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Some Remarks on the Regular Splitting of Quasi-Polynomials with Two Delays. Characterization of Double Roots in Degenerate Cases

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Abstract: This paper addresses the classification of multiple critical roots of dynamical continuous linear time-invariant systems including two constant delays in their mathematical representation. By considering the associated Weierstrass polynomial and its algebraic properties, we investigate the splitting behavior of such critical roots when the delays are subject to small variations. Some degenerate cases are also considered. The effectiveness of the proposed approach is illustrated through several numerical examples.

Keywords: Delay systems, Characteristic Roots, Stability, Weierstrass Polynomial, Asymptotic Behavior.

1. INTRODUCTION

The stability analysis of linear time-invariant (LTI) systems with time-delay have been extensively studied in the past decades and there exists an abundant literature covering the subject (see, for instance, Niculescu (2001); Gu et al. (2003); Michiels and Niculescu (2014); Li et al. (2015) and the references therein).

As discussed in Chen et al. (2010a,b), even in the the simplest case of a single constant delay, the stability tests are not easy to perform. Such a difficulty arises from the fact that that the delay systems are infinite-dimensional and the corresponding characteristic function is, in fact, a quasi-polynomial that always have an *infinite number* of *characteristic roots* (see, for instance, Michiels and Niculescu (2014) and the references therein). In the retarded case, by using an appropriate continuity argument [1], the problem of (exponential) stability can be reduced to (i) the detection and (ii) the analysis of the behavior of the characteristic roots located on the imaginary axis [2] in the case when such roots exist [3].

[1] a direct consequence of the Rouché's lemma, see, e.g., Michiels and Niculescu (2014)
[2] Such roots are simply called *critical characteristic roots.*
[3] In fact, in the case when there are no characteristic roots on the imaginary axis, the stability/instability of the dynamical system free of delays is preserved for any delay value, i.e. the *delay-independent* stability/instability property, see, for instance, Niculescu (2001); Gu et al. (2003); Michiels and Niculescu (2014) for further arguments.

In this context, one the problem particularly treated in the literature was to understand how the delay parameters affect the behavior of the *critical characteristic roots*, and to explicitly compute the stability domains in the delay-parameter space. In the case of a single or commensurate delays, such domains reduce to a *finite number* of *delay intervals.* For more insights on the existing methods, one may refer to Niculescu (2001); Olgac and Sipahi (2002); Gu et al. (2003); Michiels and Niculescu (2014); Li et al. (2015). To the best of the authors' knowledge, the complete characterization of the stability domains in the incommensurate delays case is still an *open problem.*

Next, it is well known that the roots of polynomials are continuous functions of the coefficients as long as the leading coefficient does not vanish, see, e.g., Knopp (1996). Furthermore, in the case of simple roots, these functions are also differentiable. However, in the case of multiple roots, such a property does not necessarily hold and Puiseux series may be used to perform the analysis.

In the commensurate delays case, these conclusions are also valid for quasi-polynomials of retarded type, where the characteristic roots are seen as functions of one variable - the *delay parameter* (see, for instance, Michiels and Niculescu (2014); Li et al. (2015)). By using an operator-based approach, such an idea was exploited by Chen et al. (2010b), where the authors characterized the asymptotic behavior of multiple critical characteristic roots in *semi-simple* [4] and some *not semi-simple* [5] cases.

[4] the same algebraic and geometric multiplicity
[5] more precisely, the complete regular splitting case

Based on the remarks above, the *asymptotic behavior of multiple critical characteristic roots* with respect to *several (incommensurate) delays* needs a deeper understanding (degenerate cases classification and characterization).

As mentioned before, one of the natural and standard approaches to handle such a problem is to study of the properties of the corresponding Puiseux series solutions. Such an idea, exploited in the commensurate delays case (for some insights, see Li et al. (2013)) cannot be extended straightforwardly to the incommensurate delays. In fact, as in the multivariate polynomial case, such a method has several drawbacks and, in some situations, such Puiseux series solutions do not always exist (see Subsection 3.1 below as well as various examples proposed by Alonso et al. (1992); Aroca (2004)). In order to avoid such situations, the problem needs to be *well-posed* (see, e.g., Monforte and Kauers (2013)).

