

Cyclo-Dissipativity and Thermodynamics

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Abstract: The dissipativity formulation of both the First and Second Law of thermodynamics involve the notion of cyclo-dissipativity. This motivates to revisit the, rather scarce, literature on cyclo-dissipativity. It turns out that by 'symmetrizing' the basic definitions of dissipativity theory as introduced in the seminal 1972 paper of Jan Willems some novel results can be obtained which have direct consequences for cyclo-dissipativity. A related contribution, also motivated by thermodynamics, is the notion of one-port cyclo-passivity, which provides a new angle to the Second Law of thermodynamics.

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1. INTRODUCTION

Consider a macroscopic thermodynamic system. There are two external ports¹. The first port is the *mechanical* one, with port variables being the pressure² $-P$ and the rate of volume change $u_V := \dot{V}$, where V is the volume. The instantaneous power exerted by the environment on the thermodynamic system is thus given by

$$-Pu_V = \text{rate of mechanical work} \quad (1)$$

Second port is the *thermal port*, where the thermodynamic system is connected to a heat source, with port variables the temperature T and the heat flow q from the heat source into the system.

The *First Law of thermodynamics* is expressed by assuming the existence of a function E of the state x of the thermodynamic system, satisfying along all the trajectories of the thermodynamic system

$$\begin{aligned} E(x(t_2)) - E(x(t_1)) &= \int_{t_1}^{t_2} q(t) - P(t)u_V(t) dt \\ &\left(= \int_{t_1}^{t_2} q(t) dt - \int_{t_1}^{t_2} P(t) dV(t) \right) \end{aligned} \quad (2)$$

for all $t_1 \leq t_2$. The function E is the *total energy* of the thermodynamic system, and (2) expresses that the increase of the total energy E is equal to the incoming heat flow (through the thermal port) minus the mechanical work performed by the system on the environment (through the mechanical/hydraulic port). Often the energy function E is bounded from below, in which case it can be turned into a nonnegative storage function by adding a

¹ The discussion can be easily extended to cases where the mechanical port is replaced, or extended, by other types of ports; e.g., chemical, electrical.

² In order to stick with the usual notation in thermodynamics we follow the physics convention, where Pu_V is the mechanical work exerted by the thermodynamic system on the environment. Instead, in the system-theoretic sign convention the mechanical work exerted by the environment on the thermodynamic system is considered.

suitable constant. Then the First Law of thermodynamics can be equivalently expressed as *losslessness* with respect to the supply rate

$$s(q, P, u_V) = q - Pu_V, \quad (3)$$

with storage function E . On the other hand, for a general thermodynamic system there is no reason why E should be bounded from below. This calls for the extension to *cyclo-losslessness*, allowing for *indefinite* energy functions.

The dissipativity interpretation of the *Second Law of thermodynamics* is less clear. The standard argumentation in classical thermodynamics, see e.g. Fermi (1936), is to derive, by using the *Carnot cycle*, the inequality

$$\oint \frac{q(t)}{T(t)} dt \leq 0 \quad (4)$$

for all cyclic processes, where equality holds for so-called *reversible* cyclic processes. Furthermore, based on this, one defines the *entropy* S as a function of the state of the thermodynamic system, and derives the *Clausius inequality*

$$S(x(t_2)) - S(x(t_1)) \geq \int_{t_1}^{t_2} \frac{q(t)}{T(t)} dt. \quad (5)$$

This leads to the dissipativity formulation of the Second Law as given in Willems (1972), see also Haddad (2019). Indeed, if the entropy S can be assumed to be bounded from above, then it follows that the thermodynamic system is *dissipative* with respect to the supply rate $-\frac{q}{T}$ and storage function $-S$. However, similarly to the previous dissipativity interpretation of the First Law, there is no reason why in general the entropy is bounded from above, and thus again we have to take recourse to *cyclo-dissipativity*. It should be also noted that the standard definition of the entropy function is based on the assumption Fermi (1936); Kondepudi & Prigogine (2015) that any state can be reached from a certain ground-state using *reversible* transformations; i.e., transformations where (4) is satisfied with equality. From the point of view of irre-

versible thermodynamics this is a restrictive assumption Kondepudi & Prigogine (2015).

