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Efficient Non-Conservative Realization of Dynamic Scaling-Based Output-Feedback Controllers via a Matrix Pencil Approach

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Abstract: A general matrix pencil based approach is developed for efficient non-conservative realization of dual dynamic high-gain scaling based control designs. A general class of uncertain feedforward-like nonlinear systems is considered and it is shown that the output-feedback control design procedure can be cast into a set of matrix pencil based sub-problems that capture the detailed system structure, state dependence structure of uncertain terms, and the precise roles of the design freedoms in the context of the detailed structure of the Lyapunov inequalities. The design freedoms in the dynamic high-gain scaling based design are extracted in terms of generalized eigenvalues of the formulated matrix pencil structures. It is seen that the proposed matrix pencil based approach greatly reduces design conservatism and algebraic complexity compared to prior results on dynamic high-gain scaling based control designs.

Keywords: Nonlinear control systems, Uncertain dynamic systems, Matrix methods, High gain feedback, Robust control, Dynamic output feedback.

1. INTRODUCTION

We consider the class of feedforward-like (i.e., triangular-like structure) uncertain nonlinear systems given by:

$$\dot{x}_i = \psi_i(y)x_{i+1} + \phi_i(y, \overline{x}_i, u) \quad , i = 1, \ldots, n-2$$

$$\dot{x}_{n-1} = \psi_{n-1}(y)x_n + \phi_{n-1}(y, u)\,;\; \dot{x}_n = u\,;\; y = [x_1, x_n]^T \quad (1)$$

where[1] $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ is the state of the system, $u \in \mathcal{R}$ is the input, $y \in \mathcal{R}^2$ is the measured output, and $\overline{x}_i$ denotes $[x_{i+2}, \ldots, x_n]^T$ for $i = 1, \ldots, n-1$. $\psi_i : \mathcal{R}^2 \to \mathcal{R}, i = 1, \ldots, n-1$, are known continuous functions. $\phi_i : \mathcal{R}^{n+2-i} \to \mathcal{R}, i = 1, \ldots, n-1$, are uncertain continuous functions. The control objective is asymptotic stabilization of the system via a dynamic output-feedback control law for u using the measurement of the output y. The "upper triangular" feedforward structure of state dependence of $\dot{x}_i$ has been studied in, for example, Kaliora and Astolfi (2001); Teel (1992); Mazenc and Praly (1996); Sepulchre et al. (1997). The particular structure (1), which is a slightly generalized version wherein $\dot{x}_i$ can also involve x_1, has been considered in Krishnamurthy and Khorrami (2004b, 2008) and dual dynamic high-gain scaling based output-feedback control designs have been developed.

High gain as a design methodology (Khalil and Saberi (1987); Ilchmann (1996)) has been studied in the literature for various classes of systems, both for observer and controller designs, and both using static and dynamic scaling terms. Static high-gain scaling based observers have been studied in Teel and Praly (1994); Khalil (1996); Atassi and Khalil (1999); Khalil (2008) based on observer gains $r, \ldots, r^n$ with a constant r to obtain semiglobal results. State-dependent scaling techniques for control of nonlinear systems are also addressed in Ito (2006). A combination of a high-gain observer (with dynamics of high gain parameter r in the form of a scalar differential Riccati equation) and a backstepping controller was developed in Praly (2003); Krishnamurthy et al. (2003). A dual observer/controller dynamic high-gain scaling technique was introduced in Krishnamurthy and Khorrami (2002, 2004a) that combined dynamic scaling based observer and controller structures to address uncertain strict-feedback-like systems including uncertain terms dependent on all states and uncertain Input-to-State Stable (ISS) appended dynamics with nonlinear gains from all the system states and the input. The dynamic high-gain scaling technique provides a unified design methodology applicable to both state-feedback and output-feedback control of both strict-feedback (Krishnamurthy and Khorrami (2004a); Kaliora et al. (2006); Krishnamurthy and Khorrami (2007b)) and feedforward (Krishnamurthy and Khorrami (2004b, 2008)) systems as well as state-feedback control of nontriangular polynomially-bounded systems (Krishnamurthy and Khorrami (2007a)) and is also applicable to systems with state and input time delays (Krishnamurthy and Khorrami (2010)) and systems with input unmodeled dynamics (Krishnamurthy and Khorrami (2013)).

However, while the scaling based methodology provides a flexible design approach, application of the methodology poses challenges due to significant algebraic complexity in computing upper bounds used on various terms during Lyapunov analysis to compute the design freedoms (such as dynamics of scaling parameter r). Due to this algebraic complexity, it is often difficult to compute tight bounds for a given specific system, necessitating utilization of conser-

[1] $\mathcal{R}$, $\mathcal{R}^+$, and $\mathcal{R}^k$ denote the sets of real numbers, non-negative real numbers, and real k-dimensional column vectors, respectively.