The migration of double characteristic roots depending on two parameters is studied in Gu et al. (2015); Irofti et al. (2018), where, without invoking Puiseux series, under appropriate assumptions[6], the authors proposed a more conventional geometric approach to classify the asymptotic behavior of the critical characteristic roots in the parameter space. Such a method allows covering some particular asymptotic behavior in the delay-parameter space when the parameters change in the neighborhood of the critical point. Next, in the multiple delays case, Li et al. (2019) proposed an iterative frequency-sweeping approach, by leaving one delay parameter free, while the others are fixed. Thus, the critical characteristic roots depend on only one parameter, and the analysis is performed by using the method developed by the same authors in Li et al. (2015). Finally, in the case of two parameters, some analysis was proposed by Maurer (1980); Lipman (2017), by using an appropriate parametrization of a given surface. This idea was exploited by Martínez-González et al. (2018) for the computation of roots of quasi-polynomials with two incommensurate delays without any discussion on multiple critical characteristic roots.

Taking into consideration the previous discussion, the main contribution of this paper is threefold:

- Give conditions for the existence of multi-parameter Puiseux series and give a classification of its solutions;
- Relax the non-degeneracy assumption proposed by Irofti et al. (2018);
- Propose an appropriate algorithm for studying the corresponding asymptotic behavior and related its splitting properties.

More precisely, under some particular conditions that guarantee the existence of well-defined multi-parameter solutions, we propose to relax the assumption for the existence of a parametrization around the multiple roots as well as its splitting properties. To complete the presentation, an explicit computation of the associated Puiseux series solution is proposed. To the best of the authors' knowledge, all these contributions represent a novelty in the open literature.

The remaining paper is organized as follows: Section 2 includes some preliminary results. In Section 3 some motivating example is outlined and the problem formulation is stated. Section 4 is devoted to the main results. In Section 5, several numerical examples are presented. Finally, some concluding remarks end the paper.

Throughout the paper, the following notations will be adopted: for $z \in \mathbb{C}$, $\arg(z) \in [0, 2\pi)$, $\Re(z)$ ($\Im(z)$) denote the argument, real (imaginary) part of z, respectively. Next, $\mathbb{R}_+$ denotes the set of positive real values, $\mathbb{C}[x]$ the ring of polynomials and $\mathbb{C}\{x\}$ the ring of convergent power series.

2. PRELIMINARY RESULTS

In the sequel, a dynamical LTI system of retarded type including two delays τ_1 and τ_2 is considered. For the sake of brevity and simplicity, its characteristic function f is given by the following quasi-polynomial:

$$f(s, \tau_1, \tau_2) := p_0(s) + p_1(s)e^{-\tau_1 s} + p_2(s)e^{-\tau_2 s}, \quad (1)$$

where p_k are polynomials given as,

$$p_0(s) = s^n + \sum_{\ell=0}^{n-1} a_{0\ell}s^\ell, \; p_k(s) = \sum_{\ell=0}^{n-1} a_{k\ell}s^\ell, \quad k \in \{1, 2\}.$$

2.1 Local Representation Around Multiple Roots

It is possible to reduce the analytic properties of $f(z, \mathbf{x})$ to some algebraic ones. To this purpose, consider the following result Mailybaev and Grigoryan (2001):

Theorem 1. (Weierstrass Preparation Theorem). Suppose that $f(z, \mathbf{x})$ is an analytic function vanishing at the singular point $z_0 \in \mathbb{C}$, $\mathbf{x}_0 \in \mathbb{C}^n$, where $z = z_0$ is an $m-$multiple root of the equation $f(z, \mathbf{x}) = 0$, i.e.,

$$f(z_0, \mathbf{x}_0) = \frac{\partial f}{\partial z} = \cdots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \quad \frac{\partial^m f}{\partial z^m} \neq 0, \quad (2)$$

where derivatives are evaluated at $(z_0, \mathbf{x}_0)$.