The theory of dissipative systems in systems and control theory originates from the seminal paper Willems (1972). The notion of cyclo-dissipativity was first explicitly formulated in Willems (1973); although implicitly the notion was already present in Willems (1971). In the technical report Hill & Moylan (1975) cyclo-dissipativity was further explored, extending the fundamental results for ordinary dissipativity obtained in Willems (1972). Since then the notion of cyclo-dissipativity has not received much detailed attention, although the concept regularly appears in passivity-based control (e.g. Ortega et al. (2008)) and stability analysis of interconnected systems Moylan (2014). In Section 2 we will revisit the notions of dissipativity and cyclo-dissipativity, by unifying earlier developments and thus arriving at some new results, including an external characterization of cyclo-dissipativity and a characterization of the set of (indefinite) storage functions. More details, proofs and extensions can be found in van der Schaft (2020). Direct applications to thermodynamics will be formulated at the end of Section 2. Another application to thermodynamic (as well as other physical) systems concerns the notion of one-port cyclo-passivity which will be introduced in Section 3; see van der Schaft & Jeltsema (2020) for a full treatment.

2. (CYCLO-)DISSIPATIVITY REVISITED

Consider a nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & x \in \mathcal{X}, u \in \mathbb{R}^m \\ y = h(x, u), & y \in \mathbb{R}^p \end{cases} \quad (6)$$

for some n -dimensional state space manifold \mathcal{X} . Consider furthermore a *supply rate*

$$s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \quad (7)$$

Throughout it will be assumed that for all solutions of Σ finite integrals $\int_{t_1}^{t_2} s(u(t), y(t)) dt$ are well-defined for all t_1, t_2 .

A, possibly extended, function³ $S : \mathcal{X} \rightarrow -\infty \cup \mathbb{R} \cup \infty$ satisfies the *dissipation inequality*

$$S(x(t_2)) \leq S(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t)) dt, \quad (8)$$

if (8) holds for all $t_1 \leq t_2$, all input functions $u : [t_1, t_2] \rightarrow \mathbb{R}^m$, and all initial conditions $x(t_1)$, where $y(t) = h(x(t), u(t))$, with $x(t)$ denoting the solution of $\dot{x} = f(x, u)$ for initial condition $x(t_1)$ and input function $u : [t_1, t_2] \rightarrow \mathbb{R}^m$. In particular this implies that if the left-hand side of (8) equals ∞ , then so does the right-hand side, and if the right-hand side of (8) equals $-\infty$, then so does the other side.

A *non-extended* function $S : \mathcal{X} \rightarrow \mathbb{R}$ is called a *storage function*⁴ for system Σ with supply rate s if it satisfies (8). This leads to the following standard definition of

³ In this section we will follow the notation from (cyclo-)dissipativity theory denoting storage functions by S ; not to be confused with the entropy function!

⁴ Note that we do not yet require S to be nonnegative or bounded from below.

dissipativity as pioneered in the seminal paper Willems (1972); see also Hill & Moylan (1980).

Definition 2.1. The system Σ is *dissipative* with respect to the supply rate s if there exists a *nonnegative* storage function S . If the nonnegative storage function S satisfies (8) with equality then the system is called *lossless*.

In case of the supply rate $s(u, y) = y^T u$, $u, y \in \mathbb{R}^m$ 'dissipativity' is usually referred to as 'passivity'.

In order to characterize dissipativity, and subsequently the weaker property of *cyclo-dissipativity*, let us define the following, possibly extended, functions⁵ $S_a : \mathcal{X} \rightarrow \mathbb{R} \cup \infty$ and $S_r : \mathcal{X} \rightarrow -\infty \cup \mathbb{R}$

$$\begin{aligned} S_a(x) &= \sup_{u, T \geq 0 | x(0)=x} - \int_0^T s(u(t), y(t)) dt \\ S_r(x) &= \inf_{u, T \geq 0 | x(0)=x} \int_{-T}^0 s(u(t), y(t)) dt \end{aligned} \quad (9)$$