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vative upper bounds instead, which then results in effective control gains being much larger than required hampering practical performance of the closed-loop system. While the scaled state vector is handled in the Lyapunov analysis in a matrix structure (via coupled Lyapunov inequalities), the computation of upper bounds of uncertain terms often requires dropping down, for algebraic tractability, to conservative effectively scalar bounds that do not capture specific structure of where uncertain terms appear in the system dynamics and the specific state dependence structure. To address the challenges outlined above, a new matrix pencil based framework is developed in this paper that casts the overall design problem into a sequence of matrix pencil based subproblems that capture the precise roles of the design freedoms in the context of the detailed structure of the Lyapunov inequalities appearing during the control design taking into account the detailed state dependence structure of uncertain terms. The proposed approach reduces conservatism of the resulting design and also reduces algebraic complexity by replacing hand computations of upper bounds in previous scaling based designs with direct matrix pencil based specifications of the design freedoms.

2. NOTATIONS

Given a vector a, $|a|$ denotes its Euclidean norm. The diagonal matrix with diagonal elements $T_1, \ldots, T_m$ is denoted by $\text{diag}(T_1, \ldots, T_m)$. $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote the maximum and minimum eigenvalues, respectively, of a symmetric positive-definite matrix P. I_m denotes the $m \times m$ identity matrix. Given a matrix M, $\text{diag}(M)$ denotes the diagonal matrix whose diagonal elements are equal to the diagonal elements of M. Given a vector $a = [a_1, \ldots, a_m]^T$, $|a|_e$ denotes $[|a_1|, \ldots, |a_m|]^T$. Given vectors $a = [a_1, \ldots, a_m]^T$ and $b = [b_1, \ldots, b_m]^T$, $a \leq_e b$ indicates the element-wise inequalities $|a_i| \leq |b_i|, i = 1, \ldots, m$. With $M_1, \ldots, M_m$ being matrices, $\text{diag}(M_1, \ldots, M_m)$ denotes the block diagonal matrix formed by the matrices $M_1, \ldots, M_m$ as the blocks on the diagonal. Given a matrix M, $||M||$ denotes its Frobenius norm. Given a square matrix M, $\det(M)$ denotes its determinant. Given square matrices M_1 and M_2, the generalized eigenvalues of the matrix pencil $M_1 - sM_2$ with scalar s are defined as the values of s that make $\det(M_1 - sM_2) = 0$. The set of generalized eigenvalues of the matrix pencil $M_1 - sM_2$ are denoted as $\sigma(M_1, M_2)$. The subset of these eigenvalues that are finite in magnitude are denoted as $\sigma_f(M_1, M_2)$. When M_1 and M_2 are symmetric and at least one of M_1 or M_2 is positive-definite (or negative-definite), it is seen that the generalized eigenvalues are real numbers; under such condition, denote $\sigma_{\min}(M_1, M_2) \stackrel{\triangle}{=} \min(\sigma(M_1, M_2))$ and $\sigma_{\max,f}(M_1, M_2) \stackrel{\triangle}{=} \max(\sigma_f(M_1, M_2))$.

3. ASSUMPTIONS ON CLASS OF SYSTEMS

Assumption A1: A positive constant σ exists such that $\psi_i(y) \geq \sigma, 1 \leq i \leq n-1$, for all $y \in \mathcal{R}^2$.

Assumption A2: The functions ϕ_i, can be bounded as [2] $|\phi_i| \leq \Gamma_a(y,u)\left[\sum_{j=i+2}^{n} \Gamma_{(i,j)}(y)|x_j| + \gamma_u(y)|u|\right]$ for $i = 1, \ldots, n-2$, and $|\phi_{n-1}| \leq \Gamma_a(y,u)\gamma_u(y)|u|$, for all $x \in \mathcal{R}^n$ and $u \in \mathcal{R}$ with $\Gamma_a(y,u) = \psi_1(y)\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big)$ where γ_1, γ_2, γ_u, and $\Gamma_{(i,j)}, i = 1, \ldots, n-2, j = i+2, \ldots, n$, are known continuous non-negative functions.

Assumption A3: Positive constants ρ_i and $\tilde{\rho}_i$, $i = 2, \ldots, n-1$, exist such that the inequalities

$$\psi_i(y) \leq \rho_i \psi_{i-1}(y) \;\; ; \;\; \psi_i(y) \geq \tilde{\rho}_i \psi_{i-1}(y) \tag{2}$$

are satisfied for all $y \in \mathcal{R}^2$ and $i = 2, \ldots, n-1$.

Assumption A4: A continuous function $\gamma_o : \mathcal{R} \to \mathcal{R}^+$ exists such that $\psi_{n-1}(y)\gamma_u(y) \leq \gamma_o(x_n)$ for all $y = [x_1, x_n]^T \in \mathcal{R}^2$. The functions γ_1 and γ_o are polynomially upper bounded functions of x_n, i.e., nonnegative constants p_1, p_2, α_1, p_3, p_4, and α_2 exist such that $\gamma_1(x_n) \leq p_1 + p_2|x_n|^{\alpha_1}$ and $\gamma_o(x_n) \leq p_3 + p_4|x_n|^{\alpha_2}$ for all $x_n \in \mathcal{R}$.

Remark 1: Assumptions A1-A4 are analogous to Krishnamurthy and Khorrami (2004b) except that Assumption A2 involves a more detailed structure of the state dependence of each ϕ_i (specifically, different "weighting" coefficients $\Gamma_{(i,j)}$ instead of a single "worst-case" bound with $\Gamma_{(i,j)} = \Gamma_0$) The matrix pencil based approach developed in this paper provides an efficient approach to address the detailed structure of the bounds in Assumption A2.

