Presented as late breaking results contribution
21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

High-Order State-Derivative Controller Design for Nonlinear Systems ⋆

Jorge Álvarez * **David Vázquez** * **Alán Tapia** * **Miguel Bernal** *

* *Sonora Institute of Technology, 5 de Febrero 818 Sur, Obregon, Sonora, Mexico (e-mail: miguel.bernal@itson.edu.mx).*

Abstract: A novel controller design for nonlinear systems based on inducing a set of differential algebraic equations is presented. The method generalises the state-derivative feedback control by employing an arbitrary number of derivatives of the state vector pre-multiplied by nonlinear gains. To determine these gains, an intermediary control law is indirectly synthesised via convex modelling and linear matrix inequalities through an induced singular system subject to the Pantelides algorithm. An example illustrates the effectiveness of the proposal.

Keywords: Differential Algebraic Equations, Linear Matrix Inequalities, Descriptor Systems, Polytopic Representation, Controller Design, Nonlinear Systems.

1. INTRODUCTION

Control laws can be classified according to the number of derivatives or integrals of the state that they include, being the linear feedback the most elementary case (Bass and Gura, 1965). Some control laws include the first time derivative of the state vector in addition to the state feedback (Cardim et al., 2007); pole placement formulae for this case have been developed (Abdelaziz and Valášek, 2004); robust generalisations can be found in Faria et al. (2009).

Feeding back the time derivatives of the state vector may be better described by leaving expressions in the left-hand side of the system equation, i.e., having a descriptor form which might be singular (Lewis, 1986). A control law called proportional-plus-derivative state feedback which includes the time derivative of the state vector has been employed for stabilisation of descriptor linear systems in Duan and Zhang (2003); generalisations for polytopic (Da Silva et al., 2011) and nonlinear systems (González et al., 2017) have appeared. It has been shown in the latter that instead of feeding back the time derivatives of the states it is possible to solve the control law from an algebraic loop.

State derivatives might be obtained from accelerometers as in some practical problems like the car wheel suspension system (Reithmeier and Leitmann, 2003), the suppression of vibration in mechanical systems (Abdelaziz and Valášek, 2004), and the control of bridge cable vibration (Duan et al., 2005). Alternatively, state derivatives can be obtained with an arbitrary degree of precision, under mild conditions, by employing a Levant's robust differentiator (Levant, 2003); this is the preferred option for sliding mode methodologies. This work proposes an LMI generalisation of previously appeared state-derivative control laws to any differentiation order. The resulting control law can be applied to any sufficiently smooth nonlinear system with bounded nonlinearities by means of the Levant's robust differentiator.

Notation as well as a short introduction to convex modelling, singular systems, the Pantelides algorithm, and Levant's robust differentiator, are presented in section 2; section 3 develops the novel methodology for high-order state-derivative controller design by inducing differential algebraic equations; the effectiveness of the approach is put at test with two nonlinear examples in section 4; finally, conclusions are discussed in section 5.

2. PRELIMINARIES

Nonlinear systems can be subsumed in convex structures that share the characteristics of linear parameter varying (LPV) or Takagi-Sugeno (TS) models: $\dot{x}(t) = \sum_{i=1}^{r} h_i(x)(A_i x(t) + B_i u(t))$, where $0 \leq h_i(x) \leq 1$, $\sum_{i=1}^{r} h_i(x) = 1$, within a compact set of the state space that includes the origin. This rewriting allows using a quadratic Lyapunov function $V(x) = x^T P x$ to synthesise a PDC control law $u(t) = \sum_{j=1}^{r} h_j(x) F_j x(t)$ via LMIs $X = X^T > 0$ and $A_i X + B_i M_j + (*) < 0$, where $P = X^{-1}$, $F_j = M_j X^{-1}$ (Tanaka and Wang, 2001); this work follows this outline for a system subject to the Pantelides algorithm.

