L_2([t_0,t_N], \mathbb{R}^{n_v}_2) \times l_2([0,N-1], \mathbb{R}^{n_w}_2)$ as

$$\big\|(x_0, \xi_0, v, w)\big\|_{(R,2)} \triangleq \sqrt{\zeta_0^\top R\,\zeta_0 + \|v\|^2_{L_2} + \|w\|^2_{l_2}}, \tag{3}$$

here $\zeta_0 = \mathrm{column}(x_0, \xi_0)$ and $R = \mathrm{diag}(R_x, R_\xi)$, where $R_x = R_x^\top > 0$, $R_\xi = R_\xi^\top > 0$ are the weighting matrices.

For the system (1) we define the finite-horizon generalized $\mathcal{H}_2$ norm with transients as the induced norm of the operator $\mathcal{S}$, i.e.,

$$\big\|\mathcal{S}\big\|_{\infty/(R,2)} = \sup\big\{\big\|z\big\|_{l_\infty} : \big\|(x_0, \xi_0, v, w)\big\|_{(R,2)} \leq 1\big\} \tag{4}$$

One can show that this is equivalent to

$$\big\|\mathcal{S}\big\|_{\infty/(R,2)} = \sup_{x_0,\xi_0,v,w} \frac{\max\limits_{k=0,\ldots,N} |z_k|_{\infty,2}}{\sqrt{\zeta_0^\top R\,\zeta_0 + \|v\|^2_{L_2} + \|w\|^2_{l_2}}},$$

where the supremum is taken over all elements (x_0, ξ_0, v, w) satisfying $\zeta_0^\top R\,\zeta_0 + \|v\|^2_{L_2} + \|w\|^2_{l_2} \neq 0$.

Note, that the weighting matrices R_x and R_ξ can be interpreted as a measure of the relative importance of the uncertainty in initial conditions versus the uncertainty in the disturbances. A "smaller" size of R_x, R_ξ reflect greater uncertainty in the initial conditions.

In the specific cases, when the system (1) contains no continuous part, that is $v(t) \equiv 0$ and $x(t) \equiv 0$, the proposed finite-horizon generalized $\mathcal{H}_2$ norm is the maximal output deviation, which was introduced in Balandin et al. (2019b).

Let us introduce the following matrices

$$\widehat{A}_k = \begin{bmatrix} A_{c,k} & \Delta_{c,k} \\ \Delta_{d,k} & A_{d,k} \end{bmatrix}, \quad \widehat{Q}_k = \begin{bmatrix} Q_{c,k} & 0 \\ 0 & B_{d,k}B_{d,k}^\top \end{bmatrix},$$

$$\widehat{C}_k = \big(C_{c,k},\, C_{d,k}\big),$$

where

$$A_{c,k} = \Phi(t_{k+1}, t_k), \quad \Delta_{c,k} = \int\limits_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)\Delta_c(\tau)d\tau,$$

$$Q_{c,k} = \int\limits_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B_c(\tau)B_c^\top(\tau)\Phi^\top(t_{k+1}, \tau)d\tau,$$

and the matrix $\Phi(t, \tau)$ is the state transition matrix of the equation $\dot{x} = A_c(t)x$. It is straightforward to check that

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A_c(t)\Phi(t, \tau), \qquad \Phi(\tau, \tau) = I.$$

The following result holds.

Theorem 1. The finite-horizon generalized $\mathcal{H}_2$ norm with transients of the system (1) is given by the formula

$$\|\mathcal{S}\|_{\infty/(R,2)} = \max_{k=0,\dots,N} \lambda_{\mathrm{gmax}}^{1/2}(\widehat{C}_k Y_k \widehat{C}_k^\top), \tag{5}$$

where $Y_k = Y_k^\top \geq 0$ is the solution to the equation

$$Y_{k+1} = \widehat{A}_k Y_k \widehat{A}_k^\top + \widehat{Q}_k, \quad Y_0 = R^{-1}. \tag{6}$$

