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A Nonlinear Small Phase Theorem ⋆

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Abstract: In this paper, we propose a definition of phase for a class of stable nonlinear systems called semi-sectorial systems from a pure input-output perspective. The definition involves Hilbert transform as a key tool for the purpose of complexifying real-valued signals. The proposed nonlinear system phase, serving as a counterpart of $\mathcal{L}_2$-gain, quantifies the passivity and is highly related to the dissipativity. A nonlinear small phase theorem is then established for feedback stability analysis of semi-sectorial systems. It generalizes a version of the passivity theorem and the linear time-invariant small phase theorem.

Keywords: Small Phase Theorem, Nonlinear Systems, Passivity, Hilbert Transform.

1. INTRODUCTION

In the classical frequency-domain analysis of single-input single-output (SISO) linear time-invariant (LTI) systems, the gain and the phase go hand in hand and are treated on an equal footing in many applications. For more general systems, the gain theory is rich and well established while the phase counterpart somehow is inadequate and ambiguous. It is natural to wonder what a suitable phase definition is for those systems beyond SISO LTI systems. The attempt to answer this question dates back for decades (Postlethwaite et al., 1981; Owens, 1984). Recently, for multi-input multi-output (MIMO) LTI systems, a suitable definition of system phase on the basis of the numerical range was proposed in Chen et al. (2019). The authors further formulated an LTI small phase theorem which provides a stability condition in terms of the "loop phase" less than π. We refer the readers to Chen et al. (2019) for more details of the MIMO LTI system phase.

For nonlinear systems, $\mathcal{L}_2$-gain is a fundamental quantity used in the stability analysis and control of feedback systems from a pure input-output perspective. The classical small gain theorem (Zames, 1966) conveys that a feedback system maintains stability provided that its "loop $\mathcal{L}_2$-gain" is less than one. However, the notion of nonlinear system phase is not well understood. Passivity has been considered as a phase-type counterpart of $\mathcal{L}_2$-gain for a long time. A good reference on $\mathcal{L}_2$-gain and passivity is van der Schaft (2017). A SISO LTI passive system, as is well-known, provides a phase-shift of an input sinusoid being at most $\pi/2$. The passivity theorem (Zames, 1966; Vidyasagar, 1993) ensures the stability of feedback interconnected passive and strictly passive systems. The passivity theorem thereupon is treated as a "small phase theorem" by some researchers (Rantzer, 2019). Nevertheless, the passivity theorem is conservative in the sense of requiring open-loop systems, roughly speaking, to have their "phases" inside $[-\pi/2, \pi/2]$. Another important class of nonlinear systems from a phasic point of view are counterclockwise systems (Angeli, 2006) with their "phases" inside $[-\pi, 0]$. The main purpose of this paper is to explore the notion of nonlinear system phase and utilize this quantity in stability analysis of feedback systems. We look forward to bringing up the phase notion to the same footing as $\mathcal{L}_2$-gain in nonlinear systems.

In this paper, we first define the phase for a class of stable nonlinear systems called semi-sectorial systems hereinafter. Second, a nonlinear small phase theorem is established for stability analysis of feedback semi-sectorial systems. Two essential tools are utilized in the nonlinear system phase definition, i.e., the angular numerical range and Hilbert transform. Using these tools in phase study is inherited from our previous works (Chen et al., 2019; Wang et al., 2020). The nonlinear system phase itself has a nice physical interpretation. In brief, it undertakes a significant role as a tradeoff between the active energy and reactive energy. The nonlinear system phase generalizes the MIMO LTI system $\mathcal{H}_\infty$-phase (Chen et al., 2019). It also admits a strong connection with static nonlinearity (Vidyasagar, 1993), passivity, counterclockwise dynamics, dissipativity theory (Willems, 1972; Hill and Moylan, 1980) and integral quadratic constraints theory (Megretski and Rantzer, 1997). In our current studies, the proposed nonlinear small phase theorem extends the passivity theorem when causal stable components are considered. One common practice of reducing conservatism of the passivity theorem is to quantify passivity by using input/output passivity indices (Cho and Narendra, 1968). These indices, used to measure

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the surplus or deficit of passivity, could be either positive or negative. This kind of characterization, however, deviates from the phasic perspective to some extent. It is also noteworthy that the multiplier approach (Zames and Falb, 1968) is often adopted to reduce conservatism of the passivity theorem by finding a suitable multiplier. This multiplier, often as a noncausal artificial operator, is required to meet a factorization condition which gives rise to difficulties in practice. Roughly speaking, the nonlinear small phase theorem provides an implicit multiplier that is much more straightforward and intuitive from a pure phasic perspective.

