High-Order State-Derivative Controller Design for Nonlinear Systems

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Abstract: A novel controller design for nonlinear systems based on inducing a set of differential algebraic equations is presented. The method generalises the state-derivative feedback control by employing an arbitrary number of derivatives of the state vector pre-multiplied by nonlinear gains. To determine these gains, an intermediary control law is indirectly synthesised via convex modelling and linear matrix inequalities through an induced singular system subject to the Pantelides algorithm. An example illustrates the effectiveness of the proposal.

Keywords: Differential Algebraic Equations, Linear Matrix Inequalities, Descriptor Systems, Polytopic Representation, Controller Design, Nonlinear Systems.

1. INTRODUCTION

Control laws can be classified according to the number of derivatives or integrals of the state that they include, being the linear feedback the most elementary case (Bass and Gura, 1965). Some control laws include the first time derivative of the state vector in addition to the state feedback (Cardim et al., 2007); pole placement formulae for this case have been developed (Abdelaziz and Valášek, 2004): robust generalisations can be found in Faria et al. (2009).

Feeding back the time derivatives of the state vector may be better described by leaving expressions in the left-hand side of the system equation, i.e., having a descriptor form which might be singular (Lewis, 1986). A control law called proportional-plus-derivative state feedback which includes the time derivative of the state vector has been employed for stabilisation of descriptor linear systems in Duan and Zhang (2003); generalisations for polytopic (Da Silva et al., 2011) and nonlinear systems (González et al., 2017) have appeared. It has been shown in the latter that instead of feeding back the time derivatives of the states it is possible to solve the control law from an algebraic loop.

State derivatives might be obtained from accelerometers as in some practical problems like the car wheel suspension system (Reithmeier and Leitmann, 2003), the suppression of vibration in mechanical systems (Abdelaziz and Valášek, 2004), and the control of bridge cable vibration (Duan et al., 2005). Alternatively, state derivatives can be obtained with an arbitrary degree of precision, under mild conditions, by employing a Levant’s robust differentiator (Levant, 2003); this is the preferred option for sliding mode methodologies. This work proposes an LMI generalisation of previously appeared state-derivative control laws to any differentiation order. The resulting control law can be applied to any sufficiently smooth nonlinear system with bounded nonlinearities by means of the Levant’s robust differentiator.

Notation as well as a short introduction to convex modelling, singular systems, the Pantelides algorithm, and Levant’s robust differentiator, are presented in section 2; section 3 develops the novel methodology for high-order state-derivative controller design by inducing differential algebraic equations; the effectiveness of the approach is put at test with two nonlinear examples in section 4; finally, conclusions are discussed in section 5.

2. PRELIMINARIES

Nonlinear systems can be subsumed in convex structures that share the characteristics of linear parameter varying (LPV) or Takagi-Sugeno (TS) models: \( \dot{x}(t) = \sum_{i=1}^{r} h_i(x)A_i x(t) + B_i u(t) \), where \( 0 \leq h_i(x) \leq 1, \sum_{i=1}^{r} h_i(x) = 1 \), within a compact set of the state space that includes the origin. This rewriting allows using a quadratic Lyapunov function \( V(x) = x^T P x \) to synthesise a PDC control law \( u(t) = \sum_{j=1}^{r} h_j(x)F_j x(t) \) via LMIs \( X = X^T > 0 \) and \( A_i X + B_i M_j + (\ast) < 0 \), where \( P = X^{-1}, F_j = M_j X^{-1} \) (Tanaka and Wang, 2001); this work follows this outline for a system subject to the Pantelides algorithm.

Feeding back high-order time derivatives of the state introduces singularities in the system. Roughly speaking, quasi-linear singular systems consist of differential algebraic equations which are amenable to the following descriptor form (Arceo et al., 2018):

\[
E(x) \dot{x}(t) = A(x)x(t), \quad \text{det}(E(x)) = 0.
\]

These systems can be reduced to ordinary differential equations (ODEs) by means of the Pantelides algorithm; this is required for many tasks, e.g., simulation; it consists of the following steps:

1. Put the singular system (1) into the dynamic decomposition form (DDF) that splits dynamical and algebraic relationships, i.e.
\[ \dot{x}(t) = A(x)x(t) + B(x)u(t), \]  
\[ (3) \]
where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the input vector, and the pair \((A(x), B(x))\) consists of sufficiently smooth entries which are bounded on a compact set \( \Omega \subset \mathbb{R}^n \) that includes the origin, are considered in this work.