Proof. Let us consider for the given linear operator (2) its dual operator $\mathcal{S}^*$, which is defined as

$$\mathcal{S}^* : \{z_k\} \mapsto \big(x_0, \xi_0, v(t), \{w_k\}\big), \tag{7}$$

where $z \in l_1([0, N], \mathbb{R}^n_{(1,2)})$ and $\big(x_0, \xi_0, v, w\big) \in \mathbb{R}_2^{n_x} \times \mathbb{R}_2^{n_\xi} \times L_2([t_0, t_N], \mathbb{R}_2^{n_v}) \times l_2([0, N-1], \mathbb{R}_2^{n_w})$. The norm of the operator $\mathcal{S}^*$ is given by

$$\|\mathcal{S}^*\|_{(R,2)/1} = \sup\big\{\|(x_0, \xi_0, v, w)\|_{(R,2)} : \|z\|_{l_1} \leq 1\big\}$$

and, according to duality, the following equality holds:

$$\|\mathcal{S}\|_{\infty/(R,2)} = \|\mathcal{S}^*\|_{(R,2)/1}. \tag{8}$$

Therefore, we can calculate the norm of the dual operator $\mathcal{S}^*$ instead of calculating the norm of the operator $\mathcal{S}$.

To determine the form of $\mathcal{S}^*$, let $z \in l_1([0, N], \mathbb{R}^n_{(1,2)})$, then

$$\langle z, \mathcal{S}y\rangle = \langle \mathcal{S}^* z, y\rangle_{(R,2)}, \tag{9}$$

where $y = (x_0, \xi_0, v, w)$ and the first two duality brackets are given by

$$\langle z, \zeta\rangle = \sum_{k=0}^{N-1} z_k^\top \zeta_k.$$

The last duality brackets $\langle\cdot,\cdot\rangle_{(R,2)}$ can be represented as an inner product of a pair of elements y_1 and y_2 as

$$\langle y_1, y_2\rangle_{(R,2)} = \zeta_{0,1}^\top R\,\zeta_{0,2} + \sum_{k=0}^{N-1} w_{1,k}^\top w_{2,k} + \int\limits_{t_0}^{t_N} v_1^\top(\tau)v_2(\tau)d\tau,$$

which consistent with the definition of the norm (3).

Rewrite the system (1) in the semi-discrete form, it means discretization only by continuous-time state x, while the continuous-time external disturbance v remains as is, i.e.,

$$\begin{aligned}\zeta_{k+1} &= \widehat{A}_k\zeta_k + \mathcal{B}_k\omega_k,\\ z_k &= \widehat{C}_k\zeta_k,\end{aligned} \tag{10}$$

where $\omega_k = \mathrm{column}(v(t), w_k)$ and

$$\mathcal{B}_k : L_2\big([t_k, t_{k+1}), \mathbb{R}_2^{n_v}\big) \times \mathbb{R}_2^{n_w} \to \mathbb{R}_2^{n_x+n_\xi}$$

$$: \begin{bmatrix} v(t) \\ w_k \end{bmatrix} \mapsto \begin{bmatrix} \int\limits_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B_c(\tau)v(\tau)d\tau \\ B_{d,k}w_k \end{bmatrix}. \tag{11}$$

Now we use the expression (10) to describe the closed-loop relation between vectors $\widetilde{y} = \mathrm{column}(\zeta_0, \omega_0, \dots, \omega_{N-1})$ and $\widetilde{z} = \mathrm{column}(z_0, \dots, z_N)$ as follows

$$\widetilde{z} = \widetilde{C}\widetilde{A}\widetilde{\mathcal{B}}\widetilde{y}, \tag{12}$$

here

$$\widetilde{C} = \mathrm{diag}\big(\widehat{C}_0, \widehat{C}_1, \dots, \widehat{C}_N\big), \ \ \widetilde{\mathcal{B}} = \mathrm{diag}\big(I, \mathcal{B}_0, \dots, \mathcal{B}_{N-1}\big),$$

$$\widetilde{A} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \widehat{A}_0 & I & 0 & \cdots & 0 \\ \widehat{A}_1\widehat{A}_0 & \widehat{A}_1 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}_{N-1}\dots\widehat{A}_0 & \widehat{A}_{N-1}\dots\widehat{A}_1 & \widehat{A}_{N-1}\dots\widehat{A}_2 & \cdots & I \end{bmatrix}.$$

