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Fixed-Time Estimators of Derivatives of Unknown Maps ⋆

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Abstract: A systematic and generalized asymptotic derivative estimator design method is first presented for unknown maps by adding a sinusoidal excitation signal to the argument of the map. Then, based on the proposed asymptotic derivative estimator approach and based on the existing design methods for both finite-time and fixed-time state observers, finite-time and fixed-time derivative estimators are designed. The sufficient conditions for finite-time and fixed-time input-to-state stable of the finite-time and fixed-time derivative estimators are given respectively when a bounded disturbance input exists.

Keywords: Derivative estimator; Extremum seeking; Finite-time; Fixed-time; Input-to-state stability.

1. INTRODUCTION

Extremum seeking (ES) is a kind of adaptive control which can drive and maintain the input and output of the controlled object to their respective extrema. Extremum seeking control work without any explicit knowledge about the input-output characteristics as long as the extrema exist, which is its greatest advantage Wang et al. (2016). This advantage stems from its special gradient estimation method, which is an important part of the perturbation based extremum seeking. The most of these existing perturbation based extremum seeking are asymptotically convergent. However, compared to asymptotic convergence, finite-time convergence reaches exactly zero in a finite time Bhat (2000), and the so-called fixed-time convergence does it with a uniform upper bound on the settling-time estimate Andrieu et al. (2008), meaning that there exists a maximum convergence time to zero, irrespective of the system's initial condition. In practice, finite-time or fixed-time convergence is much more desirable because the closed-loop systems under finite-time or fixed-time control law usually demonstrate higher accuracy and better disturbance rejection properties Bhat (2000).

To develop finite-time or fixed-time perturbation based extremum seeking, the design of the finite-time or fixed-time gradient estimator is crucial. However, the gradient estimation method of classical extremum seeking is difficult to achieve finite-time or fixed-time estimation due to the use of filters. Based on fixed-time gradient-flow scheme, Poveda and Krstić. (2019a) proposed the first averaging-based extremum seeking controller able to achieve fixed-time convergence to an arbitrarily small neighborhood of the optimal operating set. Furthermore, Poveda and Krstić. (2019b) designed a Newton-based practical fixed-time extremum seeking scheme. However, the fixed-time (FX) ES or even fixed-time input-to-state stable (FX-ISS) ES has not appeared yet.

The detailed basic principles for classical ES gradient (derivative for single input) estimation can be found in Mills and Krstić. (2018), which presented a generalization of the scalar Newton-based ES to maximize the map's higher derivatives. By properly demodulating the map output corresponding to the manner in which it is perturbed, the ES algorithm maximizes the n-th derivative asymptotically only through measurements of the map. In earlier work, Nešić et al. (2010) proposed a derivative estimator which can generate an approximation of higher derivatives of the unknown map. However, like most perturbation based ES, the demodulated signal is used directly to demodulate map output. These derivative estimation methods can theoretically only obtain a periodic signal, the average value of which is an approximate derivative of the map. In the study of combustion instability in gas turbine, Banaszuk et al. (2000) and Moase et al. (2010) applied similar ideas for derivative estimation, which converts the estimation of derivative into the design of a state observer. Then, the approximate derivative value instead of a periodic signal can be obtained by demodulating the corresponding observer state. Unfortunately, systematic and generalized design methods are absent. On the other hand, Lopez-Ramirez et al. designed finite-time and fixed-time state observers and presented a theoretical framework to study finite-time (FT) and FX input-to-state stability of nonlinear systems using the implicit Lyapunov function Lopez-Ramirez et al. (2018a,b); Lopez-Ramirez (2019). Although only the sufficient conditions of ISS for the state

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observers are given instead of FT-ISS or FX-ISS in Lopez-Ramirez et al. (2018b); Lopez-Ramirez (2019), which made it possible for the design of finite-time and fixed-time derivative estimators. And as a potential application of the finite-time and fixed-time derivative estimators, finite-time and fixed-time ES schemes might be implemented.

In this paper, we propose an asymptotic derivative estimator for unknown maps. The systematic and generalized design method are also presented. Furthermore, the FT and FX derivative estimators are designed based on the asymptotic one and the design method of FT and FX state observers proposed in Lopez-Ramirez et al. (2018b). For the FT and FX derivative estimators, we also give the sufficient conditions of FT-ISS and FX-ISS with respect to a bounded disturbance input. The sufficient conditions also work for the FT and FX state observers.

The remainder of the paper is organized as follows: some preliminaries are given in Section 2. In Section 3, the systematic design method and an asymptotic derivative estimator are proposed. In Section 4, the FT and FX derivative estimators are designed based on the design method of the asymptotic one. Furthermore, the sufficient conditions for FT-ISS and FX-ISS of the FT and FX derivative estimators with respect to bounded disturbance are presented. In Section 5, the simulation example of above derivative estimators are given. Finally, a brief conclusion and future work is presented in Section 6.