4. DUAL DYNAMIC SCALING-BASED DESIGN

As in Krishnamurthy and Khorrami (2004b, 2008), a scaling-based observer and controller (Section 4.1) are combined and the various design freedoms are designed based on a Lyapunov analysis (Sections 4.2–4.4).

4.1 Observer and Controller Designs

An observer with state $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^T$ is designed as

$$\dot{\hat{x}}_i = \psi_i(y)\hat{x}_{i+1} + r^{-i}g_i(y)(\hat{x}_1 - x_1), 1 \leq i \leq n-1 \tag{3}$$

$$\dot{\hat{x}}_n = u + r^{-n}g_n(y)(\hat{x}_1 - x_1) \tag{4}$$

where $g_i(y)$ are functions that will be designed in Section 4.3 based on a pair of coupled Lyapunov inequalities. The dynamics of the high-gain scaling parameter r to be designed in Section 4.4 will be such that $r(t) \geq 1$ for all time $t \geq 0$. The observer error variables are defined as $e_i = \hat{x}_i - x_i, i = 1, \ldots, n$ and the scaled observer errors are defined as $\epsilon = [\epsilon_1, \ldots, \epsilon_n]^T$ with $\epsilon_i = \frac{e_i}{r^{n-i}}, i = 1, \ldots, n$ The dynamics of scaled observer error vector ϵ are

$$\dot{\epsilon} = \frac{1}{r}A_o\epsilon - \frac{\dot{r}}{r}(D_o - bI)\epsilon - \Phi \tag{5}$$

where $A_o(y)$ is the $n \times n$ matrix function with $(i,j)^{th}$ entry given by $A_{o_{(i,i+1)}} = \psi_i$ for $i = 1, \ldots, n-1$, $A_{o_{(i,1)}} = g_i$ for $i = 1, \ldots, n$, and zeros everywhere else; $D_o = \text{diag}(n-1+b, n-2+b, \ldots, 1+b, b)$; $\Phi = \left[\frac{\phi_1}{r^{n-1}}, \ldots, \frac{\phi_{n-1}}{r}, 0\right]^T$. Defining $\xi_i = \frac{\hat{x}_i}{r^{n-i}}$, $i = 1, \ldots, n$, a dynamic extension and control input transformation are defined as follows with ρ_n and b_v being any positive constants:

$$u = \frac{\psi_n(y)\xi_{n+1}}{r} \;\; ; \;\; \psi_n(y) \stackrel{\triangle}{=} \rho_n\psi_{n-1}(y) \tag{6}$$

$$\dot{\xi}_{n+1} = v - b_v\frac{\dot{r}}{r}\xi_{n+1}. \tag{7}$$

The new control input v is defined as

$$v = r^{-1}[k_1(y), k_2(y), k_3(y), \ldots, k_{n+1}(y)]\xi \tag{8}$$

with $k_i, i = 1, \ldots, n+1$, being functions of y that will be designed in Section 4.3 based on a pair of coupled

[2] For notational convenience, we drop the arguments of functions when no confusion will result.

Lyapunov inequalities involving the upper diagonal terms ψ_i. The dynamics of $\xi = [\xi_1, \ldots, \xi_{n+1}]^T$ are given by

$$\dot{\xi} = \frac{1}{r} A_c \xi - \frac{\dot{r}}{r}(D_c - bI)\xi + \frac{1}{r} g\epsilon_1 \tag{9}$$

where b is any positive constant and $A_c(y)$ is the $(n+1)\times(n+1)$ matrix with $(i,j)^{th}$ element $A_{c_{(i,i+1)}} = \psi_i$ for $i = 1, \ldots, n-1$, $A_{c_{(n,n+1)}} = \rho_n \psi_{n-1}$, and $A_{c_{(n+1,j)}} = k_j$ for $j = 1, \ldots, n+1$ with zeros everywhere else. Also, $D_c = \mathrm{diag}(n{-}1{+}b, n{-}2{+}b, \ldots, 1{+}b, b, b_v{+}b)$ and $g = [g_1, \ldots, g_n, 0]^T$.

4.2 Lyapunov Functions

With P_o and P_c being symmetric positive definite matrices to be designed in Section 4.3, define

$$V_o = \epsilon^T P_o \epsilon \ \ ; \ \ V_c = \xi^T P_c \xi \tag{10}$$

$$V = \frac{1}{r^{2b}}[cV_o + V_c] \tag{11}$$

where $c > 0$ is picked during the control design (Section 4.4). From (5), (9), (10), and (11), we have

$$\begin{aligned}\dot{V} &= \frac{c}{r^{1+2b}}\epsilon^T[P_oA_o + A_o^TP_o]\epsilon + \frac{1}{r^{1+2b}}\xi^T[P_cA_c + A_c^TP_c]\xi \\ &- 2\frac{c}{r^{2b}}\epsilon^TP_o\Phi + \frac{2}{r^{1+2b}}\xi^TP_cg\epsilon_1 \\ &- \frac{\dot{r}}{r^{1+2b}}\Big\{c\epsilon^T[P_oD_o{+}D_oP_o]\epsilon + \xi^T[P_cD_c{+}D_cP_c]\xi\Big\}. \end{aligned} \tag{12}$$