Therefore, the operator $\mathcal{S}$ can be written as $\mathcal{S} = \widetilde{C}\widetilde{A}\widetilde{\mathcal{B}}$. By using the equation (9) it is easily seen that the dual operator $\mathcal{S}^*$ is given by

$$\mathcal{S}^* = \widetilde{\mathcal{B}}^*\widetilde{A}^\top\widetilde{C}^\top, \tag{13}$$

where $\widetilde{\mathcal{B}}^* = \mathrm{diag}\big(R^{-1}, \mathcal{B}_0^*, \dots, \mathcal{B}_{N-1}^*\big)$ and

$$\mathcal{B}_k^* : \mathbb{R}_2^{n_x+n_\xi} \to L_2\big([t_k, t_{k+1}), \mathbb{R}_2^{n_v}\big) \times \mathbb{R}_2^{n_w}$$

$$: \begin{bmatrix} x \\ \xi \end{bmatrix} \mapsto \begin{bmatrix} B_c^\top(t)\Phi^\top(t_{k+1}, t)x \\ B_{d,k}^\top\xi \end{bmatrix}.$$

It is not difficult to exploit (13) for checking the following expression

$$\|\mathcal{S}^* z\|_{(R,2)}^2 = \|\mathcal{S}^*\widetilde{z}\|_{(R,2)}^2 = \widetilde{z}^\top\widetilde{C}\,W\widetilde{C}^\top\widetilde{z},$$

where

$$W = \widetilde{A}\widetilde{Q}\widetilde{A}^\top, \quad \widetilde{Q} = \mathrm{diag}\big(R^{-1}, \widehat{Q}_0, \dots, \widehat{Q}_{N-1}\big).$$

Since the matrix $\widetilde{C}$ is a block diagonal matrix, we consider an auxiliary block diagonal matrix $\widetilde{Y} = \mathrm{diag}\big(Y_0, \dots, Y_N\big)$ which has the same main-diagonal blocks as matrix W. The blocks Y_k satisfies the linear recurrence relations

$$Y_0 = R^{-1}, \quad Y_{k+1} = \widehat{A}_k Y_k \widehat{A}_k^\top + \widehat{Q}_k$$

which coincide with equations (6) and, moreover, the following equalities hold

$$\begin{aligned}\|\mathcal{S}^*\|_{(R,2)/1}^2 &= \sup\big\{\|\mathcal{S}^* z\|_{(R,2)}^2 : \|z\|_{l_1} \leq 1\big\} \\ &= \sup\big\{\widetilde{z}^\top\widetilde{C}\,W\widetilde{C}^\top\widetilde{z} : \|\widetilde{z}\|_{l_1} \leq 1\big\}.\end{aligned} \tag{14}$$

It should be pointed out that only the main-diagonal blocks of the matrix $\widetilde{C}\,W\widetilde{C}^\top$ are needed for further calculating, thus we can replace the matrix W by the block diagonal matrix $\widetilde{Y}$. Then

$$\begin{aligned}\|\mathcal{S}^*\|_{(R,2)/1}^2 &= \sup\big\{\widetilde{z}^\top\widetilde{C}\,\widetilde{Y}\widetilde{C}^\top\widetilde{z} : \|\widetilde{z}\|_{l_1} \leq 1\big\} \\ &= \max_{k=0,\dots,N} \lambda_{\mathrm{gmax}}\big(\widehat{C}_k Y_k \widehat{C}_k^\top\big).\end{aligned} \tag{15}$$

The last expression is the same as (5). This concludes the proof of theorem.