Consider a nonlinear control law of the form:
\[ u(t) = \sum_{i=0}^{q} F_i^r(x) x^{(i)}(t), \]  
\[ (4) \]
which, defining \( \dot{x} = [x^T \; \dot{x}^T \; \cdots \; (x^{(q-1)})^T]^T \), can be further transformed into a \((q \times n)\)-th order descriptor of a possibly nonlinear singular system:
\[ \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & B(x)F^0(x) \\ \end{bmatrix} \begin{bmatrix} \dot{x} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ \vdots \\ 0 & 0 \\ -B(x)F^2(x) & \cdots & -B(x)F^{q-1}(x) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \]  
\[ (5) \]

Most plants have fewer inputs than states, so \( B(x)F^r(x) \) is generally not full rank. Assuming \( \text{rank}(B(x)F^r(x)) = r \) the last block row has \( n-r \) implicit algebraic restrictions and \( r \) dynamics, all of which can be determined via the following orderly adaptation of the Pantelides algorithm (Pantelides, 1988):

(1) Solve for \( \ddot{x}_{\text{qtr}}(n-r+1), \dot{x}_{\text{qtr}}(n-r+2), \ldots, \ddot{x}_{\text{qtr}} \) in terms of the remaining entries of \( \dot{x} \).

(2) Determine \( \dddot{x}_{\text{qtr}}(n-r+1), \dddot{x}_{\text{qtr}}(n-r+2), \ldots, \dddot{x}_{\text{qtr}} \) by taking the time derivatives of the solved states in the previous step and substituting any occurrence of \( \dddot{x}_{\text{qtr}}(n-r+1), \dddot{x}_{\text{qtr}}(n-r+2), \ldots, \dddot{x}_{\text{qtr}} \) by their equivalence found in the previous step.

(3) Substitute the new expressions for \( \dddot{x}_{\text{qtr}}(n-r+1), \dddot{x}_{\text{qtr}}(n-r+2), \ldots, \dddot{x}_{\text{qtr}} \) into the remaining dynamics. Notice that only those from the last block row may require this substitution.

(4) Solve dynamics \( \dddot{x}(q-1)n+1, \dddot{x}(q-1)n+2, \ldots, \dddot{x}(n+r), \) in terms of \( \dot{x}_1, \dot{x}_2, \ldots, \dot{x}(n+r). \)

After performing the previous algorithm the system becomes the following standard state-space representation:
\[ \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{(q-1)n} \\ \dot{x}_{(q-1)n+1} \\ \ddots \\ \dot{x}_{(n+r)} \\ \end{bmatrix} = \begin{bmatrix} \dddot{x}_{n+1} \\ \vdots \\ \dddot{x}_{2} \\ 0 \\ \vdots \\ g_1(x_1, \ldots, \dot{x}_{(q-1)n+r}) \\ \end{bmatrix}, \]  
\[ (6) \]
where \( f_i(\cdot), i \in \{1, \ldots, n+r\} \), are possibly nonlinear functions of the states, \( g_i(\cdot), i \in \{1, \ldots, n+r\} \), are possibly nonlinear functions of the states and gain entries \( F_{jk}(x), i \in \{0, 1, \ldots, q\}, j \in \{1, \ldots, m\}, k \in \{1, \ldots, n\} \).

In order to determine gains \( F_i(x), i \in \{0, 1, \ldots, q\} \), such that the origin is asymptotically stable, (6) should be written as follows:
\[ \dot{x}(t) = \begin{bmatrix} 0_{(q-2)n} & f_0(x) \\ \tilde{A}_2^3(x) & B(x) \end{bmatrix} \tilde{x}(t) + 0_{(q-2)n}, \]  
\[ (7) \]
Asymptotically stable. As a final step, gains $F^i_{jk}(\bar{x})$ of the actual control law will be obtained from the equivalence $\hat{K}(\bar{x}) = \hat{F}(\bar{x}, F^i_{jk}(\bar{x}))$.

An equivalent polytopic representation of (7) within $\Omega$ can be systematically constructed via the sector nonlinearity approach in González et al. (2017):

1. Create a list of different bounded non-constant terms $z_i(\bar{x}) \in [a_i, z_i^2]$, $i \in \{1, 2, \ldots, p\}$, in $\hat{A}_1(\bar{x})$, $\hat{A}_2(\bar{x})$, $\hat{A}_3(\bar{x})$, and $\hat{B}(\bar{x})$, by taking into account similar terms in different entries 1.

