Optimal Control: Pontryagin Minimum Principle

Lev Semenovich Pontryagin (1908-1988)
Lecture 4

Last lecture:
Existence and uniqueness of solutions
• Dynamic Programming
  • Principle of optimality
  • Linear Quadratic Regulator (LQR)
  • Continuous Dynamic Programming
  • Hamilton-Jacobi-Bellman equation

Today:
• Pontryagin Minimum Principle
• Robustness and Optimal Control
Motivation

Dynamic Programming
• Characterizes optimality along optimal solution trajectories
• Considers “all solutions”, i.e. all possible initial conditions ➤ PDE

\[ x(t_0) \xrightarrow{} x(t_1) \]

“End pieces of optimal trajectories are optimal”
• Sufficient conditions, but very strong assumptions

Potryagin Minimum (Maximum) Principle
• Characterizes optimality around optimal solution (like first order cond.)

\[ x(t_0) \xrightarrow{} \]

➤ two-point boundary value problem

• Weaker results, only hold for one initial point!
Pontryagin Minimum Principle – sketch of idea

\[
\min_{u(\cdot)} \int_{t_0}^{t_F} F(t, x(t), u(t)) dt + E(x(t_F)) \quad \text{s.t.} \quad \dot{x}(t) = f(t, x(t), u(t)) \quad x(t_0) = x_0 \\
E(x(t_F)), f(t, x(t), u(t)) \in C^1
\]

Optimality conditions?

first-order variations of \( \tilde{J} \) must be zero

Consider

\[
\begin{align*}
\delta u & \quad \delta x \\
\downarrow & \quad \downarrow \\
u(t) & = u^*(t) + \delta u(t) \quad \text{small} \\
x(t) & = x^*(t) + \delta x(t)
\end{align*}
\]

First-order optimality condition:

\[
\tilde{J}(u^* + \delta u, x^* + \delta x) - \tilde{J}(u^*, x^*) = 0
\]
Pontryagin Minimum Principle – the unconstrained case

Special case: no constraints

$$\min_{u(\cdot)} \int_{t_0}^{t_F} F(t, x(t), u(t)) dt + E(x(t_F))$$

s.t.  
$$\dot{x}(t) = f(t, x(t), u(t)) \quad x(t_0) = x_0$$
$$E(x(t_F)), f(t, x(t), u(t)) \in \mathcal{C}^1$$

Hamiltonian defined as:

$$\mathcal{H}(t, x(t), u(t), \lambda(t)) = F(t, x(t), u(t)) + \lambda^T f(t, x(t), u(t))$$

Co-states or adjoint variables

The optimal cost can be written as:

$$\Rightarrow \tilde{J} = E(x(t_F)) + \int_{t_0}^{t_F} \left[ F(t, x(t), u(t)) + \lambda^T (f(t, x(t), u(t)) - \dot{x}(t)) \right] dt$$

Hamiltonian $$\mathcal{H}(t, x(t), u(t), \lambda(t))$$

• To derive PMP: Expansion of the optimal cost and

$$\delta \tilde{J}(u^* + \delta u, x^* + \delta x) - \tilde{J}(u^*, x^*) \overset{!}{=} 0$$
Pontryagin Minimum Principle – the unconstrained case

Special case: no constraints

\[
\min_{u(\cdot)} \int_{t_0}^{t_F} F(t, x(t), u(t)) dt + E(x(t_F))
\]

\[\text{s.t. } \quad \dot{x}(t) = f(t, x(t), u(t)) \quad x(t_0) = x_0 \]

\[E(x(t_F)), f(t, x(t), u(t)) \in C^1\]

Pontryagin Minimum Principle:

Let \( u^*(\cdot), x^*(\cdot) \) be the optimal solution:

Then \( \exists \lambda(\cdot) \) such that

a) \( \dot{\lambda}(t) = -\mathcal{H}_x(t, x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_F) = E_x(x^*(t_F)) \)

b) \( u^*(t) = \arg\min \mathcal{H}(t, x^*(t), u(t), \lambda(t)) \)

\[\Rightarrow \mathcal{H}_u(t, x^*(t), u^*(t), \lambda(t)) = 0\]
Theorem:

Let $u^*(\cdot), x^*(\cdot)$ be an optimal solution. Then $\exists$ an adjoint-/co-state $\lambda(\cdot)$ s.t.

\begin{align*}
\text{a)} & \quad \dot{\lambda}(t) = -\mathcal{H}_x(t, x^*(t), u^*(t)\lambda(t)), \quad \lambda(t_F) = E_x(x^*(t_F)) \\
\text{b)} & \quad \mathcal{H}_u(t, x^*(t), u^*(t), \lambda(t)) = 0, \quad \forall t \in [t_0, t_F] \\
& \quad \Downarrow \\
& \quad \min_{u^*(t)} \mathcal{H}(t, x^*(t), u^*(t), \lambda(t)) \\
\text{c)} & \quad \mathcal{H}^*(t) = \mathcal{H}^*(t_F) - \int_{t_0}^{t_F} \frac{\partial \mathcal{H}}{\partial t}(t, x^*(t), u^*(t), \lambda(t))|_{t=s} ds, \quad \forall t \in [t_0, t_F]
\end{align*}
Pontryagin Minimum Principle – the unconstrained case