2. PRELIMINARIES

The definitions of FT and FX-ISS rely on generalized class $\mathcal{KL}$ function, which is defined as follows:

Definition 1. (Generalized class $\mathcal{KL}$ function). A continuous function $\bar{\beta}: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to belong to finite-time class $\mathcal{KL}$ function if

1)for each fixed $t \geqslant 0$, there exists a corresponding $s_t \geqslant 0$, such that the mapping $\bar{\beta}(s,t)$ is strictly increasing with respect to s when $s > s_t$, $\bar{\beta}(s,t) = 0$ when $s \leqslant s_t$. In addition, $s_t = 0$, when $t = 0$

2)for each fixed $s \geqslant 0$, the mapping $t \mapsto \beta(s,t)$ is continuous, decreases to zero and there exists some $T(s) \in [0, +\infty)$, such that $\beta(s,t) = 0$ for all $t \geqslant T(s)$.

In particular, if there exists a $T_{\max} = \max\{T(s) : s \in \mathbb{R}_{\geqslant 0}\} < +\infty$, the function $\bar{\beta}$ is said to belong to fixed-time class $\mathcal{KL}$ function with fixed-time T_{max}.

Consider the following nonlinear system

$$\dot{x}(t) = f\left(x(t), d(t)\right), \ t \geqslant 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous and ensures forward existence of the system solutions, at least locally, and $f(0,0) = 0$.

Definition 2. Hong et al. (2010); Lopez-Ramirez et al. (2018a) The system (1) is said to be finite-time ISS (FT-ISS), if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, there exist a finite-time class $\mathcal{KL}$ function $\bar{\beta}$ and a class $\mathcal{K}$ function γ such that

$$\|\psi_{x_0}(t,d)\| \leqslant \bar{\beta}(\|x_0\|, t) + \gamma(\|d\|_{[0,\infty)}), \ \forall t \geqslant 0. \tag{2}$$

The system (1) is said to be fixed-time ISS (FX-ISS), if the function $\bar{\beta}$ is a fixed-time class $\mathcal{KL}$ function.

3. ASYMPTOTIC GRADIENT ESTIMATOR

In this section, an asymptotic convergence derivative estimator will be proposed by adding a sinusoidal excitation signal to the argument of an unknown function. The systematic and generalized design method of it will also be presented. In fact, the proposed derivative estimator can be regarded as a generalization of the derivative estimation method in classical extremum seeking.

Consider a smooth function

$$y = v(\theta), \tag{3}$$

the purpose of the model-free derivative estimator is to estimate the derivative information of the map $v(\theta)$, i.e. $\frac{dv(\theta)}{d\theta}$, and without any knowledge of the function. For this purpose, a sinusoidal excitation signal $a\sin\omega t$ (or cosine excitation signal) needs to be added on the argument of the map $v(\theta)$. For example, if we want to estimate the derivative information of the map at $\hat{\theta}$, then let

$$\theta = \hat{\theta} + a\sin\omega t, \tag{4}$$

where a, ω are small positive numbers and represent the amplitude and the frequency of the excitation signal, respectively. Substituting (4) into (3), we can get

$$y = v(\hat{\theta} + a\sin\omega t). \tag{5}$$

Furthermore, apply the Taylor expansion formula to get

$$\begin{aligned} y =& v(\hat{\theta}) + av'(\hat{\theta})\sin\omega t + \frac{a^2}{2}v''(\hat{\theta})\sin^2\omega t + \cdots \\ &+ \frac{a^n}{n!}D^n v(\hat{\theta})\sin^n\omega t + O(a^{n+1}) \end{aligned} \tag{6}$$

where $D^n v(\hat{\theta})$ represents the n-th order derivative of function $v(\theta)$ at $\hat{\theta}$. Furthermore, applying trigonometric formula, the (6) can be represented as follows.

$$\begin{aligned} y =& \sum_{j\in\mathbb{N}_{\rm odd}}^{[n]_{\rm odd}} (-1)^{\frac{j-1}{2}}\sin(j\omega t) \sum_{i\in\mathbb{N}_{\rm odd}\geqslant j}^{[n]_{\rm odd}} \frac{D^i v(\hat{\theta})a^i}{i!2^{i-1}} \begin{pmatrix} i \\ \frac{i-j}{2} \end{pmatrix} \\ &+ \sum_{j\in\mathbb{N}_{\rm even}}^{[n]_{\rm even}} (-1)^{\frac{j}{2}}\cos(j\omega t) \sum_{i\in\mathbb{N}_{\rm even}\geqslant j}^{[n]_{\rm even}} \frac{D^i v(\hat{\theta})a^i}{i!2^{i-1}} \begin{pmatrix} i \\ \frac{i-j}{2} \end{pmatrix} \\ &+ \sum_{i\in\mathbb{N}^0_{\rm even}}^{[n]_{\rm even}} \frac{D^i v(\hat{\theta})a^i}{i!2^i} \begin{pmatrix} i \\ \frac{i}{2} \end{pmatrix} \end{aligned} \tag{7}$$