The outline of this paper is as follows. Preliminaries of signal spaces, systems and Hilbert transform are introduced in Section 2. In Section 3, we define the phase of semi-sectorial systems, and further develop a nonlinear small phase theorem. Moreover, a discussion on the nonlinear system phase and the dissipativity is provided.

2. PRELIMINARIES AND MOTIVATIONS

Let $\mathbb{R}$ ($\mathbb{C}$, resp.) and $\mathbb{R}^n$ ($\mathbb{C}^n$, resp.) denote real (complex, resp.) numbers and n-dimensional real (complex, resp.) vectors, respectively. For real or complex vectors x, denote $|x|$ as the Euclidean norm. For a nonzero $x = |x|\,e^{j\angle x} \in \mathbb{C}$ in the polar form, its angle or phase is denoted by $\angle x$. If $x = 0$, then $\angle x$ is undefined. Denote $\bar{\mathbb{C}}_+$ as the closed right half-plane. Denote $\mathcal{RH}_\infty$ as the space consisting of proper real rational matrix functions with no poles in $\bar{\mathbb{C}}_+$.

2.1 Signal spaces, operators and systems

The input-output analysis of nonlinear systems is often built on a real signal space. We start with the $\mathcal{L}_2$ space, the set of all energy-bounded $\mathbb{R}^n$-valued signals

$$\mathcal{L}_2 := \left\{ u : \mathbb{R} \to \mathbb{R}^n \,\middle|\, \|u\|_2^2 := \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty \right\}.$$

The causal subspace of $\mathcal{L}_2$ is denoted by $\mathcal{L}_2[0,\infty) := \{u \in \mathcal{L}_2 | u(t) = 0 \text{ for } t < 0\}$. For $T \in \mathbb{R}$, define the truncation $\mathbf{\Gamma}_T$ on all $u : \mathbb{R} \to \mathbb{R}^n$ by $(\mathbf{\Gamma}_T u)(t) := u(t)$ for $t \le T$; $(\mathbf{\Gamma}_T u)(t) := 0$ for $t > T$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{R}$. An operator $\boldsymbol{P} : \mathcal{H} \to \mathcal{H}$ is said to be causal if $\mathbf{\Gamma}_T \boldsymbol{P} = \mathbf{\Gamma}_T \boldsymbol{P} \mathbf{\Gamma}_T$ for all $T \in \mathbb{R}$, and is said to be noncausal if it is not causal. We always assume that an operator $\boldsymbol{P}$ satisfies $\boldsymbol{P}0 = 0$. We view a system as an operator from real-valued input signals to real-valued output signals. A practical nonlinear system is represented by a causal operator $\boldsymbol{P} \in \mathrm{dom}(\boldsymbol{P}) \subset \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ where $\mathrm{dom}(\boldsymbol{P})$ denotes the domain of the operator. Such a system (an operator, resp.) is said to be stable (bounded, resp.) if $\mathrm{dom}(\boldsymbol{P}) = \mathcal{L}_2[0,\infty)$ and

$$\|\boldsymbol{P}\| := \sup_{0 \ne u \in \mathcal{L}_2[0,\infty)} \frac{\|\boldsymbol{P}u\|_2}{\|u\|_2} < \infty.$$

Here $\|\boldsymbol{P}\|$ is called $\mathcal{L}_2$-gain of $\boldsymbol{P}$ and it is the key quantity used in the gain-based input-output nonlinear system control theory. For a bounded linear operator $\boldsymbol{P} : \mathcal{H} \to \mathcal{H}$, denote $\boldsymbol{P}^*$ as the adjoint operator of $\boldsymbol{P}$.