Obviously S_a, S_r satisfy

$$S_a(x) \geq 0, \quad S_r(x) \leq 0 \quad (10)$$

Furthermore, assuming *reachability* from x^* and *controllability* to x^* , we define the, possibly extended, functions $S_{ac}, S_{rc} : \mathcal{X} \rightarrow \mathbb{R} \cup \infty, S_{rc} : \mathcal{X} \rightarrow -\infty \cup \mathbb{R}$ as⁶

$$\begin{aligned} S_{ac}(x) &= \sup_{u, T \geq 0 | x(0)=x, x(T)=x^*} - \int_0^T s(u(t), y(t)) dt \\ S_{rc}(x) &= \inf_{u, T \geq 0 | x(-T)=x^*, x(0)=x} \int_{-T}^0 s(u(t), y(t)) dt \end{aligned} \quad (11)$$

Clearly for all $x \in \mathcal{X}$

$$\begin{aligned} -\infty < S_{ac}(x) \leq S_a(x) \\ S_r(x) \leq S_{rc}(x) < \infty \end{aligned} \quad (12)$$

Furthermore, it is straightforward⁷ to check that the above four functions are related by

$$\begin{aligned} S_a(x^*) &= \sup_x -S_{rc}(x) (= \\ \sup_x \sup_{u, T \geq 0 | x(-T)=x^*, x(0)=x} &- \int_{-T}^0 s(u(t), y(t)) dt) \end{aligned} \quad (13)$$

$$\begin{aligned} S_r(x^*) &= \inf_x -S_{ac}(x) (= \\ \inf_x \inf_{u, T \geq 0 | x(0)=x, x(T)=x^*} &\int_0^T s(u(t), y(t)) dt) \end{aligned}$$

In particular it follows that

$$\begin{aligned} S_a(x^*) < \infty &\Leftrightarrow \inf_x S_{rc}(x) > -\infty \\ S_r(x^*) > -\infty &\Leftrightarrow \sup_x S_{ac}(x) < \infty \end{aligned} \quad (14)$$

By using the definitions of infimum and supremum it is easily verified (see Willems (1971, 1972); van der Schaft

⁵ Here a refers to 'available storage', and r to 'required storage'. Note that we deviate from the standard notation, where S_r refers to the function S_{rc} as defined next. In fact, in the standard treatments of dissipativity Willems (1972, 1971); Hill & Moylan (1975, 1980); van der Schaft (2017) only the functions S_a and S_{rc} (there denoted as S_r) are used. The present set-up aims at 'symmetrizing' the picture; also with a view on cyclo-dissipativity.

⁶ Here c stands for 'constrained', since either $x(T) = x^*$ or $x(-T) = x^*$.

⁷ The first equality already figures in van der Schaft (2017); the second one is similar.

(2017)) that all four functions S_a, S_r, S_{ac}, S_{rc} satisfy the dissipation inequality (8).

The following theorem summarizes some of the main findings of dissipativity theory as formulated in van der Schaft (2017), extending the fundamental results of Willems (1972); see also Hill & Moylan (1980).

Theorem 1. Σ is dissipative if and only if $S_a(x) < \infty$ for all $x \in \mathcal{X}$ (that is, $S_a : \mathcal{X} \rightarrow \mathbb{R}$). If Σ is reachable from x^* then Σ is dissipative if and only if $S_a(x^*) < \infty$, or equivalently (see (14)) $\inf_x S_{rc}(x) > -\infty$. Furthermore, if Σ is dissipative then S_a is a nonnegative storage function satisfying $\inf_x S_a(x) = 0$, and all other nonnegative storage functions S satisfy

$$S_a(x) \leq S(x) - \inf_x S(x), \quad x \in \mathcal{X} \quad (15)$$

Moreover, if Σ is dissipative then $S_{rc}(x) - \inf_x S_{rc}(x)$ is a nonnegative storage function, and all other nonnegative storage functions S satisfy

$$S(x) - S(x^*) \leq S_{rc}(x) \quad (16)$$

Remark 2.2. In case the system is reachable from x^* and $S_a(x^*) = 0$ dissipativity is equivalent to the *external characterization*

$$\int_0^T s(u(t), y(t)) dt \geq 0, \quad (17)$$

for all $u : [0, T] \rightarrow \mathbb{R}^m, T \geq 0$, where $y(t) = h(x(t), u(t))$ with $x(t)$ the solution of $\dot{x} = f(x, u)$ for initial condition $x(0) = x^*$.