Feeding back high-order time derivatives of the state introduces singularities in the system. Roughly speaking, quasi-linear singular systems consist of differential algebraic equations which are amenable to the following descriptor form (Arceo et al., 2018):

$$E(x)\dot{x}(t) = A(x)x(t), \quad \det(E(x)) = 0. \tag{1}$$

These systems can be reduced to ordinary differential equations (ODEs) by means of the Pantelides algorithm; this is required for many tasks, e.g., simulation; it consists of the following steps:

(1) Put the singular system (1) into the dynamic decomposition form (DDF) that splits dynamical and algebraic relationships, i.e.:

⋆ This work has been supported by CONACYT scholarships 731289 and 731421, the ITSON PROFEXCE 2020-21, and the PROFAPI CA-18 Project.

$$\begin{bmatrix}\tilde{E}(x) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}A_{11}(x) & A_{12}(x)\\ A_{21}(x) & A_{22}(x)\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}, \quad (2)$$

where $\det(\tilde{E}(x)) \neq 0$, $x = \begin{bmatrix}x_1^T & x_2^T\end{bmatrix}^T$.

(2) Take successive derivatives of the algebraic equations $A_{21}(x)x_1 + A_{22}(x)x_2 = 0$ until the missing dynamics $\dot{x}_2(t)$ are found.
(3) Any simulation will use the dynamical part of (2), the missing dynamics found in the previous step, and proper initialisation holding the algebraic restrictions in (2) and those derived during the previous step.

Feeding back high-order time derivatives of the state requires a way to obtain these signals. Besides obvious options such as sensors and solving the algebraic loop for order 1, $x^{(i)}(t)$, $i \geq 1$, can be obtained via a finite-time convergent Levant's robust differentiator Levant (2003).

3. MAIN RESULTS

Nonlinear affine-in-control systems of the form

$$\dot{x}(t) = A(x)x(t) + B(x)u(t), \quad (3)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, and the pair $(A(x), B(x))$ consists of sufficiently smooth entries which are bounded on a compact set $\Omega \subset \mathbb{R}^n$ that includes the origin, are considered in this work.

Consider a nonlinear control law of the form:

$$u(t) = \sum_{i=0}^{q} F^i(x)x^{(i)}(t), \quad (4)$$

where $F^i(x) \in \mathbb{R}^{m\times n}$ are nonlinear gains to be determined, $x^{(i)}$, $i \in \{0,1,\ldots,q\}$ denote the time derivatives of the state vector $x(t)$. Note that $q = 0$ produces an ordinary state feedback control law $u(t) = F^0x(t)$ while $q = 1$ leads to an algebraic loop that has been considered in the literature (González et al., 2017).

Substituting (4) in (3) yields:

$$(A(x)+B(x)F^0(x))x+(B(x)F^1(x)-I)\dot{x}+\sum_{i=2}^{q}B(x)F^i(x)x^{(i)}=0,$$

which, defining $\bar{x} = \begin{bmatrix}x^T & \dot{x}^T & \cdots & (x^{(q-1)})^T\end{bmatrix}^T$, can be further transformed into a $(q \times n)$-th order descriptor of a possibly nonlinear singular system:

$$\begin{bmatrix}I & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0\\ 0 & 0 & \cdots & 0 & B(x)F^q(x)\end{bmatrix}\dot{\bar{x}}(t)=\begin{bmatrix}0 & I & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I\\ -A(x)-B(x)F^0(x) & I-B(x)F^1(x) & -B(x)F^2(x) & \cdots & -B(x)F^{q-1}(x)\end{bmatrix}\bar{x}(t). \quad (5)$$

Most plants have fewer inputs than states, so $B(x)F^q(x)$ is generally not full rank. Assuming $\text{rank}(B(x)F^q(x)) = r$ the last block row has $n-r$ implicit algebraic restrictions and r dynamics, all of which can be determined via the following orderly adaptation of the Pantelides algorithm (Pantelides, 1988):