4.3 Coupled Lyapunov Inequalities

The choice of functions $g_1, \ldots, g_n$, $k_1, \ldots, k_{n+1}$, and matrices P_o and P_c are based on pairs of coupled Lyapunov inequalities detailed below, the key to the solvability of which is the set of inequalities (the *cascading dominance* conditions Krishnamurthy and Khorrami (2004a, 2006)) in Assumption A3 on the relative "sizes" (in a nonlinear function sense) of $\psi_i, i = 2, \ldots, n-1$. Under the first set of inequalities in (2), the constructive procedure in Krishnamurthy and Khorrami (2004a, 2006) enables finding of functions $g_1, \ldots, g_n$ and a symmetric positive definite matrix P_o such that, with $C = [1, 0, \ldots, 0]$, the pair of state-dependent Lyapunov inequalities

$$\left.\begin{array}{c} P_oA_o + A_o^TP_o \le -\nu_{1o}\psi_1 I - \nu_{1o}^*\psi_1 C^TC \\ \underline{\nu}_{2o} I \le P_oD_o + D_oP_o \le \overline{\nu}_{2o} I. \end{array}\right\} \tag{13}$$

is satisfied for all $y \in \mathcal{R}^2$ with ν_{1o}, ν_{1o}^* $\underline{\nu}_{2o}$, and $\overline{\nu}_{2o}$ being positive constants. Also, under Assumption A1 and the second set of inequalities in (2), functions $k_1, \ldots, k_{n+1}$ and a symmetric positive definite matrix P_c can be constructed such that the following pair of state-dependent Lyapunov inequalities are satisfied (for all $y \in \mathcal{R}^2$)

$$P_cA_c + A_c^TP_c \le -\nu_{1c}\psi_1 I \ ; \ \underline{\nu}_{2c} I \le P_cD_c + D_cP_c \le \overline{\nu}_{2c} I \tag{14}$$

with ν_{1c}, $\underline{\nu}_{2c}$, and $\overline{\nu}_{2c}$ being some positive constants. Furthermore, from Theorem 2 in Krishnamurthy and Khorrami (2006), $g_1, \ldots, g_n$ can be chosen to be linear constant-coefficient combinations of $\psi_2, \ldots, \psi_{n-1}$. Hence, using Assumption A3, a positive constant $\overline{g}$ can be found such that

$$(\sum_{i=1}^n g_i^2)^{\frac{1}{2}} \le \overline{g}\psi_1. \tag{15}$$

Similarly, the functions $k_1, \ldots, k_{n+1}$ can be chosen to be linear constant-coefficient combinations of $\psi_1, \ldots, \psi_{n-1}$.

4.4 Design Freedoms

After picking the functions $g_1, \ldots, g_n$, and $k_1, \ldots, k_{n+1}$, as discussed in Section 4.3, the remaining design freedoms appearing in the Lyapunov inequality (12) are the constant c and the dynamics of r. The basic strategy in dynamic scaling-based designs is to design the dynamics of r in such a way that $\dot{r}$ is "large" until r itself becomes "large," largeness for both $\dot{r}$ and r being defined in terms of appropriately designed nonlinear functions of the available state variables. To determine the required structure of the dynamics of r, note that the terms in the first line of (12) are negative, the terms in the second line are sign indefinite, and the signs of the terms in the third line can be made non-positive by picking $\dot{r} \ge 0$. Hence, c and $\dot{r}$ need to be picked such that the terms in the first and third lines dominate over the terms in the second line of (12).

To compute an upper bound for the first term in the second line of (12), note from Assumptions A2 and A4 that

$$\frac{|\phi_i|}{r^{n-i}} \le \frac{\Gamma_a}{r^2}\left[\sum_{j=i+2}^n \Gamma_{(i,j)}[|\epsilon_j| + |\xi_j|] + \gamma_o\rho_n|\xi_{n+1}|\right]. \tag{16}$$

Hence, it is seen that the first term in the second line of (12) can be upper bounded as

$$-2\frac{c}{r^{2b}}\epsilon^TP_o\Phi \le \frac{1}{r^{2+2b}}Q(y,u)[|\epsilon|^2 + |\xi|^2] \tag{17}$$

with $Q(y,u) = 4c\lambda_{max}(P_o)\Gamma_a(y,u)[\sum_{i=1}^{n-1}\sum_{j=i+2}^n \Gamma_{(i,j)} + n\gamma_o(x_n)\rho_n]$. Using (13) and noting that $|g|$ can be upper bounded by $\overline{g}\psi_1$ from (15), it is seen that the second term in the second line of (12) can be dominated by the negative terms in the first line of (12) if c is chosen large enough. For example, by picking

$$c \ge \frac{4}{\nu_{1c}\nu_{1o}^*}\lambda_{max}^2(P_c)\overline{g}^2, \tag{18}$$

it is seen that

$$\frac{2}{r^{1+2b}}\xi^TP_cg\epsilon_1 \le \frac{\nu_{1o}^*}{r^{1+2b}}\psi_1\epsilon_1^2 + \frac{\nu_{1c}}{4r^{1+2b}}\psi_1|\xi|^2. \tag{19}$$