Passivity is another key notion for input-output analysis of nonlinear systems. A causal stable system $\boldsymbol{P}$ is called passive (van der Schaft, 2017) if $\langle u\,, \boldsymbol{P}u \rangle \ge 0$ for all $u \in \mathcal{L}_2[0,\infty)$. In particular, for a SISO LTI system $P(s) \in \mathcal{RH}_\infty$, this condition is equivalent to $\mathrm{Re}P(j\omega) \ge 0$ for all $\omega \in \mathbb{R}$. In such a case, $\angle P(j\omega)$ lies in $[-\pi/2, \pi/2]$. One can see that passivity is phase-related but qualitative.

We aim at a quantifiable phasic notion of nonlinear systems. When doing this, we encounter a problem that a practical nonlinear system can only accept and generate real-valued signals. However, phases are usually defined with complex numbers. Moreover, phases of SISO LTI systems are introduced in terms of frequency responses. These two facts reveal that complex numbers are essential to phase definitions. Therefore, the fundamental nontrivial question behind a nonlinear system phase definition is that how we can appropriately complexify real-valued signals. Our answer is to utilize Hilbert transform and the corresponding analytic signals which are complex-valued and commonly used in signal processing.

When working with complexified signals, it is insufficient to restrict ourselves to the real signal space $\mathcal{L}_2$. A complex signal space $\mathcal{L}_2^{\mathbb{C}}$ is supposed to be equipped. Here $\mathcal{L}_2^{\mathbb{C}}$ is the set of all energy-bounded $\mathbb{C}^n$-valued signals

$$\mathcal{L}_2^{\mathbb{C}} := \left\{ u : \mathbb{R} \to \mathbb{C}^n \,\middle|\, \|u\|_2^2 := \int_{-\infty}^{\infty} |u(t)|^2 dt < \infty \right\}.$$

From now on, we develop the phase definition using $\mathcal{L}_2^{\mathbb{C}}$.

2.2 Hilbert transform

The Hilbert transform $\boldsymbol{H}$ of a complex-valued signal $u(t)$ is defined by the integral (King, 2009)

$$(\boldsymbol{H}u)(t) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau = \frac{1}{\pi t} * u(t)$$

provided that the integral exists, where $*$ denotes the convolution operation. The integral above is improper in the sense of the Cauchy principal value. A simple example is that $\sin(t)$ is the Hilbert transform of $\cos(t)$. This example gives us an intuition that the Hilbert transform provides a $\pi/2$ phase-shift which can be clarified conveniently using the frequency-domain language. Specifically, the Fourier transform of the convolution kernel $1/(\pi t)$ is given by $-j\mathrm{sgn}(\omega)$ where $\mathrm{sgn}(\cdot)$ denotes the signum function. This gives $(\widehat{\boldsymbol{H}u})(j\omega) = -j\mathrm{sgn}(\omega)\hat{u}(j\omega)$ where $\hat{\ }$ denotes the Fourier transform of a signal. Hence the Hilbert transform provides a $-\pi/2$ phase-shift for positive frequencies while a $\pi/2$ phase-shift for negative frequencies.

In the rest of the paper, we restrict the Hilbert transform on $\mathcal{L}_2^{\mathbb{C}}$. For $u \in \mathcal{L}_2^{\mathbb{C}}$, the Hilbert transform $\boldsymbol{H} : \mathcal{L}_2^{\mathbb{C}} \to \mathcal{L}_2^{\mathbb{C}}$ is well-defined and it is a noncausal linear bounded operator possessing three favorable properties: isometry $\|\boldsymbol{H}u\|_2 = \|u\|_2$; anti-self-adjointness $\boldsymbol{H}^* = -\boldsymbol{H}$; anti-involution: $\boldsymbol{H}(\boldsymbol{H}u) = -u$. It follows that the Hilbert transform on $\mathcal{L}_2^{\mathbb{C}}$ preserves the inner product and in particular it is a unitary operator. Moreover, notice that $\langle u\,, \boldsymbol{H}u \rangle = 0$ for $u \in \mathcal{L}_2^{\mathbb{C}}$ which can be deduced from the well-known Plancherel's theorem. This implies $u(t)$ and its Hilbert transform $(\boldsymbol{H}u)(t)$ are orthogonal. The Hilbert transform is a lossless process on account of merely generating a phase-shift to an original signal. Therefore, it is often utilized to generate a complex-valued signal from a real-valued signal in signal processing. A complex-valued signal whose imaginary part is the Hilbert transform of its real part is called an analytic signal. Specifically, for $u \in \mathcal{L}_2^{\mathbb{C}}$, its analytic representation is denoted by

$$u_a(t) := \frac{1}{2}(u(t) + j(\boldsymbol{H}u)(t)). \tag{1}$$

In connection with the complexification of real-valued signals in nonlinear systems, for $u \in \mathcal{L}_2$, using (1) leads to $u_a \in \mathcal{L}_2^{\mathbb{C}}$. Equipped with analytic signals, now we are ready to define the nonlinear system phase from a pure input-output view.