2. Write each $z_i(\bar{x})$ as a convex sum of its bounds

$$z_i(\bar{x}) = \frac{z_i^1 - z_i(\bar{x})}{z_i^1 - z_i^2} z_i^0 + \frac{z_i(\bar{x}) - z_i^0}{z_i^1 - z_i^0} z_i^1,$$

with $w_i^0(\bar{x}) + w_i^1(\bar{x}) = 1$, $w_i^0(\bar{x}), w_i^1(\bar{x}) \in [0, 1]$, $i \in \{1, 2, \ldots, p\}$ for every $\bar{x} \in \Omega$.

3. Write the exact polytopic representation of (7) as:

$$\dot{\hat{x}}(t) = \sum_{i \in \mathcal{B}^p} \mathbf{w}_i(\hat{x})(\begin{bmatrix} \left[0_{(q-2)\text{non}} I_{(q-2)\text{non}} 0_{(q-2)\text{non}} \right] A_1 \hat{A}_1^2 \hat{A}_1^2 A_1^3 \\
\end{bmatrix} \hat{x}(t) + \left[0_{(q-2)\text{non}} \right] B_1 \right)$$

where $\mathcal{B} = \{0, 1\}$, $i = (i_1, i_2, \ldots, i_p) \in \mathcal{B}^p$, $\mathbf{w}_i(\hat{x}) = w_i^0(\bar{x}) w_i^0(\hat{x}) \cdots w_i^0(\hat{x})$,

$\hat{A}_1 = \hat{A}_1(\bar{x})|_{\mathbf{w}_i(\hat{x})=1}$,

$\hat{A}_1^2 = \hat{A}_1^2(\bar{x})|_{\mathbf{w}_i(\hat{x})=1}$,

$\hat{A}_1^3 = \hat{A}_1^3|_{\mathbf{w}_i(\hat{x})=1}$, and $B_1 = B(\bar{x})|_{\mathbf{w}_i(\hat{x})=1}$.

Consider the following fictitious PDC-like control law:

$$\hat{u}(t) = \hat{K}(\bar{x})\hat{x}(t) = \sum_{i \in \mathcal{B}^p} \mathbf{w}_i(\hat{x})K_i\hat{x}.$$  \hspace{1cm} (9)

**Theorem 1.** The origin of the system (7) with equivalent polytopic representation (8) within $\Omega$, under the control law (9) is asymptotically stable if $\exists X \in \mathbb{R}^{(q-1)+np+(q-1)+np}$, $M_j \in \mathbb{R}^{(n+(q-1)+np) \times (n+(q-1)+np)}$, $j \in \mathcal{B}^p$, of suitable dimensions, such that the LMIs $X = X^T > 0$ and

$$\sum_{(i,j) \in P(K, l)} \left[ \begin{bmatrix} 0_{(q-2)\text{non}} I_{(q-2)\text{non}} 0_{(q-2)\text{non}} \end{bmatrix} \hat{A}_1 \hat{A}_1^2 \hat{A}_1^2 A_1^3 \\
\end{bmatrix} + \left[0_{(q-2)\text{non}} \right] M_j + (\ast) \right] < 0, \quad \text{for all } i, j \in \mathcal{B}^p,$$  \hspace{1cm} (10)

hold for all $l, k \in \mathcal{B}^p$, with $P(K, l)$ being the set of indexes $(i, j)$ of all products $\mathbf{w}_i \mathbf{w}_j$ which are algebraically similar to $\mathbf{w}_k \mathbf{w}_l$. The gains are given by $K_j = M_j X^{-1}$, $j \in \mathcal{B}^p$. Moreover, any trajectory $\hat{x}(t)$ starting in the outermost Lyapunov level set $(\hat{x} : V(\hat{x}) \leq \kappa) \subset \Omega$, $\kappa > 0$, goes to zero as time goes to infinity.

**Proof.** Considering the quadratic Lyapunov function candidate $V(\hat{x}) = \hat{x}^T \hat{P} \hat{x}$, with $P = P^T > 0$, we have that $\dot{V}(\hat{x}) = 2 \hat{x}^T \hat{P} \hat{x}$, which is negative-definite if (omitting arguments when convenient)

1 Avoiding redundancy in the list of nonlinearities helps associating vertices of the resulting polytope, thus producing more relaxed LMI results (Arceo et al., 2018).
gain \( F_{jk} \), it is necessary to group them in \( \tilde{F}(\tilde{x}, F_{jk}(\tilde{x})) \); thus, we have:

\[
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
38 & -11 & -\tilde{x}_1 & 1 \\
10 & -10 + \tilde{x}_1 & -8/3 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \tilde{F}(\tilde{x}, F_{jk}(\tilde{x})),
\]

where no matrix \( \tilde{A}^i(\tilde{x}) \) nor the first row of blocks are present because \( q = 2 \); the remaining matrices have the following dimensions: \( \tilde{A}^1(\tilde{x}) \in \mathbb{R}^{4 \times 3} \), \( \tilde{A}^3(\tilde{x}) \in \mathbb{R}^{4 \times 1} \), and \( \tilde{B}(\tilde{x}) \in \mathbb{R}^{4 \times 1} \) with \( \tilde{F}(\cdot) \in \mathbb{R}^{4 \times 4} \).