From (13) and (14), it is seen from (19) that the second term in the second line of (12) is dominated by the negative terms in the first line of (12). From the above discussion and the form of the bound in (17), it is seen that to utilize $\dot{r}$ to help in dominating the first term in the second line of (12), we need to pick the dynamics of r in the form

$$\dot{r} = \max\left\{-a + \frac{1}{r}\Omega(y,u), 0\right\} \ \ ; \ r(0) \ge 1 \tag{20}$$

where a is to be chosen as a positive constant and $\Omega(y,u)$ is a function to be chosen taking into account the terms appearing in (12), e.g., the upper bound in (17). When r becomes "large enough" (specifically, when $r \ge \frac{\Omega(y,u)}{a}$), we have $\dot{r} = 0$. With this design of dynamics of r, (12) yields

$$\begin{aligned}\dot{V} &= \frac{c}{r^{1+2b}}\epsilon^T[P_oA_o + A_o^TP_o]\epsilon + \frac{1}{r^{1+2b}}\xi^T[P_cA_c + A_c^TP_c]\xi \\ &+ \frac{2}{r^{1+2b}}\xi^TP_cg\epsilon_1 \\ &+ \frac{a}{r^{1+2b}}\Big\{c\epsilon^T[P_oD_o{+}D_oP_o]\epsilon + \xi^T[P_cD_c{+}D_cP_c]\xi\Big\} \\ &- 2\frac{c}{r^{2b}}\epsilon^TP_o\Phi \\ &- \frac{\Omega}{r^{2+2b}}\Big\{c\epsilon^T[P_oD_o{+}D_oP_o]\epsilon + \xi^T[P_cD_c{+}D_cP_c]\xi\Big\}. \end{aligned} \tag{21}$$

The terms in (21) can be considered in terms of two parts:

- terms multiplied by $\frac{1}{r^{1+2b}}$, i.e., first three lines of (21); the sum of these terms can be made negative by choosing c large enough and a small enough.
- terms multiplied by $\frac{1}{r^{2+2b}}$, i.e., the fourth and fifth lines of (21); the term in the fifth line of (21) is considered as a term multiplied by $\frac{1}{r^{2+2b}}$ by taking into account the r dependence of the upper bound in (17). As discussed below, the sum of all these terms can be made non-positive by picking Ω appropriately.

Based on (21), the primary design considerations are summarized below:

- The positive constant c appearing in the definition of the composite Lyapunov function V in (10) must be chosen large enough so that the sign-indefinite term in the second line of (21) can be dominated by the negative terms in the first line of (21). As noted above, c can be chosen, for example, to satisfy (18).
- The positive constant a appearing in the dynamics of r in (20) must be chosen small enough to ensure that the positive terms in the third line of (21) can be dominated by the negative terms in the first line of (21). For example, using the coupled Lyapunov inequalities (13) and (14) and noting the choice of c above, a can be picked to satisfy

$$a \leq \frac{1}{2}\min\left(\frac{\nu_{1o}\sigma}{\overline{\nu}_{2o}}, \frac{\nu_{1c}\sigma}{2\overline{\nu}_{2c}}\right). \tag{22}$$

- The function Ω must be picked such that the negative term in the fifth line of (21) dominates the sign indefinite term in the fourth line of (12). For example, based on the bound computed in (17), Ω can be picked as $\Omega(y,u) = Q(y,u)\max\left(\frac{1}{c\underline{\nu}_{2o}}, \frac{1}{\underline{\nu}_{2c}}\right)$.

With the above design of c, a, and Ω, it is seen that (21) yields $\dot{V} \leq -\frac{\kappa}{r}V$ with

$$\kappa = \frac{\sigma}{4}\min(\nu_{1o}, \nu_{1c}). \tag{23}$$

While the typical design procedure in dynamic scaling-based control designs (and indeed in most other nonlinear control designs) is based on the sort of algebraic computation of scalar upper bounds as described above (e.g., (17) and (19)), these computations are algebraically complex and often result in conservative upper bounds for algebraic tractability. For example, the bounds in (17) and (19) are essentially worst-case upper bounds that do not, for instance, take into account the relative sizes of functions $\Gamma_{(i,j)}$ and the detailed structures of P_o and P_c). In contrast, the matrix pencil based approach described in Section 5 will formulate the choice of the design freedoms discussed above directly in terms of the required properties to be enforced on the Lyapunov inequalities to thereby obtain non-conservative bounds with lower algebraic complexity.

5. MATRIX PENCIL BASED DESIGN OF c, a, AND Ω

In this section, we develop a matrix pencil based formulation for picking the design freedoms c, a, ζ_1, and Ω. The main observation motivating the proposed approach is that these design freedoms are all scalar constants/functions and appear linearly in the Lyapunov inequality (21), therefore motivating an analysis of the corresponding designs via matrix pencil subproblems. For this purpose, a few almost self-evident observations, which are easy to show from the definitions of σ_{min} and $\sigma_{max,f}$, are summarized below.

Lemma 1: If M_1 is a symmetric negative definite matrix and M_2 is a symmetric negative semidefinite matrix, then $M_1 - sM_2$ is negative definite for all $s < \sigma_{min}(M_1, M_2)$.