It should be noted that we omit the $O(a^{n+1})$ term in (7) and there after. One reason is because the $O(a^{n+1})$ term is too small to be ignored in practice. In addition, due to the limited space, its theoretical impact on the derivative estimator will be given in our future work. The state variable $x = [x_1, \cdots, x_{2m}, x_{2m+1}, \cdots, x_{2n+1}]^T$ can be selected by the following rules.

$$x_1 = \sum_{i\in\mathbb{N}^0_{\rm even}}^{[n]_{\rm even}} \frac{D^i v(\hat{\theta})a^i}{i!2^i} \begin{pmatrix} i \\ \frac{i}{2} \end{pmatrix}. \tag{8}$$

When m is an odd number,

$$x_{2m} = \sum_{i\in\mathbb{N}_{\rm odd}\geqslant m}^{[n]_{\rm odd}} D^i v(\hat{\theta})a^{i-m}\frac{m!2^{m-1}}{i!2^{i-1}} \begin{pmatrix} i \\ \frac{i-m}{2} \end{pmatrix} \sin(m\omega t), \tag{9}$$

$$x_{2m+1} = \sum_{i \in \mathbb{N}_{\text{odd}} \geqslant m}^{[n]_{\text{odd}}} D^i v(\hat{\theta}) a^{i-m} \frac{m! 2^{m-1}}{i! 2^{i-1}} \begin{pmatrix} i \\ \frac{i-m}{2} \end{pmatrix} \cos(m\omega t). \tag{10}$$

When m is an even number,

$$x_{2m} = \sum_{i \in \mathbb{N}_{\text{even}} \geqslant m}^{[n]_{\text{even}}} D^i v(\hat{\theta}) a^{i-m} \frac{m! 2^{m-1}}{i! 2^{i-1}} \begin{pmatrix} i \\ \frac{i-m}{2} \end{pmatrix} \sin(m\omega t), \tag{11}$$

$$x_{2m+1} = \sum_{i \in \mathbb{N}_{\text{even}} \geqslant m}^{[n]_{\text{even}}} D^i v(\hat{\theta}) a^{i-m} \frac{m! 2^{m-1}}{i! 2^{i-1}} \begin{pmatrix} i \\ \frac{i-m}{2} \end{pmatrix} \cos(m\omega t), \tag{12}$$

where $\mathbb{N}_{\text{odd}}$ denotes the set of odd natural numbers, $\mathbb{N}_{\text{even}}$ the set of even natural numbers, $\mathbb{N}^0_{\text{even}}$ represents the union of set $\mathbb{N}_{\text{even}}$ and 0. $[n]_{\text{odd}}$ denotes an odd natural number less than or equal to n, $[n]_{\text{even}}$ denotes an even natural number less than or equal to n. In addition, when $m = n$, it can be obtained that

$$x_{2n} = D^n v(\hat{\theta}) \sin(n\omega t), \quad x_{2n+1} = D^n v(\hat{\theta}) \cos(n\omega t). \tag{13}$$

Furthermore, denoting the item containing a^2 as $O(a^2)$, the state variable can be obtained as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{2(n-1)} \\ x_{2(n-1)+1} \\ x_{2n} \\ x_{2n+1} \end{bmatrix} = \begin{bmatrix} v(\hat{\theta}) + O(a^2) \\ (v'(\hat{\theta}) + O(a^2)) \sin \omega t \\ (v'(\hat{\theta}) + O(a^2)) \cos \omega t \\ \vdots \\ D^{n-1} v(\hat{\theta}) \sin(n-1)\omega t \\ D^{n-1} v(\hat{\theta}) \cos(n-1)\omega t \\ D^n v(\hat{\theta}) \sin n\omega t \\ D^n v(\hat{\theta}) \cos n\omega t \end{bmatrix}, \tag{14}$$

Deriving the state vector, we can get the following system

$$\dot{x} = Ax + G\dot{\hat{\theta}}, \tag{15}$$

$$y = Cx, \tag{16}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (n-1)\omega & 0 & 0 \\ 0 & 0 & 0 & \cdots & -(n-1)\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n\omega \\ 0 & 0 & 0 & \cdots & 0 & 0 & -n\omega & 0 \end{bmatrix},$$

$$C = [\, 1 \;\, a \;\, 0 \; \cdots \; C_{n-1} \;\, 0 \;\, 0 \;\, C_n \,], \text{ if } n \in \mathbb{N}_{\text{even}}$$
$$C = [\, 1 \;\, a \;\, 0 \; \cdots \; 0 \;\, C_{n-1} \;\, C_n \;\, 0 \,], \quad \text{if } n \in \mathbb{N}_{\text{odd}}$$

$$C_{n-1} = \frac{(-1)^{\frac{[n-1]_{\text{even}}}{2}} a^{n-1}}{(n-1)! 2^{n-2}}, C_n = \frac{(-1)^{\frac{[n]_{\text{even}}}{2}} a^n}{n! 2^{n-1}}, G = \frac{\partial x}{\partial \hat{\theta}}.$$

It should be noted that the output (16) of the system is equal to the function (7), that is the system (15), (16) can be approximately considered as a system implementation of function (5). Furthermore, the system can be simplified as a linear system consisting of the pair (A, C) when $\dot{\hat{\theta}} = 0$. For this linear system, we have the following proposition.