Explicit expressions in \( \tilde{F}(\tilde{x}, F_{jk}(\tilde{x})) \) are logically too large; yet, for the sake of illustration, its first entry is provided:

\[
\tilde{F}_1 = -9 F_{11}^0 + 342 F_{12}^1 + 90 F_{13}^1 - 3762 F_{12}^2 - 3660 F_{13}^2 - 9 F_{12}^3 \tilde{x}_3 + 9 F_{13}^3 \tilde{x}_2 - 90 F_{12}^3 \tilde{x}_1 + 90 F_{23}^2 \tilde{x}_2 + 342 F_{23}^3 \tilde{x}_1 + 123 F_{23}^2 \tilde{x}_3 - 123 F_{23}^3 \tilde{x}_4 - 9 F_{23}^3 \tilde{x}_4 + 90 F_{32}^2 \tilde{x}_3 + 9 F_{32}^2 \tilde{x}_4 - 9 F_{23}^3 \tilde{x}_2 - 9 F_{23}^3 \tilde{x}_3 - 90 (F_{12}^0 + F_{13}^0 + 9 F_{13}^0).
\]

Considering the non-constant term \( z = \tilde{x}_1 \) and its bounds \( z \in [-20, 20] \), an exact polytopic representation is obtained with \( \tilde{x} = [\tilde{x}_1 \ \tilde{x}_2 \ \tilde{x}_3 \ \tilde{x}_4]^T \) and \( \tilde{u} = \tilde{F}(\tilde{x}, F_{jk}(\tilde{x})){\tilde{x}} \):

\[
\dot{\tilde{x}}(t) = \sum_{i=0}^{1} w_i(\tilde{x})([\tilde{A}_i^1 \ \tilde{A}_i^2] \tilde{x}(t) + \tilde{B}_i \tilde{u}(t)),
\]

where \( w_0^1 = 0.5 - \tilde{x}_1/40, w_1^1 = 1 - w_0^1, \tilde{B}_0 = \tilde{B}_1 = [0 \ 0 \ 0 \ 1]^T \), and

\[
\tilde{A}_0^1 = \begin{bmatrix}
38 & -11 & 20 \\
10 & -30 & -8/3 \\
0 & 0 & 0
\end{bmatrix}, \quad \tilde{A}_1^1 = \begin{bmatrix}
38 & -11 & -20 \\
10 & 10 & -8/3 \\
0 & 0 & 0
\end{bmatrix}, \quad \tilde{A}_0^2 = \tilde{A}_1^2 = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}.
\]

From LMI conditions (10), \( K_1 = 10^3[0.90 - 0.02 - 0.95 0.15] \) and \( K_2 = 10^3[6.20 0.05 - 0.78 0.13] \) are obtained, producing the fictitious control law (9) \( \tilde{u}(t) = (w_0^1(\tilde{x})K_1 + w_1^1(\tilde{x})K_2) \dot{\tilde{x}}(t) = \tilde{K}(\tilde{x}) \dot{\tilde{x}}(t) \). The last step is to recover the gains \( F_{jk} \) for (12) from \( \tilde{F}(\tilde{x}, F_{jk}(\tilde{x})) = \tilde{K}(\tilde{x}) \).

After simplifications, the following is obtained \( F_{jk}^0 = 22.647 x_1 + 3.666 x_2 - 23.666 x_3 + x_1 x_2 + x_1 x_3 - 18830.795, F_{jk}^1 = -4.818 x_1 - 308.4, F_{jk}^2 = 2587.078 - 12.270 x_1, F_{jk}^3 = 1.072 x_1 - x_2 + x_3 - 441.048, F_{jk}^4 = F_{jk}^5 = 0, F_{jk}^6 = F_{jk}^7 = 1 \).

Fig. 1 (left) shows the state evolution of system (11) under the control law (12) (right) for initial conditions \( x(0) = [20 -2 10] \).

5. CONCLUSIONS

A methodology for high-order state-derivative controller design has been presented; it is based on inducing a singular system by feeding back \( q \) derivatives of the state vector; this model allows transforming the original nonlinear system into a new ODE one by means of the Pantelides algorithm. An auxiliary control law has been designed for the latter system, based on which the real control law is obtained via LMI conditions.

REFERENCES