Lemma 2: If M_1 is a symmetric positive definite matrix and M_2 is a symmetric positive semidefinite matrix, then $M_1 - sM_2$ is positive definite for all $s < \sigma_{min}(M_1, M_2)$.

Lemma 3: If M_1 is a symmetric matrix and M_2 is a symmetric positive definite matrix, then $M_1 - sM_2$ is negative definite for all $s > \sigma_{max,f}(M_1, M_2)$.

Lemma 4: If M_1 is a symmetric matrix and M_2 is a symmetric negative definite matrix, then $M_1 - sM_2$ is positive definite for all $s > \sigma_{max,f}(M_1, M_2)$.

Noting that the definition of V contains an r^{2b} in the denominator while the negative terms in the right hand side of (21) contain a r^{1+2b} in the denominator, the objective that we want to achieve in the choice of design freedoms c, a, and Ω is to ensure an inequality of form

$$\dot{V} \leq -\frac{\kappa}{r}V \tag{24}$$

with κ being a positive constant. For this purpose, from the analysis of the roles of the design freedoms in Section 4.4, the picking of these design freedoms can be considered in four steps. In the first step, the design freedom c is chosen to ensure that the following matrix inequality is satisfied with $\underline{c}_1$ being any constant chosen in the interval $(0,1)$:

$$\begin{aligned} 0 \geq (1-\underline{c}_1)\{c\epsilon^T[P_oA_o + A_o^TP_o]\epsilon \\ + \xi^T[P_cA_c + A_c^TP_c]\xi\} + 2\xi^TP_cg\epsilon_1. \end{aligned} \tag{25}$$

In the second step, $\underline{c}_2$ is chosen to be any constant in the interval $(0, \underline{c}_1)$ and the design freedom a is chosen to ensure that the following matrix inequality is satisfied:

$$\begin{aligned} 0 \geq (\underline{c}_1 - \underline{c}_2)\{c\epsilon^T[P_oA_o + A_o^TP_o]\epsilon + \xi^T[P_cA_c + A_c^TP_c]\xi\} \\ + a\{c\epsilon^T[P_oD_o + D_oP_o]\epsilon + \xi^T[P_cD_c + D_cP_c]\xi\}. \end{aligned} \tag{26}$$

In the third step, the function Ω is picked such that the following matrix inequality is satisfied:

$$\begin{aligned} 0 \geq & -2rc\epsilon^TP_o\Phi \\ & - \frac{\Omega}{r}\Big\{c\epsilon^T[P_oD_o + D_oP_o]\epsilon + \xi^T[P_cD_c + D_cP_c]\xi\Big\} \end{aligned} \tag{27}$$

In the fourth step, a positive constant κ is chosen such that

$$\begin{aligned} 0 \geq \underline{c}_2\{c\epsilon^T[P_oA_o + A_o^TP_o]\epsilon + \xi^T[P_cA_c + A_c^TP_c]\xi\} \\ + \kappa\Big\{c\epsilon^TP_o\epsilon + \xi^TP_c\xi\Big\} \end{aligned} \tag{28}$$

Each of the four steps outlined above is discussed below.

Design of c: Writing the right hand side of (25) as a quadratic form in terms of $[\epsilon^T, \xi^T]^T$, (25) can be written in matrix form as

$$\begin{aligned} 0 \geq c\begin{bmatrix} (1-\underline{c}_1)(P_oA_o + A_o^TP_o) & 0 \\ 0 & 0 \end{bmatrix} \\ + \begin{bmatrix} 0 & (P_cgB_1)^T \\ (P_cgB_1) & (1-\underline{c}_1)(P_cA_c + A_c^TP_c) \end{bmatrix} \end{aligned} \tag{29}$$

where $B_1 = [1, 0, \ldots, 0]$ is a $1 \times n$ row vector so that $B_1\epsilon = \epsilon_1$. In the matrix inequality (29) and all matrix inequalities below, the parts shown as 0 denote, as per the standard notation, blocks of compatible dimensions based on the other shown parts of the matrices. The right hand side of (29) is of form $cQ_{c1}(y) + Q_{c2}(y)$ with Q_{c1} and Q_{c2} being known and completely determined matrix functions of y (with c_1 being a constant chosen to be in the interval $(0, 1)$). In Section 4.4, it was seen that c can be chosen to be large enough (e.g., similar to (18)) to satisfy an inequality analogous to (29). However, such a construction of c is conservative (i.e., larger than required) since it is computed purely in terms of "macroscopic" quantities such as $\lambda_{max}(P_c)$, $\overline{g}$, ν_{1o}^*, and ν_{1c} and ignores the detailed structure of g, P_c, P_o, etc. Alternatively, by directly addressing the actual underlying requirement that we want to satisfy (25), i.e., equivalently make $cQ_{c1}(y) + Q_{c2}(y)$ negative semidefinite as shown in (29), a much less conservative estimate of c can be found. One point however to note is that we want to find a constant c since a state-dependent c will result in additional terms in $\dot{V}$, which will substantially change the resulting Lyapunov analysis. To obtain a constant c, we use the bi-directional cascading dominance conditions from Assumption A3 that the upper diagonal terms ψ_i are comparable (in a nonlinear function sense) and the fact noted in Section 4.3 that $g_1, \ldots, g_n$, and $k_1, \ldots, k_{n+1}$ can be picked to be linear constant-coefficient combinations of $\psi_1, \ldots, \psi_{n-1}$. Hence, dividing (29) throughout by ψ_1, each matrix appearing in the resulting inequality varies in a polytopic set whose vertices can be computed in terms of constants ρ_i and $\tilde{\rho}_i, i = 2, \ldots, n-1$ from Assumption A3, and coefficients in the designs of functions $g_1, \ldots, g_n$ and $k_1, \ldots, k_{n+1}$ in terms of $\psi_1, \ldots, \psi_{n-1}$. In particular, when all upper diagonal terms ψ_i are identical (up to scaling by constant coefficients), the polytope reduces to a single point. In either case, the resulting system of equations (diagonally concatenated over vertices of the polytope or from the single value when the polytope reduces to a single point) can be written in the form $0 \geq c\overline{Q}_{c1} + \overline{Q}_{c2}$ with $\overline{Q}_{c1}$ and $\overline{Q}_{c2}$ being constant matrices. From (18), we know that a large enough constant c definitely exists that will ensure that this inequality is satisfied. Analogous to Lemma 3, it is seen that the large enough constant c can be picked as