Then, there exists a neighborhood $U_0 \subset \mathbb{C}^{n+1}$ of the point $(z_0, \mathbf{x}_0) \in \mathbb{C}^{n+1}$ in which the function $f(z, \mathbf{x})$ can be expressed as

$$f(z, \mathbf{x}) = W(z, \mathbf{x})\, b(z, \mathbf{x}), \quad (3)$$

where $W(z, \mathbf{x})$ is given by

$$(z - z_0)^m + w_{m-1}(\mathbf{x})(z - z_0)^{m-1} + \cdots + w_0(\mathbf{x}), \quad (4)$$

and $w_0(\mathbf{x})$,...,$w_{m-1}(\mathbf{x})$, $b(z, \mathbf{x})$ are analytic functions uniquely defined by the function $f(z, \mathbf{x})$, and $w_i(\mathbf{x}_0) = 0$, $b(z_0, \mathbf{x}_0) \neq 0$.

In Martínez-González et al. (2019b) the authors proposed an explicit method for computing the associated Weierstrass polynomial of (1).

Remark 1. It can be seen from Theorem 1 that, since $b(z, \mathbf{x})$ is an holomorphic non vanishing function at $(0, \mathbf{0})$. Then, there must exist some neighborhood $U \subset \mathbb{C}^{n+1}$ at which $b(z, \mathbf{x})$ preserves the same property. Hence, based on this observation, we can ensure that the roots behavior of a given quasi-polynomial f in the neighborhood U will be completely described by the roots behavior of $W(\mathbf{x}, x)$, see, for instance, Hörmander (1973).

2.2 Splitting Properties

In the case of one parameter, the quasi-polynomial f defines an appropriate plane curve given by $f = 0$ (see Wall

[6] *"least degenerate"* double critical characteristic roots

(2004); Walker (1978)), and the root locus of a multiple root of f is characterized by its branches which explicitly describe a solution curve $\mathcal{C} \in \mathbb{C}^2$. This characterization is composed by a finite union of r-branches $s_j(\tau^{1/m_j})$ which can be parametrized as $(s, \tau) = (\varphi(\tau), \tau^{m_j})$. In the form of Puiseux series, the root locus gives rise to the following branching:

$$s_{j\sigma}(\tau) = c_{j\sigma}\tau_j^m + o\left(|\tau|^{m_j}\right), \quad j = 0, \ldots, r-1, \sigma = 1, \ldots, m_j \tag{5}$$

where each branch has multiplicity m_j such that $m = m_1 + m_2 + \cdots + m_r$.

Definition 2.1. The trivial solution $s^* = 0$ has a *Complete Regular Splitting (CRS) property at* $\tau^* = 0$ if $c_{j\sigma} \neq 0$, $\forall j$. Next, if some of the coefficients $c_{j\sigma}$ for which $m_j = 1$ may be equal to zero, then the trivial solution has a *Regular Splitting (RS) property at* $\tau^* = 0$. Finally, in the remaining cases of the coefficients $c_{j\sigma}$, the trivial solution has a *Non-Regular Splitting (NRS) property at* $\tau^* = 0$.

In Martínez-González et al. (2019a), the authors proposed a methodology based on the first partial derivatives to analyzed the splitting behavior, see Figure 1. It is worth mentioning that these results cover only the commensurate delays case and the extension to incommensurate delays is not straightforward.

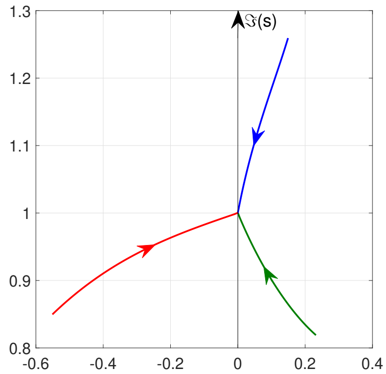

Fig. 1. Illustration of Completely Regular Splitting (CRS) property in the case of a triple critical characteristic root $s^* = j$

2.3 Quasi-ordinary Roots

In order to extend the above approach to the multi-parameter case, the so-called Abhyankar-Jung Theorem must be taken into consideration (see, for instance,Abhyankar (1955)). To this end, the notion of quasi-ordinary polynomials must be introduced.

Definition 2.2. Let $W(z, x_1, x_2)$ be a Weierstrass polynomial (3). Then W is a *quasi-ordinary* polynomial with respect to z, if the discriminant Δ is of the form

$$\Delta(x_1, x_2) = x_1^{p_1} x_2^{p_2} V(x_1, x_2), \quad x_1, x_2 \in \mathbb{N}, \tag{6}$$

where V is an analytic function such that $V(0, 0) \neq 0$.