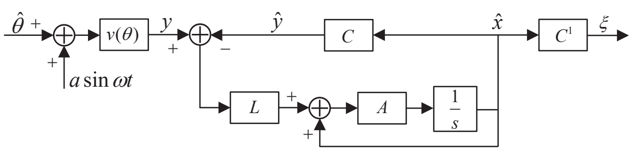

Fig. 1. Asymptotic derivative estimator

Proposition 1. The linear system consisting of the pair (A, C) is observable.

Thus, we can design the following state observer for the linear system (A, C).

$$\hat{y} = C\hat{x}, \tag{17}$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + L(y - \hat{y}). \tag{18}$$

where $L \in \mathbb{R}^{2n+1}$ is gain vector, which makes $A - LC$ is Hurwitz. Furthermore, we can get

$$\hat{D}^i v(\hat{\theta}) + O(a^2) = C^i \hat{x}, \tag{19}$$

where $C^i = [\, 0 \cdots 0 \; \sin i\omega t \; \cos i\omega t \; 0 \cdots]$, $\hat{D}^i v(\hat{\theta})$ is an estimate of the i-th order derivative of map $v(\theta)$ at $\hat{\theta}$, $\sin i\omega t$ and $\cos i\omega t$ are the $2i$-th and $(2i+1)$-th elements of vector C^i, respectively. From (19), it can be seen that the estimator consisting of (17), (18), (19) can be used to estimate any order (i-th) derivative of map $v(\theta)$ at $\hat{\theta}$ theoretically. Specially, let $i = 1$ and from (19), we can get

$$\xi := C^1 \hat{x} = \hat{D}^1 v(\hat{\theta}) + O(a^2), \tag{20}$$

where $\hat{D}^1 v(\hat{\theta})$ denotes an estimate of the derivative of map $v(\theta)$ at $\hat{\theta}$. The estimate $\hat{D}^1 v(\hat{\theta})$ converges to the real value of the derivative asymptotically as the state estimate $\hat{x}$ asymptotically converges to the actual state x. Furthermore, the output ξ of the derivative estimator converges to a $O(a^2)$-neighbourhood of the derivative asymptotically. Thus, ξ can be approximated as an estimate of the derivative. So far, we have obtained a derivative estimator consisting of (17), (18), and (20) for the unknown map (3) as shown in Fig.1. In particular, the state observer (17), (18) of linear system (A, C) is ISS when there exists a bounded disturbance input or $\dot{\hat{\theta}} \neq 0$.

4. FINITE-TIME AND FIXED-TIME GRADIENT ESTIMATORS

Based on the principle of the proposed asymptotic derivative estimator and the design method of finite-time and fixed-time state observers provided in Lopez-Ramirez et al. (2018b), we will design the finite-time and fixed-time derivative estimators in this section. In addition, the sufficient conditions of FT-ISS and FX-ISS for the finite-time and fixed-time derivative estimators w.r.t a bounded disturbance input will be provided, too. First, we introduce the following lemma from Lopez-Ramirez et al. (2018b).

Lemma 1. Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{21}$$

with the pair (A, C) being observable and $\text{rank}(C) = k$. Then there exists a nonsingular transformation Φ such that

$\Phi A \Phi^{-1} = F\tilde{C} + \tilde{A}, C\Phi^{-1} = [C_0 \, 0 \cdots 0], \tilde{C} = [I_k \; 0] \in \mathbb{R}^{k \times n},$

$$\tilde{A} = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{m-1\,m} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; C_0 \in \mathbb{R}^{k\times k}, F \in \mathbb{R}^{n\times k},$$

where m is an integer, $A_{j-1,j} \in \mathbb{R}^{n_{j-1}\times n_j}$, $n_j = \operatorname{rank}(A_{j-1,j})$, $j = 2, ..., m$, so that $n_1 = \operatorname{rank}(C) = k$ and $\sum_{i=1}^{m} n_i = n$.

4.1 Finite-Time Gradient Estimator

From the principle of the asymptotic derivative estimator, it is known that the finite-time design of the derivative estimator requires a finite-time state observer of system (15), (16). Then combining with (19), the derivative estimation can be obtained in finite-time. To this end, the design method of finite-time state observer will be introduced from Lopez-Ramirez et al. (2018b) to design the finite-time derivative estimator as follows.