$$c = \sigma_{\max,f}(\overline{Q}_{c2}, -\overline{Q}_{c1}). \tag{30}$$

Design of a**:** Writing the right hand side of (26) as a quadratic form in terms of $[\epsilon^T, \xi^T]^T$, the inequality (26) can be equivalently written as the matrix inequality

$$0 \geq a \begin{bmatrix} c(P_oD_o + D_oP_o) & 0 \\ 0 & (P_cD_c + D_cP_c) \end{bmatrix} + (\underline{c}_1 - \underline{c}_2) \begin{bmatrix} c(P_oA_o + A_o^TP_o) & 0 \\ 0 & (P_cA_c + A_c^TP_c) \end{bmatrix}. \tag{31}$$

which is of the form $0 \geq aQ_{a1}(y) + Q_{a2}(y)$. Both Q_{a1} and Q_{a2} are known and completely determined matrix functions of y given a choice of $\underline{c}_2$ in the interval $(0, \underline{c}_1)$. While the choice of c was required to be a constant, the choice of a can indeed be a function of y as long as it is lower bounded by a constant, i.e., $a(y) \geq \underline{a}$ with some positive $\underline{a}$. While c appears in the definition of the Lyapunov function in (10) and time variation in c results in additional terms in $\dot{V}$, a is simply a term appearing in the dynamics of r and dependence of a on y does not result in new terms in $\dot{V}$ and does not affect the stability analysis. Instead, allowing a to be a function of y could make utilization of larger values of a possible. Since a acts as a stabilizing term in the dynamics of r (and $\dot{r} = 0$ if $r \geq \frac{\Omega(y,u)}{a}$), larger values of a tend to reduce values of r, benefiting the transient response of the closed-loop system. From the coupled Lyapunov inequalities (13) and (14), it can be seen that both Q_{a2} and $-Q_{a1}$ are symmetric negative definite matrices. Hence, from Lemma 1, it is seen that $Q_{a2} + sQ_{a1}$ is negative definite for all $s < \sigma_{min}(Q_{a2}, -Q_{a1})$. Hence, to satisfy (26), a can be picked to be a function of y defined as

$$a(y) = \sigma_{\min}(Q_{a2}(y), -Q_{a1}(y)). \tag{32}$$

Design of Ω**:** Similar to the discussion in Section 4.4, the design of Ω to satisfy (27) entails finding a large enough function $\Omega(y, u)$ to ensure that the negative terms in the second line of (27) dominate over the sign indefinite term in the right hand side of the first line of (27). While an upper bound for $-2rc\epsilon^TP_o\Phi$ can be written using (17), a less conservative upper bound can be written from (16) retaining the structure of the state dependence in the bounds in Assumption A2 as

$$-2rc\epsilon^TP_o\Phi \leq 2rc|\epsilon|_e^T|P_o|_e|\Phi|_e \tag{33}$$

$$|\Phi|_e \leq_e \frac{1}{r^2}\Gamma_a(y,u)[\overline{\Phi}_\epsilon(y), \overline{\Phi}_\xi(y)] \begin{bmatrix} |\epsilon|_e \\ |\xi|_e \end{bmatrix} \tag{34}$$

- $\overline{\Phi}_\epsilon$ is an $n \times n$ matrix with $(i, j)^{th}$ elements given by
$$\overline{\Phi}_{\epsilon,(i,j)} = \Gamma_{(i,j)}, i = 1, \ldots, n-1, j = i+2, \ldots, n \tag{35}$$
and zeros everywhere else.
- $\overline{\Phi}_\xi$ is an $n \times (n+1)$ matrix with $(i, j)^{th}$ elements
$$\overline{\Phi}_{\xi,(i,j)} = \Gamma_{(i,j)}, i = 1, \ldots, n-1, j = i+2, \ldots, n$$
$$\overline{\Phi}_{\xi,(i,n+1)} = \gamma_o\rho_n, i = 1, \ldots, n-1. \tag{36}$$