The *discriminant* Δ of a given polynomial is a function that depends on the coefficients and it can be defined up to some constant factor as the product of the square of its roots. In general, it can be expressed by means of the m-solutions $z_i = \phi(\mathbf{x})$ as follows

$$\Delta(\mathbf{x}) = \prod_{i<j} (\phi_i - \phi_j).$$

The discriminant of a quadratic Weierstrass polynomial is given by $\Delta(\mathbf{x}) = w_1^2(\mathbf{x}) - 4w_0(\mathbf{x})$ (see, for instance, Wall (2004)).

Now, the existence of parametric equation that satisfies the equation $W = 0$ for a quasi-ordinary polynomials, i.e., the existence of Puiseux series solutions in the multi-parameter case is given by the following result (see, for instance Zurro (1993); Lipman (2017)).

Theorem 2. (Abhyankar-Jung Theorem). Let $f \in \mathbb{C}[z]$ be a quasi-ordinary polynomial in z with analytic coefficients in (x_1, x_2). Then, there exist a natural number r such that the roots of f are given by convergent fractional power series that belong to $\mathbb{C}\{x_1^{1/r}, x_2^{1/r}\}$.

2.4 Regularity Condition

Consider now the quasi-polynomials $f(s, \tau_1, \tau_2)$ described in (1), with a multiple root $s = 0$ at (τ_1^*, τ_2^*). Furthermore, we restrict the analysis to the case when f satisfies the following assumption:

Assumption 2.1. (Regularity Condition). Let $s = 0$ be a m-multiple of the quasi-polynomial f at (τ_1^*, τ_2^*). Then, the following condition

$$\left.\frac{\partial f}{\partial \tau_i}\right|_{(0,\mathbf{0})} \neq 0 \tag{7}$$

holds for at least one τ_i with $i = 1, 2$.

Remark 2. As a consequence of the Implicit Function Theorem, if the Regularity Condition (7) holds for both τ_1 and τ_2, then, the characteristic equation $f = 0$ defines the parameters (τ_1, τ_2) as continuous functions of s, in some sufficiently small neighborhood of $s = 0$. This implies that the regularity condition presented in (7), relax the conditions imposed by the so-called *non-degeneracy* condition proposed by Irofti et al. (2018).

Lemma 3. If the following condition:

$$\left.\frac{\partial f}{\partial \tau_i}\right|_{s_0} \neq 0, \quad i = 1, 2.$$

holds, then the equation $f(s, \tau_1, \tau_2) = 0$ is satisfied by the unique solutions τ_1 and τ_2 defined by continuous functions $\tau_i(s_0)$ with $s_0 \in \mathcal{V}_\delta^*(\boldsymbol{i}\omega)$ excluding the double root $s^* = \boldsymbol{i}\omega$.

3. PROBLEM FORMULATION

The present work focuses on the analysis of the asymptotic behavior of multiple imaginary roots under multi-parameter perturbations. More precisely, we consider as variable parameters the pair (τ_1, τ_2).

In this vein,for the quasi-polynomial $f(s, \tau_1, \tau_2)$, we will focus in the following problems:

(1) derive the splitting behavior of a double root at $\boldsymbol{i}\omega$ under small variations of the delays (τ_1, τ_2);
(2) characterize conditions on the delay-parameter space (τ_1, τ_2) guaranteeing the existence of Puiseux series

$s(\tau_1, \tau_2)$ of multiple roots, together with its corresponding region of convergence, of the form

$$s(\boldsymbol{\tau}) = c_1\tau_1^{1/d} + c_2\tau_1^{1/d} + c_3\tau_1^{1/d}\tau_1^{1/d} + o(\tau_1^{1/d}\tau_n^{1/d}),$$

where $d \in \{1, 2\}$;

(3) characterize the existing links between the assumptions made in previous works devoted to the same problem and the condition for quasi-ordinary singularities.

3.1 Insights on Multivariate Polynomial Perturbations

In the sequel, we present some difficulties that arise when considering multi-parameter functions. In order to illustrate such arguments, we will present two examples that motivate the proposed approach.