Now, let us consider a question concerning the worst-case disturbances and initial states. The following result holds.

Theorem 2. If the finite-horizon generalized $\mathcal{H}_2$ norm with transients of the system (1) is equal to γ^* and attained at the moment $k = k^*$, the worst-case initial and external disturbances are determined as

$$\begin{aligned} \zeta_0^* &= (\gamma^*)^{-1} R^{-1} \Psi_{k^*,0}^\top \widehat{C}_{k^*}^\top e, \\ \begin{bmatrix} v^*(t) \\ w_k^* \end{bmatrix} &= \frac{1}{\gamma^*} \begin{bmatrix} B_c^\top(t) \Phi^\top(t_{k+1}, t) x \\ B_{d,k}^\top \xi \end{bmatrix} \Psi_{k^*,k+1}^\top \widehat{C}_{k^*}^\top e, \\ k &= 0, \ldots, k^* - 1, \end{aligned} \quad (16)$$

where $e = e_{\text{gmax}}\big(\widehat{C}_{k^*} Y_{k^*} \widehat{C}_{k^*}^\top\big)$ denotes the normilized generalized eigenvector corresponding to the generalized maximal eigenvalue, $\Psi_{i,j}$ is the transient matrix of the system $\zeta_k = \widehat{A}_k \zeta_k$, i.e.,

$$\Psi_{0,0} = I, \quad \Psi_{i,j} = \widehat{A}_{i-1} \widehat{A}_{i-2} \ldots \widehat{A}_j, \quad i \geq j+1.$$

Proof. To prove the expressions (16), let us assume that the generalized $\mathcal{H}_2$ norm is attained at the moment $k = k^*$ and equal to γ^*. In this case, there exists an element $\widetilde{z}^* = \text{column}(0, \ldots, 0, z_{k^*}, 0, \ldots, 0)$, $\|\widetilde{z}^*\|_{l_1} = 1$, such that

$$\gamma^* = \big\|\mathcal{S}^* \widetilde{z}^*\big\|_{(R,2)}.$$

The equality $\|\widetilde{z}^*\|_{l_1} = 1$ means $z_{k^*} = e_{\text{gmax}}\big(\widehat{C}_{k^*} Y_{k^*} \widehat{C}_{k^*}^\top\big)$, hence $\widetilde{y}^* = \mathcal{S}^* \widetilde{z}^* = \text{column}(\zeta^*, \omega_0^*, \ldots, \omega_{N-1}^*)$ is the vector which is composed of the worst-case initial and external disturbances, moreover, $\|\widetilde{y}^*\|_{(E,2)} = \gamma^*$. To calculate $\widetilde{y}^*$, the k^*th column should be select from the matrix representation of the operator $\mathcal{S}^*$:

$$\widetilde{y}^* = \begin{bmatrix} \zeta^* \\ \omega_0^* \\ \vdots \\ \omega_{k^*}^* \\ \vdots \\ \omega_{N-1}^* \end{bmatrix} = \begin{bmatrix} R^{-1} \widehat{A}_0^\top \ldots \widehat{A}_{k^*}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \mathcal{B}_0 \widehat{A}_1^\top \ldots \widehat{A}_{k^*}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \vdots \\ \mathcal{B}_{k^*} \widehat{A}_{k^*}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} R^{-1} \Psi_{k^*,0}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \mathcal{B}_0 \Psi_{k^*,1}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \vdots \\ \mathcal{B}_{k^*} \Psi_{k^*,k^*}^\top \widehat{C}_{k^*}^\top z_{k^*} \\ \vdots \\ 0 \end{bmatrix}.$$

At last, to get the equation (16), we should scale the vector $\widetilde{y}^*$ by $1/\gamma^*$ because from the definition of the generalized $\mathcal{H}_2$ norm (4) the vector of the worst-case initial and external disturbances have to satisfy the condition $\|\widetilde{y}^*\|_{(R,2)} = 1$. This completes the proof.