While the upper bound from (33) and (34) is much less conservative than the upper bound from (17), it is to be noted that (33) and (34) involve the element-wise magnitudes $|\epsilon|_e$ and $|\xi|_e$ instead of just ϵ and ξ. This reflects the fact that the signs of elements of Φ are not known (since the only information on the uncertain ϕ_i is the structure of upper bounds in Assumption A2). Hence, the negative term in the second line of (27) must also be written in terms of $|\epsilon|_e$ and $|\xi|_e$. While conservative estimates can be written such as $\epsilon^T(P_oD_o + D_oP_o)\epsilon \geq -\underline{\nu}_{2o}|\epsilon|_e^T|\epsilon|_e$, relatively non-conservative estimates can be obtained by considering the diagonal elements of matrices $P_oD_o + D_oP_o$ and $P_cD_c + D_cP_c$, etc. Defining $\overline{D}_o = \text{diag}(P_oD_o + D_oP_o)$, we have $\epsilon^T\overline{D}_o\epsilon = |\epsilon|_e^T\overline{D}_o|\epsilon|_e$. Considering the matrix pencil of form $P_oD_o + D_oP_o - s\overline{D}_o$ and defining $\delta_{D_o} = \sigma_{\min}(P_oD_o + D_o^TP_o, \overline{D}_o)$, we have $P_oD_o + D_oP_o \geq \delta_{D_o}\overline{D}_o$. Similarly, defining $\overline{D}_c = \text{diag}(P_cD_c + D_cP_c)$ and $\delta_{D_c} = \sigma_{\min}(P_c\tilde{D}_c + \tilde{D}_c^TP_c, \overline{D}_c)$, we have $P_cD_c + D_c^TP_c \geq \delta_{D_c}\overline{D}_c$. Using (33) and (34) and noting that $\overline{D}_o$ and $\overline{D}_c$ are diagonal matrices, (27) reduces to

$$0 \geq 2\frac{1}{r}c|\epsilon|_e^T|P_o|_e\Gamma_a(y,u)[\overline{\Phi}_\epsilon(y), \overline{\Phi}_\xi(y)] \begin{bmatrix} |\epsilon|_e \\ |\xi|_e \end{bmatrix} - \frac{\Omega}{r}\left\{c\delta_{D_o}|\epsilon|_e^T\overline{D}_o|\epsilon|_e + \delta_{D_c}|\xi|_e^T\overline{D}_c|\xi|_e\right\}. \tag{37}$$

Hence, writing (37) as a quadratic form in terms of $[|\epsilon|_e^T, |\xi|_e^T]^T$, (37) can be written equivalently as

$$0 \geq \begin{bmatrix} 2c\Gamma_a|P_o|_e\overline{\Phi}_\epsilon & c\Gamma_a|P_o|_e\Phi_\xi \\ c\Gamma_a(|P_o|_e\Phi_\xi)^T & 0 \end{bmatrix} - \Omega \begin{bmatrix} c\delta_{D_o}\overline{D}_o & 0 \\ 0 & c\delta_{D_c}\overline{D}_c \end{bmatrix} \tag{38}$$

The matrix inequality (38) is of the form $0 \geq Q_{\Omega 1}(y,u) - \Omega(y,u)Q_{\Omega 2}$. Noting that $Q_{\Omega 2}$ is a symmetric positive definite matrix, it is seen from Lemma 3 that Ω can be picked as $\Omega(y,u) = \sigma_{max,f}(Q_{\Omega 1}(y,u), Q_{\Omega 2})$.

Design of κ***:*** Writing the right hand side of (28) as a quadratic form in terms of $[\epsilon^T, \xi^T]^T$, (28) can be written equivalently as

$$0 \geq \underline{c}_2 \begin{bmatrix} c(P_o A_o + A_o^T P_o) & 0 \\ 0 & (P_c A_c + A_c^T P_c) \end{bmatrix} + \kappa \begin{bmatrix} cP_o & 0 \\ 0 & P_c \end{bmatrix} \quad (39)$$

which is of the form $0 \geq Q_{\kappa 1}(y) + \kappa Q_{\kappa 2}$. Noting that both $Q_{\kappa 1}$ and $-Q_{\kappa 2}$ are symmetric negative definite matrices and using Lemma 1, it is seen that κ can be chosen as

$$\kappa(y) = \sigma_{min}(Q_{\kappa 1}(y), -Q_{\kappa 2}). \quad (40)$$

As with the choice of a, it is acceptable for κ to be a function of y as long as it is lower bounded by a positive constant since a state dependence of κ will not create any additional terms in $\dot{V}$. It is known that a positive constant κ can be chosen as (23) from the analysis in Section 4.4; hence, the choice of κ in (40) is lower bounded by a positive constant. Once the inequality (24) is achieved through the above choices of the design freedoms, it can be shown as in Krishnamurthy and Khorrami (2004b, 2008) that the closed-loop signals are all uniformly bounded over time interval $[0, \infty)$ and x, $\hat{x}$, and u converge to 0 as $t \to \infty$.

6. CONCLUSION

A matrix pencil based approach was developed for efficient non-conservative realization of dual dynamic high-gain scaling based control designs based on capturing the detailed structure of the closed-loop system and associated Lyapunov inequalities within matrix pencil structures that explicitly show the roles of each of the design freedoms.

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