Example 4. Consider the following polynomial:

$$P(z, \epsilon) = z^2 + 3\epsilon_1 z + 2\left(\epsilon_1^2 + 2\epsilon_2^2\right),$$

where $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{R}^2$. Here, ϵ_1 and ϵ_2 are considered as perturbation parameters. It is clear to see, that if $\epsilon_1 = \epsilon_2 = 0$, $z = 0$ is a root of multiplicity two. We can see that P is a Weierstrass polynomial (3) such that the regularity condition (7) is not satisfied. In this case, the solutions $z_{1,2}(\boldsymbol{\epsilon})$ are not continuous at $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2) = (0, 0)$. Furthermore, $z_{1,2}(\boldsymbol{\epsilon})$ does not have a *unique* representation as a power series which is convergent in some punctured neighborhood of the origin. In order to illustrate this assertion on P, let us consider the case when $|\epsilon_1| < |\epsilon_2|$. In this region the solutions admit the following representation:

$$z_{1,2}(\boldsymbol{\epsilon}) = -\frac{1}{2}\left(3\epsilon_1 \pm \boldsymbol{i}4\epsilon_2\right) + \frac{1}{16}\epsilon_1\left(\pm\boldsymbol{i}\frac{\epsilon_1}{\epsilon_2} \pm \frac{\boldsymbol{i}}{64}\left(\frac{\epsilon_1}{\epsilon_2}\right)^3 + \pm\frac{\boldsymbol{i}}{2048}\left(\frac{\epsilon_1}{\epsilon_2}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_1}{\epsilon_2}\right)^5\right)\right).$$

Now, if instead of the previous region, we consider the region $|\epsilon_2| < |\epsilon_1|$. Then, for $k \in \{1, 2\}$ the solutions admit the following representation

$$z_k(\boldsymbol{\epsilon}) = -2^{k-1}\epsilon_1 + (-1)^k 4\epsilon_2\left(\frac{\epsilon_2}{\epsilon_1} + 4\left(\frac{\epsilon_2}{\epsilon_1}\right)^3 + 32\left(\frac{\epsilon_2}{\epsilon_1}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_2}{\epsilon_1}\right)^5\right)\right).$$

The previous example shows the difficulties inherent in the study of the migration of the solution for small perturbation of the parameters.

Example 5. Consider now the perturbed polynomial Q given by

$$Q(z, \boldsymbol{\epsilon}) = z^3 + (\epsilon_1 + \epsilon_1\epsilon_2)z^2 + (\epsilon_1 + \epsilon_2 + \epsilon_1^2\epsilon_2)z + (\epsilon_1^2 + \epsilon_1\epsilon_2). \quad (8)$$

It is easy to see that one of the roots is given by $z_1(\epsilon_1) = -\epsilon_1$, and the polynomial $(z + \epsilon_1)^{-1}Q(z, \boldsymbol{\epsilon})$ posses a double root at $z_{2,3} = 0$ if $(\boldsymbol{\epsilon}) = (0, 0)$. The two remaining roots can be found using the binomial expansion, with expansion given as follows

$$z_{2,3} = \pm\boldsymbol{i}\epsilon_2^{1/2} + \mathcal{O}\left(\epsilon_1^{1/2}\epsilon_2^{1/2}\right)$$

These two solutions posses the form given in (2) in the problem formulation, with $c_1 = c_2 = 0$ and $c_3 = \boldsymbol{i}$. A particular case of fractional power series solution, known as quasi-ordinary singularities, can be expressed as $z_{1,2}(\boldsymbol{\epsilon}) = \epsilon_1^{u/2}\epsilon_2^{v/2}\varphi(\epsilon_1^{1/2}, \epsilon_2^{1/2})$ such that $\varphi(0, 0)$. This situation is guaranteed by the structure of the corresponding discriminant of Q.