Now we reformulate the problem of computation the finite-horizon generalized $\mathcal{H}_2$ norm with transients as a convex semidefinite programming problem which will be useful for further application.

Theorem 3. The finite-horizon generalized $\mathcal{H}_2$ norm with transients can be computed as a solution to the problem $\inf \gamma$ subject to constraints

$$\begin{aligned} &\begin{bmatrix} Y_k & * \\ \widehat{A}_k Y_k & Y_{k+1} - \widehat{Q}_k \end{bmatrix} \geq 0, \quad k = 0, \ldots, N-1, \\ &\begin{bmatrix} Y_k & * \\ \widehat{C}_{k,j} Y_k & \gamma^2 I \end{bmatrix} \geq 0, \quad \begin{matrix} k = 0, \ldots, N, \\ j = 1, \ldots, m, \end{matrix} \end{aligned} \quad (17)$$

with respect to the variables γ and $Y_0, \ldots, Y_N$. Here the matrices C_k are partitioned on blocks $\widehat{C}_{k,j} \in \mathbb{R}^{n_j \times n}$, $n_1 + \ldots + n_m = n$.

The proof of this theorem can be obtained by following the same guidelines of Theorem 2 in Balandin et al. (2019b).

4. SYNTHESIZING GENERALIZED $\mathcal{H}_2$ CONTROL

Consider a linear time-varying hybrid controlled system described by the set of differential and difference equations

$$\begin{aligned} \dot{x} &= A_c(t) x + \Delta_c(t) \xi_k + B_c(t) v + H_c(t) u(t), \\ \xi_{k+1} &= A_{d,k} \xi_k + \Delta_{d,k} x(t_k) + B_{d,k} w_k + H_{d,k} u(t_k), \\ z_k &= C_{c,k} x(t_k) + C_{d,k} \xi_k + D_k u(t_k), \end{aligned} \quad (18)$$

here $t \in [t_k, t_{k+1})$, $t_k < t_{k+1}$, $k = 0, \ldots, N-1$, and $u_k \in \mathbb{R}^{n_u}$ is the control input.

The problem of interest is to synthesize for the system (18) a piecewise constant state-feedback control law

$$u(t) = u_k, \quad t_k \leq t < t_{k+1}, \quad k = 0, \ldots, N-1, \quad (19)$$

and

$$u_k = \Theta_{c,k} x(t_k) + \Theta_{d,k} \xi_k \quad (20)$$

minimizing the generalized $\mathcal{H}_2$ norm of the closed-loop system

$$\begin{aligned} \zeta_{k+1} &= \big(\widehat{A}_k + \widehat{H}_k \widehat{\Theta}_k\big) \zeta_k + \mathcal{B}_k \omega_k, \\ z_k &= \big(\widehat{C}_k + D_k \widehat{\Theta}_k\big) \zeta_k, \end{aligned} \quad (21)$$

where functionals $\mathcal{B}_k$ are defined by (11),

$$\widehat{H}_k = \begin{bmatrix} H_{c,k} \\ H_{d,k} \end{bmatrix}, \quad \widehat{\Theta}_k = \big(\Theta_{c,k}, \, \Theta_{d,k}\big),$$

and

$$H_{c,k} = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau) H_c(\tau) d\tau.$$

From Theorem 3 it follows that the finite-horizon generalized $\mathcal{H}_2$ norm of the closed-loop system (21) is computed as a solution to the problem (17), in which the matrices $\widehat{A}_k$ and $\widehat{C}_k$ should be replaced by the matrix $\widehat{A}_k + \widehat{H}_k \widehat{\Theta}_k$ and $\widehat{C}_k + D_k \widehat{\Theta}_k$, respectively. By introducing variables $Z_k = \widehat{\Theta}_k Y_k$, we arrive at the following result.