3. A NONLINEAR SMALL PHASE THEOREM

3.1 Phase of nonlinear systems

In this paper, we consider a causal stable nonlinear system $\boldsymbol{P} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$. Based on analytic signals (1), the angular numerical range of $\boldsymbol{P}$ is defined to be

$$W'(\boldsymbol{P}) := \{\langle u_a\,, \boldsymbol{P}u\rangle \in \mathbb{C} \mid 0 \neq u \in \mathcal{L}_2[0,\infty)\}\,.$$

Such a system $\boldsymbol{P}$ is said to be semi-sectorial if $W'(\boldsymbol{P})$ is contained in a closed complex half-plane. Geometrically for a semi-sectorial system $\boldsymbol{P}$, we can have two unique supporting rays of $W'(\boldsymbol{P})$. Denote $\phi_c(\boldsymbol{P}) \in (-\pi,\pi]$ as the angle from the positive real axis of the interior angle bisector of these two rays. Then, the phase of $\boldsymbol{P}$, denoted by $\Phi(\boldsymbol{P})$, is defined to be the phase sector

$$\Phi(\boldsymbol{P}) := [\underline{\phi}(\boldsymbol{P}), \overline{\phi}(\boldsymbol{P})]$$

where $\underline{\phi}(\boldsymbol{P})$ and $\overline{\phi}(\boldsymbol{P})$ are called the smallest phase and the largest phase of $\boldsymbol{P}$, respectively, which are given by

$$\underline{\phi}(\boldsymbol{P}) := \inf_{0\neq z\in W'(\boldsymbol{P})} \angle z \in [\phi_c(\boldsymbol{P}) - \pi/2, \phi_c(\boldsymbol{P}) + \pi/2],$$
$$\overline{\phi}(\boldsymbol{P}) := \sup_{0\neq z\in W'(\boldsymbol{P})} \angle z \in [\phi_c(\boldsymbol{P}) - \pi/2, \phi_c(\boldsymbol{P}) + \pi/2].$$

Moreover, it is easy to see $\overline{\phi}(\boldsymbol{P}) - \underline{\phi}(\boldsymbol{P}) \in [0,\pi]$ and $\phi_c(\boldsymbol{P}) = 1/2\left(\underline{\phi}(\boldsymbol{P}) + \overline{\phi}(\boldsymbol{P})\right)$.

Each $\langle u_a\,, \boldsymbol{P}u\rangle \in W'(\boldsymbol{P})$ in $\mathbb{C}$ is associated with the point $(\mathrm{Re}\langle u_a\,, \boldsymbol{P}u\rangle, \mathrm{Im}\langle u_a\,, \boldsymbol{P}u\rangle)$ in $\mathbb{R}^2$. When $\boldsymbol{P}$ is semi-sectorial, there exists a real number $\alpha \in [-\pi,\pi]$ such that $\boldsymbol{\Phi}(\boldsymbol{P}) \subseteq [-\pi/2-\alpha, \pi/2-\alpha]$ which is equivalent to

$$\cos\alpha \mathrm{Re}\langle u_a\,, \boldsymbol{P}u\rangle - \sin\alpha \mathrm{Im}\langle u_a\,, \boldsymbol{P}u\rangle \geq 0$$

for all $0 \neq u \in \mathcal{L}_2[0,\infty)$. Geometrically this inequality means that the set of all points $(\mathrm{Re}\langle u_a\,, \boldsymbol{P}u\rangle, \mathrm{Im}\langle u_a\,, \boldsymbol{P}u\rangle)$ is contained in a closed half-plane in $\mathbb{R}^2$ with its normal vector $[\cos\alpha \;\; -\sin\alpha]^T$. In addition, $\boldsymbol{P}$ is said to be sectorial if there exists $\delta > 0$ such that

$$\cos\alpha \mathrm{Re}\langle u_a\,, \boldsymbol{P}u\rangle - \sin\alpha \mathrm{Im}\langle u_a\,, \boldsymbol{P}u\rangle \geq \delta \left\|u\right\|_2^2$$

for all $0 \neq u \in \mathcal{L}_2[0,\infty)$. That is to say, the set of all points is required to deviate from the origin along the normal vector.