(1) Solve for $\bar{x}_{qn-(n-r)+1}$, $\bar{x}_{qn-(n-r)+2}$, …, $\bar{x}_{qn}$ in terms of the remaining entries of $\bar{x}$.
(2) Determine $\dot{\bar{x}}_{qn-(n-r)+1}$, $\dot{\bar{x}}_{qn-(n-r)+2}$, …, $\dot{\bar{x}}_{qn}$ by taking the time derivatives of the solved states in the previous step and substituting any occurrence of $\bar{x}_{qn-(n-r)+1}$, $\bar{x}_{qn-(n-r)+2}$, …, $\bar{x}_{qn}$ by their equivalence found in the previous step.
(3) Substitute the new expressions for $\dot{\bar{x}}_{qn-(n-r)+1}$, $\dot{\bar{x}}_{qn-(n-r)+2}$, …, $\dot{\bar{x}}_{qn}$ into the remaining dynamics. Notice that only those from the last block row may require this substitution.
(4) Solve dynamics $\dot{\bar{x}}_{(q-1)n+1}$, $\dot{\bar{x}}_{(q-1)n+2}$, …, $\dot{\bar{x}}_{(q-1)n+r}$, in terms of $\bar{x}_1$, $\bar{x}_2$, …, $\bar{x}_{(q-1)n+r}$.

After performing the previous algorithm the system becomes the following standard state-space representation:

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \vdots\\ \dot{\bar{x}}_{(q-2)n}\\ \dot{\bar{x}}_{(q-2)n+1}\\ \vdots\\ \dot{\bar{x}}_{(q-2)n+n+r}\end{bmatrix}=\begin{bmatrix}\bar{x}_{n+1}\\ \vdots\\ \bar{x}_{(q-2)n+n}\\ f_1(\bar{x}_1,\ldots,\bar{x}_{(q-1)n+r})\\ \vdots\\ f_{n+r}(\bar{x}_1,\ldots,\bar{x}_{(q-1)n+r})\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\ 0\\ g_1(\bar{x}_1,\ldots,\bar{x}_{(q-1)n+r},F_{11}^0(x),\ldots,F_{mn}^q(x))\\ \vdots\\ g_{n+r}(\bar{x}_1,\ldots,\bar{x}_{(q-1)n+r},F_{11}^0(x),\ldots,F_{mn}^q(x))\end{bmatrix}, \quad (6)$$

where $f_i(\cdot)$, $i \in \{1,2,\ldots,n+r\}$, are possibly nonlinear functions of the states, $g_i(\cdot)$, $i \in \{1,2,\ldots,n+r\}$, are possibly nonlinear functions of the states and gain entries $F_{jk}^i(x)$, $i \in \{0,1,\ldots,q\}$, $j \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,n\}$.

In order to determine gains $F^i(x)$, $i \in \{0,1,\ldots,q\}$, such that the origin is asymptotically stable, (6) should be written as follows:

$$\dot{\tilde{x}}(t) = \begin{bmatrix}0_{(q-2)n\times n} & I_{(q-2)n} & 0_{(q-2)n\times r}\\ \tilde{A}^1(\tilde{x}) & \tilde{A}^2(\tilde{x}) & \tilde{A}^3(\tilde{x})\end{bmatrix}\tilde{x}(t) + \begin{bmatrix}0_{(q-2)n\times m}\\ \tilde{B}(\tilde{x})\end{bmatrix}\tilde{F}(\tilde{x}, F_{jk}^i(\tilde{x}))\tilde{x}(t), \quad (7)$$

where $\tilde{x} = \begin{bmatrix}\bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_{(q-1)n+r}\end{bmatrix}^T$, $\tilde{A}^1(\tilde{x}) \in \mathbb{R}^{(n+r)\times n}$, $\tilde{A}^2(\tilde{x}) \in \mathbb{R}^{(n+r)\times(q-2)n}$, $\tilde{A}^3(\tilde{x}) \in \mathbb{R}^{(n+r)\times r}$, and $\tilde{B}(\tilde{x}) \in \mathbb{R}^{(n+r)\times m}$ are matrices with known possibly nonlinear entries, and $\tilde{F}(\cdot) \in \mathbb{R}^{m\times((q-1)n+r)}$ is a possibly nonlinear matrix function of the states and gain entries $F_{jk}^i(\tilde{x})$, $i \in \{0,1,\ldots,q\}$, $j \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,n\}$. There is, of course, an infinite number of such factorisations.