Remarks:
- Only necessary conditions!
- Some remarks:
  a) can be also written as
  
  \[
  \dot{\lambda}(t) = -F_x(t, x^*(t), u^*(t)) - \sum_{k=1}^{n} \frac{\partial f_i}{\partial x}(t, x^*(t), u^*(t))\lambda_k(t)
  \]

  note also that \( \dot{x}(t) = H_\lambda(t, x(t), u(t), \lambda(t)) \)

  c) very valuable if problem is time invariant

  \[
  \Rightarrow H(t, x^*(t), u^*(t), \lambda(t)) = \text{const.}
  \]

  "proof" assume \( u \in C^1 \)

  \[
  \begin{align*}
  \dot{H} &= H_t^* + (H_{xx}^*)^T f + (H_u^*)^T \dot{u} + \dot{\lambda}^T f \\
  &= H_t^* + H_u^T \dot{u} + (H_{xx}^* + \dot{\lambda}) f \\
  &= H_t^*
  \end{align*}
  \]

  - remember \( \frac{\partial H}{\partial u}(t, x^*, u^*, \lambda) = 0 \iff u^* = \arg \min H(t, x^*, u, \lambda) \)
How to use and example

**Approach to solve / use:**

1. $\mathcal{H}(t, x, u, \lambda) = F(t, x, u) + \lambda^T f(t, x, u)$
2. solve $\mathcal{H}_u(t, x, u, \lambda) = 0$ for $u \Leftrightarrow u = \arg \min \mathcal{H}$
3. solve TPBV
   \[
   \begin{align*}
   \dot{x} &= f(t, x, k(t, x, \lambda)), \quad x(t_0) = x_0 \\
   \lambda &= -\mathcal{H}_x(t, x, k(t, x, \lambda), \lambda(t_F)), \lambda(t_F) = E_x(x(t_F))
   \end{align*}
   \]
4. $u = k(t, x(t), \lambda(t))$
5. Check optimality of solution
   - Automatic if system linear and cost functional is convex
   - Verify that is the only $u$ that satisfies the PMP

**Remarks**

- Constrained case is not as trivial!
- One of the greatest achievements of the 20th century
A car example: maximize end distance with minimum effort
General PMP

Extensions of the Pontryagin Minimum principle for different cases possible
• Fixed terminal state
• Free initial state
• Free terminal time
Solution is not that trivial but still possible in many cases, adding extra boundary conditions or states

For details see e.g.:
• Dynamic Programming and Optimal Control, D.P. Bertsekas, 1995
• Primer on Optimal Control Theory, J.L. Speyer and D.H. Jacobson, 2010
Summary PMP

Characterizes optimality around optimal solution (like first order cond.)

- Delivers only in rare cases sufficient conditions (e.g. linear system with convex cost)
- Solution easier to obtain (two-point boundary value problem, no PDE)
- Weaker assumptions, value function does not need to be $C^1$

- In general no feedback, only open-loop control
- Many open research questions, e.g. higher order optimality conditions
Robustness and Optimal Control

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Motivational example

Example:

\[ \dot{x} = x^2 + u \]

Cancel nonlinearity completely with

\[ u_1 = -x^2 - x \]

Any inherent robustness?
Consider perturbed

\[ u_{\epsilon 1} = (1 + \epsilon) u_1(x) \]

Closed-loop

\[ \dot{x} = x^2 + u_{\epsilon 1} = -(1 + \epsilon)x - \epsilon x^2 \]

Finite escape time for $\epsilon \neq 0$

No robustness against input perturbations
Inherent robustness and robust optimal control

Very active area of research [Scokaert et al. ’98, Mayne et al ’00, Fontes et al ’04, Findeisen ’06, Kerrigan ’07, Rakovic et al. ’05, Raimondo et al. 09]...

Inherent robustness

measurement disturbances
state estimation errors
meas. delays
input disturbances
actuator dynamics
numerical errors
model uncertainty
state disturbances

Optimal Control

\[ \dot{x} = f(x, u) + p \]

optimal controllers can be inherently robust (to small disturbances)
can tolerate: measurement disturbances, estimation errors, ...