$$\dot{\hat{x}}(t) = A\hat{x}(t) - g_{\mathrm{FT}}(y(t) - C\hat{x}(t)), \tag{22}$$
$$\xi = C^1\hat{x}, \tag{23}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector and C^1 can be obtained from (19). The function $g_{\mathrm{FT}} : \mathbb{R}^k \to \mathbb{R}^n$ is defined as

$$g_{\mathrm{FT}}(\sigma) := \Phi^{-1}[D_{\tilde{r}}(\|\tilde{P}C_0^{-1}\sigma\|^{-1})L_{\mathrm{FT}} - F]C_0^{-1}\sigma, \tag{24}$$

where $\sigma \in \mathbb{R}^k$, the matrices $\Phi \in \mathbb{R}^{n\times n}$, $C_0 \in \mathbb{R}^{k\times k}$ and $F \in \mathbb{R}^{n\times k}$ are defined in Lemma 1, $D_{\tilde{r}}$ is the dilation matrix defined as

$$D_{\tilde{r}}(\lambda) = \operatorname{diag}(\lambda^{r_1}I_{n_1}, \lambda^{r_2}I_{n_2}, ..., \lambda^{r_m}I_{n_m}), \tag{25}$$

$$\tilde{r} = \left[\frac{\mu}{1+(m-1)\mu}, \frac{2\mu}{1+(m-1)\mu}, ..., \frac{m\mu}{1+(m-1)\mu}\right]^T, \tag{26}$$

$\mu \in (0, 1]$, $L_{\mathrm{FT}} \in \mathbb{R}^{n\times k}$ and $\tilde{P} \in \mathbb{R}^{k\times k}$ are matrices of observer gains, to be determined. The error equation of the state observer as follows

$$\begin{aligned}\dot{e} &= \Phi\left(A\Phi^{-1}e + \Phi^{-1}[D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right)L_{\mathrm{FT}} - F]\tilde{C}e\right) \\ &= \left(\tilde{A} + D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right)L_{\mathrm{FT}}\tilde{C}\right)e \end{aligned}\tag{27}$$

where $e = \Phi(x - \hat{x})$, $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{C} \in \mathbb{R}^{k\times n}$ are defined in Lemma 1. Introduce the following definitions

$$H_r = \operatorname{diag}(r_1I_{n_1}, r_2I_{n_2}, ..., r_mI_{n_m}) \in \mathbb{R}^{n\times n},$$
$$r = \left[1 + \frac{\mu}{1+(m-1)\mu}\right]\mathbf{1}_m - \tilde{r}, \Xi(\lambda) = \lambda\left(D_{\tilde{r}}\left(\lambda^{-1}\right) - I_n\right),$$

where $\mathbf{1}_k$ denotes a column vector with all elements being 1. First, consider the case when $\dot{\hat{\theta}} = 0$, then the following result can be obtained.

Theorem 1. Let for some $\mu \in (0, 1]$, $\alpha > 0$, $\zeta > 0$ and $\tau \geqslant 1$, if the matrix inequalities

$$\begin{bmatrix} \bar{A} + \zeta P + \alpha(PH_r + H_rP) & P \\ P & -\zeta Z \end{bmatrix} \leqslant 0, \tag{28a}$$
$$P > 0, Z > 0, X > 0, \tag{28b}$$
$$\begin{bmatrix} \tau X & Y^T \\ Y & P \end{bmatrix} \geqslant 0, \tag{28c}$$
$$P \geqslant \tilde{C}^TX\tilde{C}, \tag{28d}$$
$$PH_r + H_rP > 0 \tag{28e}$$
$$\Xi(\lambda)Z\Xi(\lambda) \leqslant \frac{1}{\tau}P, \; \forall\lambda \in [0, 1], \tag{28f}$$

is feasible for some $P, Z \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ and $X \in \mathbb{R}^{k\times k}$, where $\bar{A} = P\tilde{A} + \tilde{A}^TP + \tilde{C}^TY^T + Y\tilde{C}$. Then the error equation (27) with $L_{\mathrm{FT}} = P^{-1}Y$, $\tilde{P} = X^{1/2}$ is globally finite-time stable with settling time $T \leqslant \frac{V^\rho(e(0))}{\alpha\rho}$, $\rho = \frac{\mu}{1+(m-1)\mu}$. Furthermore, the output ξ of the derivative estimator consisting of (16), (22), and (23) converges to a $O(a^2)$-neighbourhood of the derivative in finite-time.

Next, consider the case that $\dot{\hat{\theta}} \neq 0$ or there exists a $\mathcal{L}_\infty$ disturbance $d_x(t)$, the error equation of the state observer as follows

$$\dot{e} = \left(\tilde{A} + D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right)L_{\mathrm{FT}}\tilde{C}\right)e + d \tag{29}$$

where $d = \Phi d_x(t)$. Then the following result can be obtained.