Let us define $\tilde{u} \equiv \tilde{F}(\tilde{x}, F_{jk}^i(\tilde{x}))\tilde{x}$. The sector nonlinearity approach will be employed to find an exact polytopic representation of the 4-tuple $(\tilde{A}^1(\tilde{x}), \tilde{A}^2(\tilde{x}), \tilde{A}^3(\tilde{x}), \tilde{B}(\tilde{x}))$ within $\tilde{\Omega}$, which is defined as a compact set in $\mathbb{R}^{(q-1)n+r}$ that includes the bounds induced by $x \in \Omega$ as well as others for the additional states. Once this 4-tuple is exactly rewritten as a convex sum of constant matrices, a fictitious PDC-like control law $\tilde{u}(t) = \tilde{K}(\tilde{x})\tilde{x}(t)$ will be designed such that the origin of (7) is guaranteed to be

Presented as late breaking results contribution
21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

asymptotically stable. As a final step, gains $F^i_{jk}(\tilde{x})$ of the actual control law will be obtained from the equivalence $\tilde{K}(\tilde{x}) = \tilde{F}(\tilde{x}, F^i_{jk}(\tilde{x}))$.

An equivalent polytopic representation of (7) within $\tilde{\Omega}$ can be systematically constructed via the sector nonlinearity approach in González et al. (2017):

(1) create a list of different bounded non-constant terms $z_i(\tilde{x}) \in [z_i^0, z_i^1]$, $i \in \{1, 2, \ldots, p\}$, in $\tilde{A}^1(\tilde{x})$, $\tilde{A}^2(\tilde{x})$, $\tilde{A}^3(\tilde{x})$, and $\tilde{B}(\tilde{x})$, by taking into account similar terms in different entries [1],
(2) write each $z_i(\tilde{x})$ as a convex sum of its bounds

$$z_i(\tilde{x}) = \underbrace{\frac{z_i^1 - z_i(\tilde{x})}{z_i^1 - z_i^0}}_{w_0^i(\tilde{x})} z_i^0 + \underbrace{\frac{z_i(\tilde{x}) - z_i^0}{z_i^1 - z_i^0}}_{w_1^i(\tilde{x})} z_i^1,$$

with $w_0^i(\tilde{x}) + w_1^i(\tilde{x}) = 1$, $w_0^i(\tilde{x}), w_1^i(\tilde{x}) \in [0, 1]$, $i \in \{1, 2, \ldots, p\}$ for every $\tilde{x} \in \tilde{\Omega}$,
(3) write the exact polytopic representation of (7) as:

$$\dot{\tilde{x}}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(\tilde{x}) \Bigg(\begin{bmatrix} 0_{(q-2)n\times n} & I_{(q-2)n} & 0_{(q-2)n\times r} \\ \tilde{A}^1_{\mathbf{i}} & \tilde{A}^2_{\mathbf{i}} & \tilde{A}^3_{\mathbf{i}} \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0_{(q-2)n\times m} \\ \tilde{B}_{\mathbf{i}} \end{bmatrix} \tilde{u}(t) \Bigg), \tag{8}$$

where $\mathbb{B} = \{0, 1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_p) \in \mathbb{B}^p$, $\mathbf{w_i}(\tilde{x}) = w^1_{i_1}(\tilde{x}) w^2_{i_2}(\tilde{x}) \cdots w^p_{i_p}(\tilde{x})$, $\tilde{A}^1_{\mathbf{i}} = \tilde{A}^1(\tilde{x})|_{\mathbf{w_i}(\tilde{x})=1}$, $\tilde{A}^2_{\mathbf{i}} = \tilde{A}^2(\tilde{x})|_{\mathbf{w_i}(\tilde{x})=1}$, $\tilde{A}^3_{\mathbf{i}} = \tilde{A}^3(\tilde{x})|_{\mathbf{w_i}(\tilde{x})=1}$, and $\tilde{B}_{\mathbf{i}} = \tilde{B}(\tilde{x})|_{\mathbf{w_i}(\tilde{x})=1}$.