4. MAIN RESULTS

As mention in the Introduction, our approach is based on the properties of the associated Weierstrass polynomial (4). This allows us using the algebraic properties for the root behavior analysis. Since any critical solution $(s^*, \boldsymbol{\tau}^*)$ can always be translated to the origin by appropriate shifts $s \mapsto s - s^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$ hereinafter we will assume that $(s^*, \tau_1^*, \tau_2^*) = (0, \mathbf{0})$. In addition, we will consider that $m \in \mathbb{N}$ with $m \geq 2$ is the algebraic multiplicity of f at $(0, \mathbf{0})$, that is,

$$f(0, \mathbf{0}) = \left.\frac{\partial f}{\partial s}\right|_{(0,\mathbf{0})} = \cdots = \left.\frac{\partial^{m-1} f}{\partial z^{m-1}}\right|_{(0,\mathbf{0})} = 0 \text{ and } \left.\frac{\partial^m f}{\partial z^m}\right|_{(0,\mathbf{0})} \neq 0.$$

4.1 Puiseux Series Solutions for Quasi-Polynomials

By means of a recursive procedure, in the commensurate delays case, Martínez-González et al. (2019a) propose a method to compute the splitting behavior of multiple root for quasi-polynomials under the variation of the delay parameter. By defining an appropriate solution surface around a multiple root, such a procedure can be extended to systems with two delay parameters. In other words, the space curve $\mathcal{C}$ defined by the set $\{(s, \tau_1, \tau_2) \in \mathbb{C} \times \mathbb{R}^2 : f = 0\}$ can be parametrized by fractional power series in (τ_1, τ_2) called Puiseux series, as in the case of one parameter delay discussed in Section 2.2.

Proposition 6. Let the regularity condition (7) holds for $i = 1$. Then, the quasi-polynomial f admits Puiseux series solutions in τ_1 of the form

$$s(\boldsymbol{\tau}) = c_1(\tau_2)\tau_1^{1/m} + o\left(\boldsymbol{\tau}^{1/m}\right)$$

where the coefficients $c_k(p_2)$ can be expressed as a power fractional series in τ_2.

The above results give some explicit representation of m-solutions which determine the solution surface. These solutions are in the form of Puiseux series, which give some insights on the splitting for a fixed τ_2. In previous works, by using iterated Newton diagram procedure, the leading terms of the quasi-polynomial $f(s, \boldsymbol{\tau})$ are given in an explicit manner. Now, under the regularity condition (7) the following property guarantees the existence of Puiseux series solutions in the general case.

Proposition 7. Suppose that the Assumption 2.1 is satisfied for τ_i with $i = 1, 2$. Then, the leading terms are given as

$$s_j(\boldsymbol{\tau}) = c_1\tau_1^{1/m} + c_2\tau_2^{1/\beta} + o\left(\boldsymbol{\tau}^{1/m}\right), \quad (9)$$

where $m, \beta \in \mathbb{N}$ such that $\beta \leq m$.

Proposition 8. Assume that f satisfies (7) for τ_i with $i \in \{1, 2\}$. Let $n \in \mathbb{N}$, such that the following partial derivatives are satisfied:

$$\frac{\partial f}{\partial \tau_j} = \cdots = \frac{\partial^{n-1} f}{\partial \tau_j^{n-1}} = 0, \quad \frac{\partial^n f}{\partial \tau_j^n} \neq 0, \quad (10)$$

for $n > 1$, $j \in \{1, 2\}$ and $j \neq i$. Then, the $m-$solutions of f are characterized as,

$$s_k(\tau_1, \tau_2) = \tau_i^{1/m}\tau_j^{1/n_i}\varphi_k(\tau_i^{1/m}, \tau_j^{1/n_i}), \quad k = 1, 2, \ldots, m,$$

for some $n_i \leq m$ and $\varphi_k(0, 0) \neq 0$.

4.2 Double Root of Quasi-Polynomials

The above construction does not give any details on the characterization of the root behavior in the parameter space (τ_1, τ_2). Thus, to describe such a behavior, it will be shown the explicit relationship between the local form (quadratic polynomial in s) and the branches of the double root. Hence, by assuming that $m = 2$ and according to the previous results, the Weierstrass polynomial $W(s, \tau)$ can be expressed by:

$$W(s, \tau) = s^2 + w_1(\tau_1, \tau_2)s + w_0(\tau_1, \tau_2). \tag{11}$$

It is possible to characterize the the root behavior by means of the Discriminant D_s given by:

$$D_s(\tau_1, \tau_2) := w^2(\tau_1, \tau_2) - 4w_0(\tau_1, \tau_2). \tag{12}$$

Clearly, in the parameter space (τ_1, τ_2), the condition $D_s = 0$ guarantees the existence of a double root.