Theorem 2. Let for some $\mu \in (0, 1)$, $\alpha > 0$, $\zeta > 0$ and $\tau \geqslant 1$, if the matrix inequalities

$$\begin{bmatrix} \frac{1}{2}\bar{A} + \zeta P + \alpha(PH_r + H_rP) & P \\ P & -\zeta Z \end{bmatrix} \leqslant 0, \tag{30a}$$

with (28b-28f) is feasible for some $P, Z \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ and $X \in \mathbb{R}^{k\times k}$. Let $L_{\mathrm{FT}} = P^{-1}Y$, $\tilde{P} = X^{1/2}$, and suppose there exists $\varepsilon > \frac{1}{2\alpha}$ such that

$$\tilde{A}P^{-1} + P^{-1}\tilde{A}^T + L_{\mathrm{FT}}\tilde{C}P^{-1} + P^{-1}\tilde{C}^TL_{\mathrm{FT}}^T + 2\varepsilon I_n \leqslant 0, \tag{31}$$

then there exist a finite-time class $\mathcal{KL}$ function $\bar{\beta}$ and a class $\mathcal{K}$ function γ, such that for all $e_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, the solution of the error equation (29) satisfy

$$\|e(t)\| \leqslant \bar{\beta}(\|e_0\|, t) + \gamma(\|d\|_{[0,\infty)}) \; \forall t \geqslant 0, \tag{32}$$

i.e. the state observer (22) is FT-ISS w.r.t the $\mathcal{L}_\infty$ disturbance $d_x(t)$. Furthermore, the output of the derivative estimator consisting of (16), (22), and (23) converges to a $O(a^2) + \gamma(\|d\|_{[0,\infty)})$-neighbourhood of the derivative in finite-time.

4.2 Fixed-Time Gradient Estimator

Let's design the fixed-time derivative estimator as follows. Consider the following system

$$\dot{\hat{x}}(t) = A\hat{x}(t) - g_{\mathrm{FX}}(y(t) - C\hat{x}(t)), \tag{33}$$
$$\xi = C^1\hat{x}, \tag{34}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector and C^1 can be obtained from (19). The function $g_{\mathrm{FX}} : \mathbb{R}^k \to \mathbb{R}^n$ is defined as

$$\begin{aligned} g_{\mathrm{FX}}(\sigma) := \Phi^{-1}\Big\{\frac{1}{2}\Big[&D_{\tilde{r}}(\|\tilde{P}_1C_0^{-1}\sigma\|^{-1}) \\ &+ D_{\tilde{r}}(\|\tilde{P}_2C_0^{-1}\sigma\|^{-1})\Big]L_{\mathrm{FX}} - F\Big\}C_0^{-1}\sigma, \end{aligned}\tag{35}$$

where $\sigma \in \mathbb{R}^k$, the matrices $\Phi \in \mathbb{R}^{n\times n}$, $C_0 \in \mathbb{R}^{k\times k}$ and $F \in \mathbb{R}^{n\times k}$ are defined in Lemma 1. $L_{\mathrm{FX}} \in \mathbb{R}^{n\times k}$ and $\tilde{P}_i \in \mathbb{R}^{k\times k}, i = 1, 2$ are matrices of observer gains, to be determined. The error equation of the state observer as follows,

$$\dot{e} = \left\{\tilde{A} + \frac{1}{2}\left[D_{\tilde{r}}\left(\frac{1}{\|\tilde{P}_1\tilde{C}e\|}\right) + D_{\tilde{r}}\left(\frac{1}{\|\tilde{P}_2\tilde{C}e\|}\right)\right]L_{\mathrm{FX}}\tilde{C}\right\}e, \tag{36}$$

where $e = \Phi(x - \hat{x})$, $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{C} \in \mathbb{R}^{k\times n}$ are defined in Lemma 1. Introduce the following definitions

$$\Xi_i^{\delta}(\lambda) = \frac{\lambda_1}{2}\left\{D_{\tilde{r}}\left(\frac{\lambda_2^{i-1}}{\delta_1\lambda_1}\right) + D_{\tilde{r}}\left(\frac{\delta_2\lambda_1}{\lambda_2^{i-2}}\right) - 2I_n\right\},$$
$$\delta = (\delta_1, \delta_2),\ \lambda = (\lambda_1, \lambda_2),$$
$$r_i = (-1)^i\tilde{r} + \left[1 + \frac{(-1)^{i+1}\mu}{1+(m-1)\mu}\right]\mathbf{1}_m,$$
$$H_i = \mathrm{diag}\left\{(r_i)_1 I_{n_1}, (r_i)_2 I_{n_2}, ..., (r_i)_m I_{n_m}\right\},\ \text{for } i = 1, 2.$$

Consider the case when $\dot{\hat{\theta}} = 0$, then the following result can be obtained.

Theorem 3. Let for some $\mu \in (0, 1]$, $\alpha > 0$, $\zeta > 0$, $\tau \geqslant 1$ and $\delta = (\delta_1, \delta_2), \delta_i > 0, i = 1, 2$, if the matrix inequalities

$$\begin{bmatrix} \bar{A} + \zeta P + \alpha(PH_r + H_rP) & P \\ P & -\zeta Z_i \end{bmatrix} \leqslant 0, \tag{37a}$$
$$\begin{bmatrix} \tau X & Y^T \\ Y & P \end{bmatrix} \geqslant 0, \tag{37b}$$
$$P > 0, Z_i > 0, X > 0, \tag{37c}$$
$$PH_i + H_iP > 0 \tag{37d}$$
$$P \geqslant \tilde{C}^T X \tilde{C}, \tag{37e}$$
$$\Xi_i^{\delta}(\lambda) Z_i \Xi_i^{\delta}(\lambda) \leqslant \frac{1}{\tau}P,\ \forall\lambda \in [0, 1]\times[0, 1], \tag{37f}$$