Consider the following fictitious PDC-like control law:

$$\tilde{u}(t) = \tilde{K}(\tilde{x})\tilde{x}(t) = \sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w_i}(\tilde{x}) K_{\mathbf{j}} \tilde{x}. \tag{9}$$

Theorem 1. The origin of the system (7) with equivalent polytopic representation (8) within $\tilde{\Omega}$, under the control law (9) is asymptotically stable if $\exists X \in \mathbb{R}^{(q-1)n+r\times(q-1)n+r}$, $M_{\mathbf{j}} \in \mathbb{R}^{m\times(q-1)n+r}$, $\mathbf{j} \in \mathbb{B}^p$, of suitable dimensions, such that the LMIs $X = X^T > 0$ and

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathcal{P}(\mathbf{k},\mathbf{l})} \Bigg(\begin{bmatrix} 0_{(q-2)n\times n} & I_{(q-2)n} & 0_{(q-2)n\times r} \\ \tilde{A}^1_{\mathbf{i}} & \tilde{A}^2_{\mathbf{i}} & \tilde{A}^3_{\mathbf{i}} \end{bmatrix} X + \begin{bmatrix} 0_{(q-2)n\times m} \\ \tilde{B}_{\mathbf{i}} \end{bmatrix} M_{\mathbf{j}} + (*) \Bigg) < 0, \tag{10}$$

hold for all $\mathbf{k}, \mathbf{l} \in \mathbb{B}^p$, with $\mathcal{P}(\mathbf{k},\mathbf{l})$ being the set of indexes $(\mathbf{i},\mathbf{j})$ of all products $\mathbf{w_i}\mathbf{w_j}$ which are algebraically similar to $\mathbf{w_k}\mathbf{w_l}$. The gains are given by $K_{\mathbf{j}} = M_{\mathbf{j}} X^{-1}$, $\mathbf{j} \in \mathbb{B}^p$. Moreover, any trajectory $\tilde{x}(t)$ starting in the outermost Lyapunov level set $\{\tilde{x} : V(\tilde{x}) \leq \kappa\} \subset \tilde{\Omega}$, $\kappa > 0$, goes to zero as time goes to infinity.

Proof. Considering the quadratic Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^T P \tilde{x}$, with $P = X^{-1} = P^T > 0$, we have that $\dot{V}(\tilde{x}) = 2\tilde{x}^T P \dot{\tilde{x}}$, which is negative-definite if (omitting arguments when convenient)

$$P \sum_{\mathbf{i},\mathbf{j}\in\mathbb{B}^p} \mathbf{w_i}\mathbf{w_j} \Bigg(\begin{bmatrix} 0_{(q-2)n\times n} & I_{(q-2)n} & 0_{(q-2)n\times r} \\ \tilde{A}^1_{\mathbf{i}} & \tilde{A}^2_{\mathbf{i}} & \tilde{A}^3_{\mathbf{i}} \end{bmatrix} + \begin{bmatrix} 0_{(q-2)n\times m} \\ \tilde{B}_{\mathbf{i}} \end{bmatrix} K_{\mathbf{j}} \Bigg) + (*) < 0.$$

The desired LMI conditions follow from pre- and post-multiplication by $X = P^{-1}$, renaming of products $K_{\mathbf{j}} X$ as $M_{\mathbf{j}}$, and dropping of the products of convex functions $\mathbf{w_i}(\tilde{x})\mathbf{w_j}(\tilde{x})$ by associating similar terms.

Once the convex sum $\tilde{K}(\tilde{x})$ is obtained by means of the previous theorem, the gain entries $F^i_{jk}(\tilde{x})$ of the actual control law will be obtained from the equation $\sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w_i}(\tilde{x}) K_{\mathbf{j}} = \tilde{F}(\tilde{x}, F^i_{jk}(\tilde{x}))$.