Robust designs

\[
\min_{p(\cdot)} \max_{\hat{x}(t_i) \in \hat{X}(t_i)} \int_{t_i}^{t_i+T} L(\zeta(\tau), p(\tau)) d\tau + E(\zeta(t_i + T))
\]

• Explicit consideration of disturbances in design
• Often sup-inf or inf-sup formulation
Many classes of uncertainties and disturbances

Parametric uncertainties
\[ \dot{x} = f(x, u, p), \text{ where } p \in \mathcal{P} \]

Static uncertainties
\[ \dot{x} = f(x, u) + \phi(x), \text{ where } ax \leq \phi(x) \leq bx \]

Dynamic uncertainties
\[ \dot{x} = f(x, z, u) \]
\[ \dot{z} = \zeta(x, z, u) \]

Additive disturbances
\[ \dot{x} = f(x, u) + w \]

Many different disturbances/uncertainties. Design and analysis will normally depend on type
Inherent robustness

Question: Does the nominal optimal controller possess robustness properties?

Example: static input uncertainties
Inherent robustness II

Consider:

\[
\min_{u \in \mathbb{R}} \int_0^\infty F(x) + u^T R(x) u \, dt
\]

subject to: \( \dot{x} = f(x) + g(x)u \)

no constraints

\textbf{Theorem (Inherent robustness)}

Let (i) \( F(x) \) is positive semi-definite,

(ii) \( R(x) = \text{diag}(r(x)) \) is positive definite,

(iii) \( \dot{x} = f(x), y = F(x) \) is zero-state-detectable,

(iv) \( \Delta(u) = [\delta_1(u), \ldots, \delta_m(u)] \),

(v) HJBE admits a continuously differentiable solution \( V \) and \( u = k(x) \)

then the closed-loop is globally as. stable for all \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R} \),

that satisfy \( \frac{1}{2} u^T u \leq u^T \Delta(u) \leq \infty \).
Inherent robustness: sector margin and disc margin

single input

\[ \frac{1}{2} u^T u \leq u^T \Delta(u) \leq \infty \]

System is stable with respect to static input uncertainties that map to \([\frac{1}{2}, \infty]\)

Extension to dynamic uncertainties \(\Delta\) possible
If \(R(x) = I\), Disk margin \(D(0.5)\)
Inherent robustness: example

\[
\min_{u \in \mathbb{R}} \int_{0}^{\infty} x^T Q x + u^T R u \, dt
\]
subject to: \( \dot{x} = Ax + Bu \)

\((A, Q^{1/2})\) observable \((A, B)\) controllable

**Theorem (Gain and Phase Margin of the LQR)**

If \( Q > 0 \) and \( R = \text{diag}(r_i) \), then every input channel has gain margin \((\frac{1}{2}, \infty)\) and phase margin \( \pm 60 \)
Inherent robustness: inverse optimality

Many controllers are not based on ideas from optimal control
Are these robustness results transferable to general controllers?

Inverse Optimality:
consider a nonlinear system
\[ \dot{x} = f(x) + g(x)u \]
where a stabilizing control law is know that can be represented by
\[ u^* = -\frac{1}{2} R^{-1}(x) \left( \frac{\partial V}{\partial x} g(x) \right)^T \]
where \( V(x) \) is a suitable Lyapunov function that is continuously differentiable, and where \( R(x) > 0 \)

- Then the feedback \( u(x) \) is \textit{optimal with respect to an (unknown) cost function}
- The closed loop posses inherent robustness properties as outlined

Remarks:
- no optimal control design necessary
- for specific system classes this is easy to satisfy, e.g. Hamiltonian systems, ...
Inherent robustness: inverse optimality

Example:

\[ \dot{x} = x^2 + u \]

Optimal controller

\[
\begin{align*}
    u^*(x) &= \arg \min_{u \in \mathbb{R}} \int_0^\infty x^2 + u^2 \, dt \\
    \text{subject to: } &\dot{x} = x^2 + u \\
    &u^*(x) = -x^2 - x\sqrt{x^2} + 1
\end{align*}
\]

Closed-loop

\[ \dot{x} = x^2 + u^*(x) = -x\sqrt{x^2} + 1 \text{ asymptotically stable} \]

Same robustness properties as indicated!
And now?

What we already know:

- Formulate Optimal Control Problems (OCPs)
- Analysis of existence and uniqueness of a solution for some special cases
- Analytical approaches to solve OCPs (only for “simple” cases):
  - Dynamic Programming – HJBE
  - Pontryagin Minimum Principle

What is left?

- The most used *application* of Optimal Control
  - **Model Predictive Control**
- How to apply Model Predictive Control to real (mechatronic) systems?
  - **Embedded** Optimization
Structure

1. **Introduction**
   What is optimal control?
   Examples

2. **Static Optimization**

3. **Basic Setup of optimal control problems**
   Cost function, constraints
   Existence of solutions

4. **Analytic approaches to optimal control**
   Dynamic Programming
   Prontryagin minimum principle

5. **Numerical approaches to optimal control**
   Direct and indirect methods
   Convex optimization
   Model predictive control

6. **Embedded model predictive control**
   Embedded optimization
   Solution of QPs for MPC on embedded platforms
   Code generation and implementation aspects