Similar results can be derived for the newly defined functions S_r and S_{ac} ; however with the key difference that $S_r \leq 0$, and thus does *not* correspond to dissipativity. More details are given in van der Schaft (2020).

Next we come to the study of *cyclo-dissipativity*, as coined in Willems (1973) and explored in Hill & Moylan (1975); see already Willems (1971) for implicit statements.

Definition 2.3. Σ is *cyclo-dissipative* if

$$\oint s(u(t), y(t)) dt \geq 0 \quad (18)$$

for all $T \geq 0$ and all $u : [0, T] \rightarrow \mathbb{R}^m$ such that $x(T) = x(0)$. Assume furthermore that Σ is reachable from x^* and controllable to x^* . Then Σ is called *cyclo-dissipative with respect to x^** if

$$\oint s(u(t), y(t)) dt \geq 0 \quad (19)$$

for all $T \geq 0$ and all $u : [0, T] \rightarrow \mathbb{R}^m$ such that $x(T) = x(0) = x^*$. In case (18) or (19) holds with equality, then we speak about cyclo-losslessness.

The following proposition is obvious (substitute $x(T) = x(0)$ in (8)).

Proposition 2. If there exists a storage function for the system Σ then Σ is cyclo-dissipative.

The following theorem extends the results in Hill & Moylan (1975) in a number of directions; see van der Schaft (2020) for the proof.

Theorem 3. Assume again that Σ is reachable from x^* and controllable to x^* . Then Σ is cyclo-dissipative with respect to x^* if and only if

$$S_{ac}(x) \leq S_{rc}(x), \quad x \in \mathcal{X} \quad (20)$$

In particular, if Σ is cyclo-dissipative with respect to x^* then both S_{ac} and S_{rc} are storage functions, and thus Σ is cyclo-dissipative. Furthermore, if Σ is cyclo-dissipative with respect to x^* then

$$S_{ac}(x^*) = S_{rc}(x^*) = 0, \quad (21)$$

and any other storage function S satisfies

$$S_{ac}(x) \leq S(x) - S(x^*) \leq S_{rc}(x) \quad (22)$$

The most interesting application of the above results to thermodynamics concerns the definition of the entropy function. Indeed, assuming reachability from and controllability to a certain ground state x^* it follows from Theorem 3 that (4) implies the existence of storage functions S satisfying $S_{ac}(x) \leq S(x) - S(x^*) \leq S_{rc}(x)$. Hence for any such S the function $\tilde{S}(x) := -S(x)$ qualifies as a possible *entropy function* satisfying the Clausius inequality (5).

At the same time this leads to the question of uniqueness of the entropy function. For example, a 'maximal' entropy function could be defined as $\tilde{S}(x) := -S_{ac}(x)$.

3. ONE-PORT CYCLO-PASSIVITY

Thermodynamic systems generally have two ports, with one port being the thermal port, and the other e.g. the mechanical port. One of the consequences of the Second Law of thermodynamics is that if the temperature at the thermal port is kept constant no energy can be transferred from the thermal port to the mechanical port; at least not in a repeatable manner. This leads to the general notion of *one-port (cyclo-)passivity* defined as follows, cf. van der Schaft & Jeltsema (2020).

Definition 3.1. Consider a system with two ports u_1, y_1 and u_2, y_2 , which is cyclo-passive with storage function S ; i.e., along every solution

$$\frac{d}{dt} S(x(t)) \leq y_1^T(t) u_1(t) + y_2^T(t) u_2(t) \quad (23)$$

Then the system is *one-port cyclo-passive* (with respect to the port u_2, y_2) if for every constant \bar{y}_1 there exists F such that for all trajectories for which $y_1(t) = \bar{y}_1$

$$\frac{d}{dt} F(x(t)) \leq y_2^T(t) u_2(t) \quad (24)$$

Thermodynamic systems are one-port cyclo-passive with respect to the mechanical port; as a consequence of the Second Law. In fact, the storage function F in this case is given by the *Helmholtz energy*. A problem of current investigation is to what extent one-port cyclo-passivity in the thermodynamic case is equivalent to the Second Law of thermodynamics.

On the other hand, many other physical systems with multiple ports share this property of one-port cyclo-dissipativity; see van der Schaft & Jeltsema (2020).

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