4. EXAMPLE

Consider the 3rd-order nonlinear system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_1 \\ 0 & x_1 & -8/3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \tag{11}$$

with the control law (4) considering $q = 2$,

$$u(t) = F^0(x)x(t) + F^1(x)\dot{x}(t) + F^2(x)\ddot{x}(t). \tag{12}$$

Once (12) is substituted in (11), the closed-loop system can be rewritten in the singular form (5) (arguments are omitted for brevity):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^2_{11} & F^2_{12} & F^2_{13} \\ 0 & 0 & 0 & F^2_{11} & F^2_{12} & F^2_{13} \\ 0 & 0 & 0 & F^2_{11} & F^2_{12} & F^2_{13} \end{bmatrix} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \\ \dot{\bar{x}}_5 \\ \dot{\bar{x}}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 10 - F^0_{11} & -10 - F^0_{12} & -F^0_{13} & 1 - F^1_{11} & -F^1_{12} & -F^1_{13} \\ -28 - F^0_{11} & 1 - F^0_{12} & \bar{x}_1 - F^0_{13} & -F^1_{11} & 1 - F^1_{12} & -F^1_{13} \\ -F^0_{11} & -\bar{x}_1 - F^0_{12} & 8/3 - F^0_{13} & -F^1_{11} & -F^1_{12} & 1 - F^1_{13} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ \bar{x}_5 \\ \bar{x}_6 \end{bmatrix}, \tag{13}$$

where F^i_{jk}, $i \in \{0, 1, 2\}$, $j = 1$, $k \in \{1, 2, 3\}$, are the (j,k)-entries of gain $F^i(x)$, $\bar{x}_i = x_i$, $\bar{x}_{i+3} = \dot{x}_i$, $i \in \{1, 2, 3\}$.

Since $\text{rank}([1\ 1\ 1]^T F^2(x)) = 1$ for a 3rd-order system, there are $3 - 1 = 2$ algebraic restrictions. Applying the Pantelides algorithm in section 2, the last 3 equations allow solving the following states:

$$\bar{x}_5 = 38\bar{x}_1 - 11\bar{x}_2 + \bar{x}_4 - \bar{x}_1\bar{x}_3,$$
$$\bar{x}_6 = 10\bar{x}_1 - 10\bar{x}_2 - (8/3)\bar{x}_3 + \bar{x}_4 + \bar{x}_1\bar{x}_2,$$

whose derivatives are

$$\dot{\bar{x}}_5 = \dot{\bar{x}}_4 - 418\bar{x}_1 + 121\bar{x}_2 + 27\bar{x}_4 + 10\bar{x}_1\bar{x}_2 + (41/3)\bar{x}_1\bar{x}_3 - \bar{x}_1\bar{x}_4 - \bar{x}_3\bar{x}_4 - \bar{x}_1^2\bar{x}_2 - 10\bar{x}_1^2,$$
$$\dot{\bar{x}}_6 = \dot{\bar{x}}_4 - (1220/3)\bar{x}_1 + (410/3)\bar{x}_2 + (64/9)\bar{x}_3 - (8/3)\bar{x}_4 - (41/3)\bar{x}_1\bar{x}_2 + 10\bar{x}_1\bar{x}_3 + \bar{x}_1\bar{x}_4 + \bar{x}_2\bar{x}_4 - \bar{x}_1^2\bar{x}_3 + 38\bar{x}_1^2.$$

Then, from any of the last 3 equations in (13), it is possible to solve the single missing dynamic $\dot{\bar{x}}_4$. Thus, defining $\tilde{x}_i = \bar{x}_i$, $i \in \{1, 2, 3, 4\}$, the resulting ODE system can be written as (7). Since all terms in $\dot{\bar{x}}_4$ depend on some

[1] Avoiding redundancy in the list of nonlinearities helps associating vertices of the resulting polytope, thus producing more relaxed LMI results (Arceo et al., 2018).

Presented as late breaking results contribution
21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

gain F_{jk}^{i}, it is necessary to group them in $\tilde{F}(\tilde{x}, F_{jk}^{i}(\tilde{x}))$; thus, we have:

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\\ \dot{\tilde{x}}_3\\ \dot{\tilde{x}}_4\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1\\ 38 & -11 & -\tilde{x}_1 & 1\\ 10 & -10+\tilde{x}_1 & -8/3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{x}_4\end{bmatrix}+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\tilde{F}(\tilde{x}, F_{jk}^{i}(\tilde{x}))\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{x}_4\end{bmatrix},$$

where no matrix $\tilde{A}^2(\tilde{x})$ nor the first row of blocks are present because $q = 2$; the remaining matrices have the following dimensions: $\tilde{A}^1(\tilde{x}) \in \mathbb{R}^{4\times 3}$, $\tilde{A}^3(\tilde{x}) \in \mathbb{R}^{4\times 1}$, and $\tilde{B}(\tilde{x}) \in \mathbb{R}^{4\times 1}$ with $\tilde{F}(\cdot) \in \mathbb{R}^{1\times 4}$.