Theorem 4. The state-feedback matrices $\widehat{\Theta}_k$ of the optimal control are computed as $\widehat{\Theta}_k = Z_k Y_k^{-1}$, where Z_k, Y_k, $k = 0, \ldots, N-1$, are solutions to the following semidefinite programming problem: find $\inf \gamma$ subject to constraints

$$\begin{aligned} &\begin{bmatrix} Y_k & * \\ \widehat{A}_k Y_k + \widehat{H}_k Z_k & Y_{k+1} - \widehat{Q}_k \end{bmatrix} \geq 0, \\ &\begin{bmatrix} Y_k & * \\ \widehat{C}_{k,j} Y_k + D_{k,j} Z_k & \gamma^2 I \end{bmatrix} \geq 0, \quad \begin{matrix} k = 0, \ldots, N, \\ j = 1, \ldots, m. \end{matrix} \end{aligned} \quad (22)$$

Here the matrices C_k and D_k are partitioned on blocks $\widehat{C}_{k,j} \in \mathbb{R}^{n_j \times n_x}$, $D_{k,j} \in \mathbb{R}^{n_j \times n_u}$, $n_1 + \ldots + n_m = n$.

Now we consider the multi-objective control problem with the finite-horizon generalized $\mathcal{H}_2$ norms with transients of several outputs of the system

$$\begin{aligned} \dot{x} &= A_c(t)x + \Delta_c(t)\xi_k + B_c(t)v + H_c(t)u(t), \\ \xi_{k+1} &= A_{d,k}\xi_k + \Delta_{d,k}x(t_k) + B_{d,k}w_k + H_{d,k}u(t_k), \\ z_k^{(i)} &= C_{c,k}^{(i)}x(t_k) + C_{d,k}^{(i)}\xi_k + D_k u(t_k), \end{aligned} \tag{23}$$

where $z_k^{(i)}$, $i = 1,\dots,m$, are the controlled outputs. Let $\gamma_i(\Theta)$ be the finite-horizon generalized $\mathcal{H}_2$ norm of ith output of the system (23) closed by piecewise constant control law (19), (20) and Θ denotes the set of state-feedback matrices $\widehat{\Theta}_k$, $k = 0,\dots,N-1$. The multi-objective control problem is to find Pareto optimal set Θ^* minimizing the performance measures $\gamma_i(\Theta)$. The set Θ^* is the Pareto optimal if there is not a matrix Θ such that the inequalities $\gamma_i(\Theta) \le \gamma_i(\Theta^*)$, $i = 1,\dots,m$, with at least one of the inequalities being strict, be valid. Necessary conditions for Pareto optimality in the problem under consideration are formulated as follows (see Balandin and Kogan (2017)).

Theorem 5. Let $(\gamma_1,\dots,\gamma_m)$ be a Pareto optimal point in the space of criteria and Θ_α be a minimum of the multi-objective cost function in the form of Germeyer convolution

$$G(\Theta) = \max_{i=1,\dots,m} \frac{\gamma_i(\Theta)}{\alpha_i}, \quad \alpha_i = \frac{\gamma_i}{\max\limits_{i=1,\dots,m} \gamma_i}. \tag{24}$$

Then Θ_α belongs to Pareto optimal set and $\gamma_i(\Theta_\alpha) = \gamma_i$, $i = 1,\dots,m$.

For the multi-objective problem under consideration and in accordance with Theorem 3 Germeyer convolution (24) takes the form

$$G(\Theta) = \max_{i=1,\dots,m} \max_{k=0,\dots,N} \alpha_i^{-1} \lambda_{\mathrm{gmax}}^{1/2}\big(\widehat{C}_k^{(i)} Y_k \widehat{C}_k^{(i)\top}\big), \tag{25}$$

where $Y_k = Y_k^\top \ge 0$ is the solution to the equation (6) for the closed-loop system. It means that the Pareto optimal controls minimizing the finite-horizon generalized $\mathcal{H}_2$ norms with transients can be found solving the inequalities (22) in which the matrices $\widehat{C}_{k,i} = \alpha_i^{-1}\widehat{C}_k^{(i)}$ for all α_i, $i = 1,\dots,m$.