Proposition 9. Let $s^* = \boldsymbol{i}\omega$ be a double root of the quasi-polynomial $f(s, \tau)$. Then, there exists a change of variable such that the roots have the form

$$z^2 = \varphi(\tau_1, \tau_2),$$

where φ can be expressed in the form convergent power series of order 1 in τ_1 and τ_2.

4.3 Complete Regular Splitting

Now, to obtain information on the migration of a double root we study the properties of the branch given by the Puiseux series, but, without its explicit computation. Notice that the existence of such a series is guaranteed by Proposition 7. Hence, the migration of the double root in all cases is summarized by the following result.

Proposition 10. Let W be the Weierstrass polynomial of f for the critical point $(0, \mathbf{0})$, such that $D_s = 0$. Assume that the regularity condition (7) is satisfied for τ_1 and τ_2. Then, in a neighborhood of $(0, \mathbf{0})$ the two solutions of f possess the Completely Regular Splitting Property (CRS) with respect to τ_1 and τ_2, that is, both solutions can be expressed as:

$$s_{1,2}(\tau) = \pm c_1 \tau_1^{1/2} \pm c_2 \tau_2^{1/2} + \mathcal{O}(\boldsymbol{\tau}),$$

where $c_1 \neq 0$ and $c_2 \neq 0$.

5. NUMERICAL EXAMPLES

In this section, we consider several numerical examples encountered in the control literature, that will allow us to illustrate the effectiveness of the proposed results.

Example 11. Consider the following quasi-polynomial

$$f(s, \tau_1, \tau_2) = p_0(s) + p_1(s)e^{-\tau_1 s} + e^{-\tau_2 s} \tag{13}$$

where

$$p_0(s) := s^2 - 2s + 2,$$
$$p_1(s) := 2\cos(1)s - 2(\cos(1) + \sin(1)).$$

Simple computations show that for $(\tau_1, \tau_2) = (1, 2)$, f has a critical root at $s^* = \boldsymbol{i}$ with multiplicity two. Additionally, the first partial derivatives are given by

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{(\boldsymbol{i},1,2)} \approx 2.91 + \boldsymbol{i}0.584, \quad \left.\frac{\partial f}{\partial \tau_2}\right|_{(\boldsymbol{i},1,2)} \approx -0.91 + \boldsymbol{i}0.416.$$

By applying Proposition 6, we conclude that the solutions of the Weierstrass polynomial can be expanded as a Puiseux series.

Example 12. Consider now the quasi-polynomial:

$$f(s, \boldsymbol{\tau}) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2}$$

where

$$p_0(s) = s^5 + s^4 + \frac{4+\pi}{2}s^3 + 2s^2 + \frac{2+\pi}{2}s + 2, \tag{14a}$$
$$p_1(s) = 1, \quad p_2(s) = 2s^4 + 4s^2 + 2. \tag{14b}$$

For $(\tau_1, \tau_2) = (\pi, 1)$, f has a double root at $s = \boldsymbol{i}$. Let $\widehat{f}$ be the quasi-polynomial derived by shifting from $(\boldsymbol{i}, \pi, 1)$ to the origin. Then the first non-zero partial derivatives of the quasi-polynomial at $(0, 0, 0)$ are given by

$$\left.\frac{\partial \widehat{f}}{\partial \tau_1}\right|_{(0,\mathbf{0})} = \boldsymbol{i}, \qquad \left.\frac{\partial^n \widehat{f}}{\partial \tau_2^n}\right|_{(0,\mathbf{0})} = 0, \ \forall n \in \mathbb{N}, \tag{15}$$

$$\left.\frac{\partial^2 \widehat{f}}{\partial s \partial \tau_1}\right|_{(0,\mathbf{0})} = 1 - \boldsymbol{i}\pi, \qquad \left.\frac{\partial^{n+1} \widehat{f}}{\partial s \partial \tau_2^n}\right|_{(0,\mathbf{0})} = 0, \ \forall n \in \mathbb{N}. \tag{16}$$