A bridge between the nonlinear system phase and the passivity can be evidently built. Recall that a causal stable passive system $\boldsymbol{P}$ requires $\langle u\,, \boldsymbol{P}u\rangle \geq 0$ for all $u \in \mathcal{L}_2[0,\infty)$ which is equivalent to $W'(\boldsymbol{P}) \in \overline{\mathbb{C}}_+$ due to $\langle u\,, \boldsymbol{P}u\rangle = 2\mathrm{Re}\langle u_a\,, \boldsymbol{P}u\rangle \geq 0$. Thereupon using the phasic language, we understand that such a system $\boldsymbol{P}$ has its phase $\Phi(\boldsymbol{P}) \subseteq [-\pi/2, \pi/2]$. In particular, for some passive systems we can further estimate their phases. For instance, we can show that a static nonlinear passive sector can be transformed to a phase sector, a subset of $[-\pi/2, \pi/2]$. As a consequence, the phase of a class of sector bounded static nonlinearities is contained in the phase spread of a disk in $\mathbb{C}$. This disk is analogous to the circle in the celebrated circle criterion (Vidyasagar, 1993).

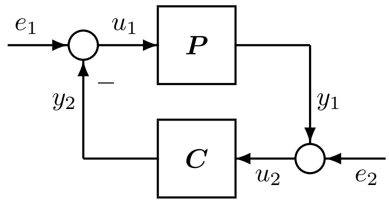

Fig. 1. A standard feedback system for stabilization.

3.2 Closed-loop stability

After defining the nonlinear system phase, we proceed to stability analysis of feedback interconnected semi-sectorial systems. Let us consider a standard feedback system shown in Fig. 1, where $\boldsymbol{P} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and $\boldsymbol{C} : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ are two causal stable systems, e_1, e_2 are external signals and u_1, u_2, y_1, y_2 are internal signals. Algebraically, we have the following equations

$$u = e - \begin{bmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{bmatrix} y \quad \text{and} \quad y = \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{C} \end{bmatrix} u \tag{2}$$

with $u = [u_1 \; u_2]^T$, $e = [e_1 \; e_2]^T$ and $y = [y_1 \; y_2]^T$.

Let $\boldsymbol{P} \,\#\, \boldsymbol{C}$ denote the feedback system. We introduce two indispensable definitions involving feedback systems. First of all, well-posedness of $\boldsymbol{P} \,\#\, \boldsymbol{C}$ is an important assumption to guarantee that the closed-loop system (2) makes sense as a model of a real system. We stipulate the following feedback well-posedness adopted from Khong et al. (2013).

Definition 1. The feedback system $\boldsymbol{P} \,\#\, \boldsymbol{C}$ is said to be well-posed if

$$\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}} : \mathcal{L}_2[0,\infty) \times \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$$
$$:= \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

has a causal inverse on the image of $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}$.

Definition 2. The well-posed feedback system $\boldsymbol{P} \,\#\, \boldsymbol{C}$ is said to be stable if $\|e \mapsto u\| < \infty$.

Given the definition of closed-loop stability, we are ready to present the following main result.

Theorem 1. (Nonlinear small phase theorem). For sectorial $\boldsymbol{P}$ and semi-sectorial $\boldsymbol{C}$, the well-posed feedback system $\boldsymbol{P} \,\#\, \boldsymbol{C}$ is stable if

$$\overline{\phi}(\boldsymbol{P}) + \overline{\phi}(\boldsymbol{C}) < \pi,$$
$$\underline{\phi}(\boldsymbol{P}) + \underline{\phi}(\boldsymbol{C}) > -\pi.$$