5. EXAMPLE

Consider the optimal shock and vibration protection problem for a mechanical single-degree-of-freedom system shown in Fig. 1. A body to be protected "2" connected to moving base "1" by means of an active isolator. This mechanical system is described by the equation

$$\begin{gathered} \ddot{x} = u + v + \sum_{k=0}^{N-1} w_k \delta(t - t_k), \\ x(0) = x_{10}, \quad \dot{x}(0) = x_{20}, \end{gathered} \tag{26}$$

where x is coordinate of the body "2" with respect to the moving base, u is the active component of the isolator, v is the continuous-time external disturbance coinciding up to a sign with acceleration (or deceleration) of the moving base, and w_k is the discrete-time external disturbance coinciding with the impact on the base at the moment t_k.

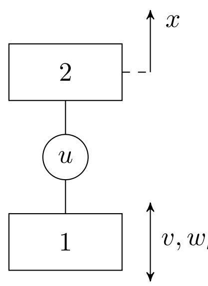

Fig. 1. Scheme of a protection from shock and vibration

This time moments are known and form a monotonically increasing sequence. Let us introduce new variables

$$x_1 = x, \qquad x_2 = \dot{x} - \sum_{k=0}^{N-1} w_k H(t - t_k),$$

where $H(t)$ is the Heaviside step function. Rewrite the equation (26) as follows

$$\begin{aligned} \dot{x}_1 &= x_2 + \xi_k, & x_1(0) &= x_{10}, \\ \dot{x}_2 &= u + v, & x_2(0) &= x_{20}, \\ \xi_{k+1} &= \xi_k + w_k, & \xi_0 &= 0, \end{aligned} \tag{27}$$

here the discrete variable ξ_k is the total impulse of the base during the time $[t_0, t_{k+1})$.

We choose two performance indices

$$\begin{aligned} J_1(u) &= \sup_{\zeta_0,\, v,\, w} \frac{\max\limits_{k=0,\dots,N} |x_1(t_k)|}{\sqrt{\zeta_0^\top R\,\zeta_0 + \|v\|_{L_2}^2 + \|w\|_{l_2}^2}}, \\ J_2(u) &= \sup_{\zeta_0,\, v,\, w} \frac{\max\limits_{k=0,\dots,N} |u(t_k)|}{\sqrt{\zeta_0^\top R\,\zeta_0 + \|v\|_{L_2}^2 + \|w\|_{l_2}^2}}. \end{aligned}$$

where $\zeta_0 = \mathrm{column}(x_{10}, x_{20}, \xi_0)$. The first functional characterizes the maximal displacement of the body to be protected with respect to the moving base, while the second one characterizes the maximal force acting on the body. These criteria are competing, i.e. the greater force counteracting the body movement, the smaller its displacement with respect to the base.

For numerical experiments we choose $R = \mathrm{diag}(10, 10, 1)$, $N = 5$, and two sets of resetting times $\mathcal{T}_1 = \{0; 4; 8; 12; 20\}$ and $\mathcal{T}_2 = \{0; 3.2; 8; 11.6; 16; 20\}$. By using the above approach we synthesized Pareto optimal controllers $\widehat{\Theta}_k$ and computed the corresponding optimal values of the criteria for both sets of resetting times. The Pareto optimal fronts on the plane of criteria $(J_1; J_2)$ are shown in Figure 2. For the set $\mathcal{T}_1$ the Pareto front and the point A_1 with coordinates (5.546; 1.564) lying on this curve is depicted by the black line. The red line corresponds the Pareto optimal front and the point A_2 with coordinates (6.222, 1.755) computed for the set $\mathcal{T}_2$. Both points corresponds to the parameter $\alpha = 0.78$. For comparison the value of the first functional for the case without control, i.e. $u = 0$, is $J_1 = 59.890$ for the set $\mathcal{T}_1$ and $J_1 = 60.1628$ for $\mathcal{T}_2$. The time histories for the optimal coefficients $\widehat{\Theta}_k$ are presented in Figure 3, the black and red curves correspond to the points A_1 and A_2, respectively.