is feasible for some $P, Z_1, Z_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ and $X \in \mathbb{R}^{k\times k}$. Then the error equation (36) with $L_{\mathrm{FX}} = P^{-1}Y$, $\tilde{P}_i = \delta_i X^{1/2}$ is globally fixed-time stable with settling time $T_{\max} \leqslant \frac{1+(m-1)\mu}{0.5\alpha\mu}$. Furthermore, the output of the derivative estimator consisting of (16), (33), and (34) converges to a $O(a^2)$-neighbourhood of the derivative in fixed-time.

Consider the case that $\dot{\hat{\theta}} \neq 0$ or there exists a $\mathcal{L}_\infty$ disturbance $d_x(t)$, the error equation as follows

$$\dot{e} = \left\{\tilde{A} + \frac{1}{2}\left[D_{\tilde{r}}\left(\frac{1}{\|\tilde{P}_1\tilde{C}e\|}\right) + D_{\tilde{r}}\left(\frac{1}{\|\tilde{P}_2\tilde{C}e\|}\right)\right]L_{\mathrm{FX}}\tilde{C}\right\}e + d, \tag{38}$$

where $d = \Phi d_x(t)$. Then, the following result can be obtained for the derivative estimator (16), (33), and (34).

Theorem 4. Let for some $\mu \in (0, 1)$, $\alpha > 0$, $\zeta > 0$, $\tau \geqslant 1$ and $\delta = (\delta_1, \delta_2), \delta_i > 0, i = 1, 2$, if the matrix inequalities

$$\begin{bmatrix} \frac{1}{2}\bar{A} + \zeta P + \alpha(PH_r + H_rP) & P \\ P & -\zeta Z_i \end{bmatrix} \leqslant 0, \tag{39a}$$

with (37b-37f) is feasible for some $P, Z_1, Z_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ and $X \in \mathbb{R}^{k\times k}$. Let $L_{\mathrm{FX}} = P^{-1}Y$, $\tilde{P}_i = \delta_i X^{1/2}$, and suppose there exists $\varepsilon > \frac{1}{2\alpha}$ such that

$$\tilde{A}P^{-1} + P^{-1}\tilde{A}^T + L_{\mathrm{FX}}\tilde{C}P^{-1} + P^{-1}\tilde{C}^T L_{\mathrm{FX}}^T + 2\varepsilon I_n \leqslant 0. \tag{40}$$

Then, there exist a fixed-time class $\mathcal{KL}$ function $\bar{\beta}$ and a class $\mathcal{K}$ function γ, such that for all $e_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, the solution of the error equation (38) satisfy

$$\|e(t)\| \leqslant \bar{\beta}(\|e_0\|, t) + \gamma(\|d\|_{[0,\infty)})\ \forall t \geqslant 0, \tag{41}$$

i.e. the state observer (34) is FX-ISS w.r.t the $\mathcal{L}_\infty$ disturbance $d_x(t)$. Furthermore, the output of the derivative estimator consisting of (16), (33), and (34) converges to a $O(a^2) + \gamma(\|d\|_{[0,\infty)})$-neighbourhood of the derivative in fixed-time.

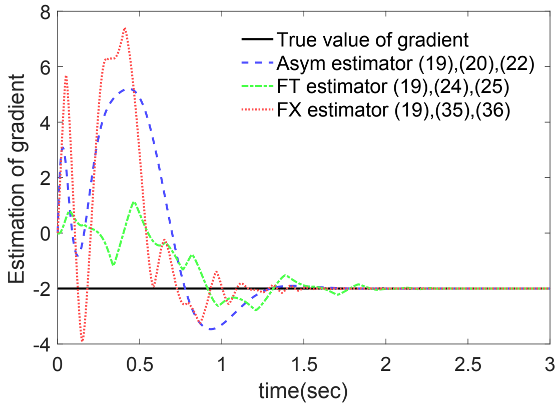

Fig. 2. Estimations of derivative by asymptotic, finite-time, and fixed-time derivative estimators: $\hat{\theta} = 2$

5. EXAMPLES

In order to verify the effectiveness of the derivative estimators proposed in this paper, the simulation results of the derivative estimators are given in this section. In addition, the simulation results of the derivative estimators when $\dot{\hat{\theta}} \neq 0$ are also presented.