Explicit expressions in $\tilde{F}(\tilde{x}, F_{jk}^{i}(\tilde{x}))$ are logically too large; yet, for the sake of illustration, its first entry is provided: $\tilde{F}_1 = -(9F_{11}^0 + 342F_{12}^1 + 90F_{13}^1 - 3762F_{12}^2 - 3660F_{13}^2 - 9F_{12}^1\tilde{x}_3 + 9F_{13}^1\tilde{x}_2 - 90F_{12}^2\tilde{x}_1 + 90F_{12}^2\tilde{x}_2 + 342F_{13}^2\tilde{x}_1 + 123F_{12}^2\tilde{x}_3 - 123F_{13}^2\tilde{x}_2 - 9F_{12}^2\tilde{x}_4 + 90F_{13}^2\tilde{x}_3 + 9F_{13}^2\tilde{x}_4 - 9F_{12}^2\tilde{x}_1\tilde{x}_2 - 9F_{13}^2\tilde{x}_1\tilde{x}_3 - 90)/(9F_{11}^2 + 9F_{12}^2 + 9F_{13}^2)$.

Considering the non-constant term $z = \tilde{x}_1$ and its bounds $z \in [-20, 20]$, an exact polytopic representation is obtained with $\tilde{x} = [\tilde{x}_1\ \tilde{x}_2\ \tilde{x}_3\ \tilde{x}_4]^T$ and $\tilde{u} = \tilde{F}(\tilde{x}, F_{jk}^{i}(\tilde{x}))\tilde{x}$:

$$\dot{\tilde{x}}(t) = \sum_{i=0}^{1} w_i^1(\tilde{x})\Big(\big[\tilde{A}_i^1\ \ \tilde{A}_i^3\big]\tilde{x}(t) + \tilde{B}_i\tilde{u}(t)\Big),$$

where $w_0^1 = 0.5 - \tilde{x}_1/40$, $w_1^1 = 1 - w_0^1$, $\tilde{B}_0 = \tilde{B}_1 = [0\,0\,0\,1]^T$,

$$\tilde{A}_0^1=\begin{bmatrix}0 & 0 & 0\\ 38 & -11 & 20\\ 10 & -30 & -8/3\\ 0 & 0 & 0\end{bmatrix}, \tilde{A}_1^1=\begin{bmatrix}0 & 0 & 0\\ 38 & -11 & -20\\ 10 & 10 & -8/3\\ 0 & 0 & 0\end{bmatrix}, \tilde{A}_0^3=\tilde{A}_1^3=\begin{bmatrix}1\\1\\1\\0\end{bmatrix},$$

From LMI conditions (10), $K_1 = 10^3[6.90\ -0.02\ -0.95\ 0.15]$ and $K_2 = 10^3[6.22\ 0.05\ -0.78\ 0.13]$ are obtained, producing the fictitious control law (9) $\tilde{u}(t) = \big(w_0^1(\tilde{x})K_1 + w_1^1(\tilde{x})K_2\big)\,\tilde{x}(t) = \tilde{K}(\tilde{x})\tilde{x}(t)$. The last step is to recover the gains F_{jk}^{i} for (12) from $\tilde{F}(\tilde{x}, F_{jk}^{i}(\tilde{x})) = \tilde{K}(\tilde{x})$. After simplifications, the following is obtained $F_{11}^0 = 22.647x_1 + 3.666x_2 - 23.666x_3 + x_1x_2 + x_1x_3 - 18830.795$, $F_{12}^0 = -4.818x_1 - 308.4$, $F_{13}^0 = 2587.078 - 12.279x_1$, $F_{11}^1 = 1.072x_1 - x_2 + x_3 - 441.048$, $F_{12}^1 = F_{13}^1 = 0$, $F_{11}^2 = F_{12}^2 = F_{13}^2 = 1$.