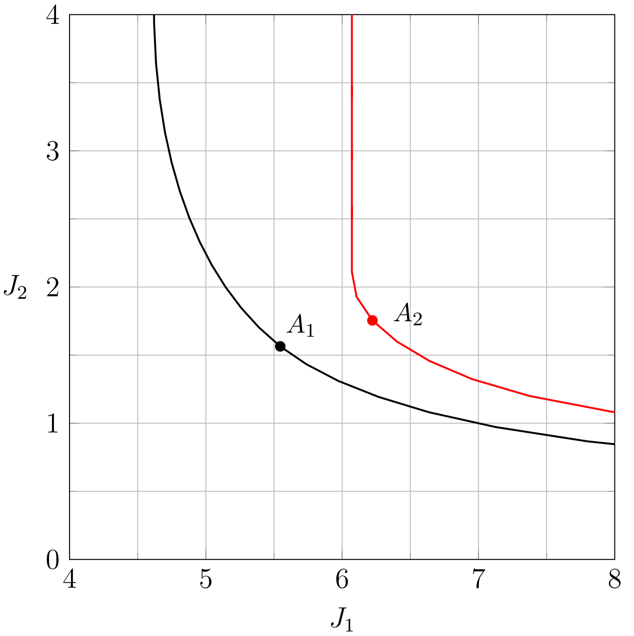

Fig. 2. Pareto optimal fronts for two sets of resetting times

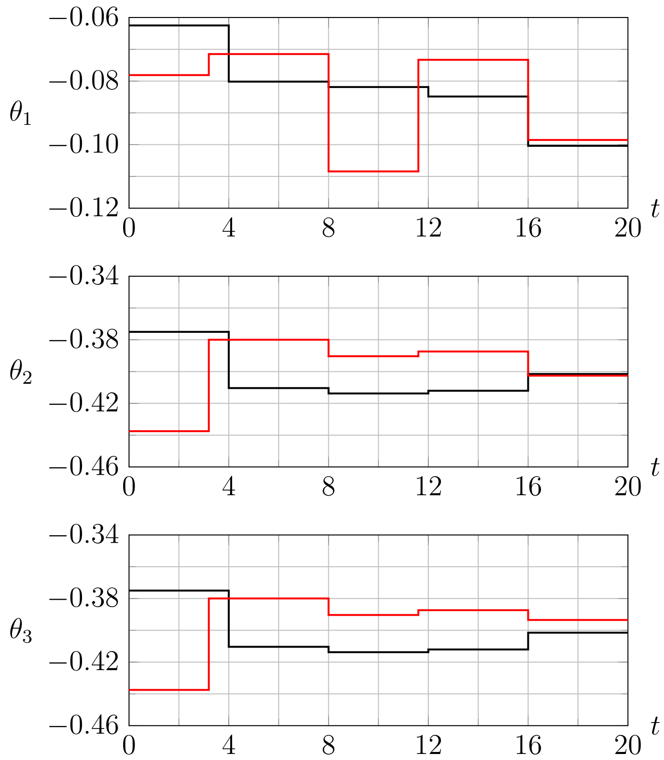

Fig. 3. Time histories of the optimal coefficients of state-feedback corresponding two sets of resetting times

6. CONCLUSION

This paper proposes a novel performance measure of the linear time-varying hybrid system over a finite horizon which characterizes the worst-case peak value of the multiple output in response to uncertain initial states and the external disturbances. This measure represents an induced operator norm named the finite-horizon generalized $\mathcal{H}_2$ norm with transients. It is characterized in terms of difference linear matrix inequalities (DLMIs) that allows to synthesize the optimal piecewise constant state-feedbacks minimizing the generalized $\mathcal{H}_2$ norm of one or several outputs. The efficiency of the proposed method was demonstrated on the problem of optimal protection from shock and vibration.

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