To simplify the simulation, let $n = 2$, then (5) can be represented as

$$\begin{aligned} y &= v(\hat{\theta}) + v'(\hat{\theta})a\sin\omega t + \frac{1}{2}v''(\hat{\theta})a^2\sin^2\omega t + O(a^3) \\ &= v(\hat{\theta}) + \frac{a^2 v''(\hat{\theta})}{4} + av'(\hat{\theta})\sin\omega t - \frac{a^2 v''(\hat{\theta})}{4}\cos 2\omega t + O(a^3) \end{aligned} \tag{42}$$

First, consider the case that $\hat{\theta}$ is a constant and the higher order term $O(a^3)$ can be ignored, then (42) can be represented by the following linear system

$$\dot{x} = Ax = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & -2\omega & 0 \end{bmatrix} x, \tag{43}$$
$$y = Cx = \begin{bmatrix} 1 & a & 0 & 0 & -a^2/4 \end{bmatrix} x. \tag{44}$$

Based on the design methods in Section 3 and 4, choosing $C^1 = [\,0\ \sin\omega t\ \cos\omega t\ 0\ 0\,]$, the asymptotical, finite-time, and fixed-time derivative estimators are designed. For asymptotic derivative estimator, the vector $L \in \mathbb{R}^{2n+1}$ is chosen as $[\,22\ 126.5\ 57.2\ 1090.6\ 465.2\,]$, which makes $A - LC$ is Hurwitz and the poles be -6, -10, -10.1, -15, -15.1, respectively. The parameters of finite-time derivative estimator are chosen as $\mu = 0.3$, $\alpha = 0.5$, $\tau = 5$, and $\zeta = 0.1$. The parameters of finite-time derivative estimator are chosen as $\mu = 0.3$, $\alpha = 0.5$, $\delta_1 = 2.5$, $\delta_2 = 0.2$, $\tau = 5$, $\beta = 8$, and $\zeta = 0.1$. In order to complete the simulation, we select the function as $y = v(\theta) = -(\theta - 1)^2 + 5$ and $a = 0.5$, $\omega = 2\pi$. Then, the simulation results, when $\hat{\theta} = 2$, are shown as Fig.2.

The Fig.2 shows that all the derivative estimators proposed in this paper can accurately estimate the derivative information of the unknown function when there is no disturbance input. Although different derivative estima-

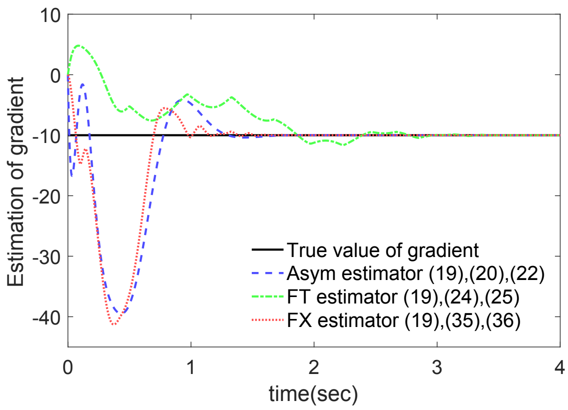

Fig. 3. Estimations of derivative by asymptotic, finite-time, and fixed-time derivative estimators: $\hat{\theta} = 6$

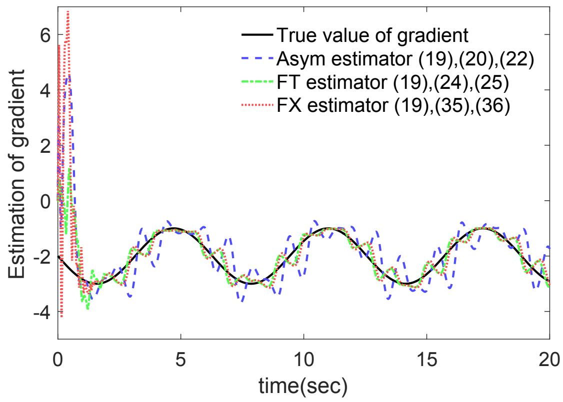

Fig. 4. Estimations of derivative by asymptotic, finite-time, and fixed-time derivative estimators: $\hat{\theta} = 2 + 0.5 \sin t$

tors have different fluctuation ranges in the convergence process.

In order to verify the advantage in convergence time of fixed-time derivative estimator compared to the other two derivative estimators with different initial conditions. Let $\hat{\theta} = 6$, the simulation results are shown in Fig.3. Compared with Fig.2, it can be seen that the convergence time of fixed-time derivative estimator is not sensitive to changes of initial conditions relative to the other two estimators.

Using the same parameters of the above derivative estimators, just replace $\hat{\theta}$ as $\hat{\theta} = 2 + 0.5 \sin t$, then the simulation results are shown as Fig.4. It can be seen from the Fig.4 that the steady-state performance of the FT and FX derivative estimators are significantly better than the asymptotic one when there is a bounded disturbance input. This is because the FT-ISS and FX-ISS characteristics of the FT and FX derivative estimators.

6. CONCLUSION AND FUTURE WORK

This paper proposed an asymptotic derivative estimator for unknown maps and the systematic design method was also presented. Based on it, the FT and FX derivative estimators were designed. For the case that there exists a bounded disturbance input, the sufficient conditions of FT-ISS and FX-ISS for the FT and FX derivative estimators were provided. However, only the derivative of static map are studied in the paper. It should be more valuable to estimate the derivative of function with dynamic. We will focus on this topic in future.

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