It is clear that the associated Weierstrass polynomial W has the following structure:

$$W(s, \boldsymbol{\tau}) := s^2 + w_1(\boldsymbol{\tau}) + w_0(\boldsymbol{\tau}).$$

Moreover, since the regularity condition (7) holds, we only need to compute $w_0(\boldsymbol{\tau})$. After simple computations, one gets:

$$w_0(\boldsymbol{\tau}) = \frac{-2\boldsymbol{i}}{(8+\pi^2) + \boldsymbol{i}(8-3\pi) + 16e^{-\boldsymbol{i}}}\tau_1 + o(\boldsymbol{\tau}).$$

Thus, its solutions are given by:

$$s(\boldsymbol{\tau}) = \boldsymbol{i} \pm \frac{\sqrt{2}\boldsymbol{i}^{3/2}}{\sqrt{(8+\pi^2) + \boldsymbol{i}(8-3\pi) + 16e^{-\boldsymbol{i}}}}(\tau_1 - \pi)^{1/2} + \mathcal{O}(|\boldsymbol{\tau} - \boldsymbol{\tau}^*|).$$

Proposition 10 guarantees that the solution $(\boldsymbol{i}, \pi, 1)$ has the CRS property. This behavior is illustrated in Figure 2.

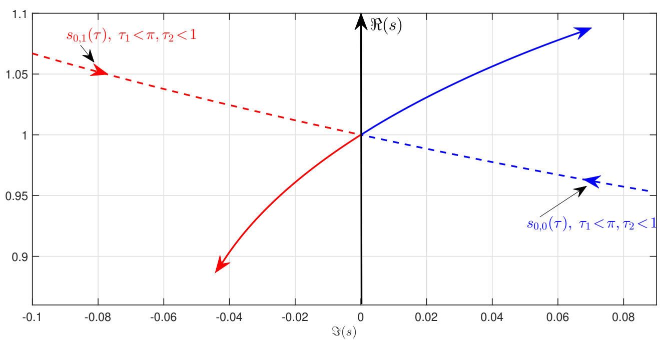

Fig. 2. Root locus of quasi-polynomial $f(s, \boldsymbol{\tau})$ (14) around $(\boldsymbol{i}, \pi, 1)$.

Example 13. Consider now the special case of a model of population dynamics discussed by Irofti et al. (2018):

$$f(s, \tau_1, \tau_2) = s - a - b\frac{1 - e^{-s\tau_1}}{s} - c\frac{1 - e^{-s\tau_2}}{s}. \tag{17}$$

Taking the same values as the ones proposed by Irofti et al. (2018), that is, $a = -0.214104$, $b = -0.996801$ and $c = 0.5$, we know that for $(\tau_1^*, \tau_2^*) \approx (3.84003026849, 10.44866732901)$, f possesses a double root at $s^* = \boldsymbol{i}$. In this case, we have the following partial derivatives:

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{(\boldsymbol{i},\boldsymbol{\tau}^*)} \neq 0, \quad \left.\frac{\partial f}{\partial \tau_2}\right|_{(\boldsymbol{i},\boldsymbol{\tau}^*)} \neq 0.$$

Thus, the regularity condition (7) holds for τ_1 and τ_2, and consequently, by using Proposition 6, its solutions can be expanded as Puiseux series with fractional power of $\frac{1}{2}$. Furthermore, by Proposition 7, its solutions behave as:

$$s(\boldsymbol{\tau})=\boldsymbol{i} \pm (0.3885 - 0.3307\boldsymbol{i})(\tau_2-\tau_2^*)^{1/2} + \mathcal{O}\left(|\boldsymbol{\tau} - \boldsymbol{\tau}^*|\right).$$

6. CONCLUDING REMARKS

The paper addresses the asymptotic behavior of multiple critical roots for quasi-polynomials of retarded-type with two incommensurate delays. The proposed approach allows identifying the structure of the solutions by only computing some partial derivatives. Thus, such results reveal if the solutions behave as a power series or as a Pusieux series. Besides, the adopted approach allows to relax some conditions imposed on the quasi-polynomial in some results previously reported in the literature, and as a consequence, our method allows capturing the behavior of any quasi-polynomial of retarded type with algebraic multiplicity two.

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