The proof of this theorem is omitted here. The result above also holds if $\boldsymbol{C}$ is sectorial and $\boldsymbol{P}$ is semi-sectorial. Basically, the nonlinear small phase theorem provides a crucial condition from a pure phasic view to guarantee the closed-loop stability. That is, analogous to that in the small gain theorem the loop $\mathcal{L}_2$-gain less than one, the loop largest phase $\overline{\phi}(\boldsymbol{P}) + \overline{\phi}(\boldsymbol{C})$ is less than π as well as the loop smallest phase $\underline{\phi}(\boldsymbol{P}) + \underline{\phi}(\boldsymbol{C})$ is greater than $-\pi$ simultaneously. Two existing results can be subsumed into the nonlinear small phase theorem. First, when $\boldsymbol{P}$ and $\boldsymbol{C}$ are both MIMO LTI systems in $\mathcal{RH}_\infty$, Theorem 1 reduces to the $\mathcal{H}_\infty$-phase version small phase theorem (Chen et al., 2019). Second, when $\boldsymbol{P}$ is input strictly passive and $\boldsymbol{C}$ is passive, then Theorem 1 spontaneously reduces to a version of the passivity theorem.

3.3 Discussions on the phase and dissipativity

Passivity and $\mathcal{L}_2$-gain can be incorporated in a unified dissipativity framework. The property of dissipativeness can be regarded as either a state-space property (Willems, 1972) or an input-output property (Hill and Moylan, 1980). We adopt the dissipativeness as a pure input-output property in phase study. The notion of supply rates, as an abstraction of the concept of input power, plays a key role in the input-output theory of dissipative systems. Two important classes of supply rates are as follows. The first one describes passive systems, namely, a system $\boldsymbol{P}$ is passive if it is dissipative with respect to the supply rate

$$s(u(t), y(t)) = u(t)^T y(t)$$

where $y = \boldsymbol{P}u$. A second important class characterizes the gain bounded systems. Concretely, a system $\boldsymbol{P}$ has its $\mathcal{L}_2$-gain no greater than γ if it is dissipative with respect to the supply rate

$$s(u(t), y(t)) = 1/2\gamma^2|u(t)|^2 - 1/2|y(t)|^2$$

where $\gamma > 0$. Likewise, the nonlinear system phase, analogous to passivity and $\mathcal{L}_2$-gain, can be understood from the dissipativity theory using a new supply rate. Before unveiling this supply rate, we present the following proposition for semi-sectorial systems, which establishes a connection between the half-plane containing $W'(\boldsymbol{P})$ and the corresponding time-domain inequality.

Proposition 1. For a semi-sectorial system $\boldsymbol{P} : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$ and a constant $\alpha \in [-\pi, \pi]$, it holds that $\angle W'(\boldsymbol{P}) \in [-\pi/2 - \alpha, \pi/2 - \alpha]$ if and only if

$$\langle u\, , \cos\alpha \boldsymbol{P}u - \sin\alpha \boldsymbol{H}\boldsymbol{P}u\rangle \geq 0, \quad \forall\, 0 \neq u \in \mathcal{L}_2[0, \infty).$$

The proof is omitted here. On the basis of the nonlinear system phase definition and Proposition 1, we discover a class of supply rates describing the phase bounded systems. We claim that a system $\boldsymbol{P}$ has its phase belongs to $[-\pi/2 - \alpha, \pi/2 - \alpha]$ if it is dissipative with respect to the dynamic supply rate

$$s(u(t), y(t)) = u(t)^T \left[\cos\alpha y(t) - \sin\alpha(\boldsymbol{H}y)(t)\right]. \tag{3}$$

In this claim, the nonlinear system phase belongs to a sector with length π. When more accurate phase information of a system is available, a generalization can be made by intersecting a few sectors. For example, a system has its phase belongs to $[-\pi/4, \pi/3]$ if it is dissipative with respect to simultaneous two supply rates, namely $\alpha = \pi/6$ and $\alpha = -\pi/4$ in (3).

The supply rate mentioned in (3) is not a memoryless supply rate which only depends on the present time. It is, in fact, a dynamic supply rate. The reason is that the Hilbert transform is a noncausal LTI operator whose output depends on the past, present and future values of the input. This kind of dynamic supply rates is new while the notion of dynamic supply rates is not new. See other dynamic supply rates such as the quadratic differential form (Willems and Trentelman, 1998), the counterclockwise dynamics (Angeli, 2006) and the general dynamic supply rate (Arcak et al., 2016).

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