Fig. 1 (left) shows the state evolution of system (11) under the control law (12) (right) for initial conditions $x(0) = [20\ -2\ 10]$.

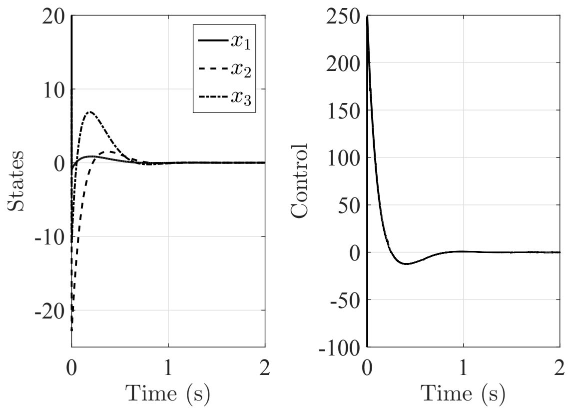

Fig. 1. States evolution (left); Control signal (right).

5. CONCLUSIONS

A methodology for high-order state-derivative controller design has been presented; it is based on inducing a singular system by feeding back q derivatives of the state vector; this model allows transforming the original nonlinear system into a new ODE one by means of the Pantelides algorithm. An auxiliary control law has been designed for the latter system, based on which the real control law is obtained via LMI conditions.

REFERENCES

Abdelaziz, T. and Valášek, M. (2004). Pole-placement for SISO linear systems by state-derivative feedback. *IEE Proceedings-Control Theory and Applications*, 151(4), 377–385.

Arceo, J.C., Sánchez, M., Estrada-Manzo, V., and Bernal, M. (2018). Convex stability analysis of nonlinear singular systems via linear matrix inequalities. *IEEE Transactions on Automatic Control*, 64(4), 1740–1745.

Bass, R. and Gura, I. (1965). High order system design via state-space considerations. In *Joint Automatic Control Conference*, 3, 311–318.

Cardim, R., Teixeira, M., Assunção, E., and Covacic, M.R. (2007). Design of state-derivative feedback controllers using a state feedback control design. *IFAC Proceedings Volumes*, 40(20), 22–27.

Da Silva, E., Assunção, E., Teixeira, M., Faria, F., and Buzachero, L. (2011). Parameter-dependent lyapunov functions for state-derivative feedback control in polytopic linear systems. *International Journal of Control*, 84(8), 1377–1386.

Duan, G. and Zhang, X. (2003). Regularizability of linear descriptor systems via output plus partial state derivative feedback. *Asian Journal of Control*, 5(3), 334–340.

Duan, Y., Ni, Y., and Ko, J. (2005). State-derivative feedback control of cable vibration using semiactive magnetorheological dampers. *Computer-Aided Civil and Infrastructure Engineering*, 20(6), 431–449.

Faria, F., Assunção, E., Teixeira, M., Cardim, R., and Da Silva, N. (2009). Robust state-derivative pole placement lmi-based designs for linear systems. *International Journal of Control*, 82(1), 1–12.

González, T., Bernal, M., Sala, A., and Aguiar, B. (2017). Cancellation-based nonquadratic controller design for nonlinear systems via takagi–sugeno models. *IEEE transactions on cybernetics*, 47(9), 2628–2638.

Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control. *International journal of control*, 76(9-10), 924–941.

Lewis, F. (1986). A survey of linear singular systems. *Circuits, Systems, and Signal Processing*, 5(1), 3–36.

Pantelides, C.C. (1988). The consistent initialization of differential-algebraic systems. *SIAM Journal on Scientific and Statistical Computing*, 9(2), 213–231.

Reithmeier, E. and Leitmann, G. (2003). Robust vibration control of dynamical systems based on the derivative of the state. *Archive of Applied Mechanics*, 72(11-12), 856–864.

Tanaka, K. and Wang, H. (2001). *Fuzzy Control Systems Design and Analysis: A linear matrix inequality approach.* John Wiley & Sons, New York.