Practical Set Invariance for Decentralized Discrete Time Systems

by

Saša V. Raković, Benjamin Kern and Rolf Findeisen

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Otto-von-Guericke University Magdeburg
Institute for Automation Engineering
Universitätsplatz 2
39106 - Magdeburg
Germany

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Abstract

This paper discusses set invariance notions for decentralized discrete time systems which are physically interconnected. We employ independent set–dynamics induced by the underlying subsystems subject to the available information in the decentralized setting. The main novelty of the approach lies within the fact that the concept of set invariance for independent set–dynamics is formalized by employing appropriate families of sets. The complexity of the exact notion is alleviated by introducing a practical set invariance notion which is then complemented with the corresponding relaxed stability analysis. Under mild assumptions, the introduced notion allows for safe, stable and independent operation of the subsystems forming the overall decentralized system.

Keywords: Decentralized Control, Interconnected Systems, Set Invariance, Stability, Constraints, Set–Dynamics

1 Introduction

The control of large scale systems has been an ongoing research area for more then four decades. The main underlying research dilemma is concerned with the trade–off between the centralized and decentralized synthesis methods. For an overview of the area see the comprehensive monograph [14], more recent publications [4,17] and references therein.

Our contribution is concerned with set invariance and stability notions for decentralized discrete time control systems, which are physically interconnected. Our prime objective is to develop both the flexible and practicable notions allowing for safe, stable and independent operation of all subsystems (and, in turn, the overall system) despite the presence of constraints as well as restrictions on the amount of information available locally (this "informational restriction" is inevitably induced by the decentralization of the original system). The approach we employ is compatible, at the conceptual level, with the notion of the *dynamics* of so–called vector Lyapunov functions [6,10,14–16]. Motivated by this notion and the recent results on set invariance under state and output feedback utilizing set–dynamics [1, 2], we consider the independent set–dynamics induced by the underlying subsystems subject to the available information in the decentralized setting and discuss both the exact and practical set invariance concepts. The exact set invariance notions are in this setting captured by considering invariant families of sets. The practical set invariance notions are, however, obtained by considering a parameterized family of sets and utilizing the approximate, outer–bounding, set–dynamics whose evolution is described by ordinary, vector–valued, dynamics. The employed family of sets is parameterized via a collection of sets $\{S_i \ : \ i = 1, 2, \ldots, N\}$ and a set Θ. The sets S_i are associated with the corresponding subsystems while the set Θ is obtained by employing the classical set invariance concepts for suitably designed vector–valued dynamics that describe the evolution of approximate, outer–bounding, set–dynamics. We show that, under mild assumptions, the introduced notion allows for safe, stable and independent operation of the subsystems forming the overall decentralized system. Motivated by computational tractability and simplicity of necessary analysis, we focus on the case of constrained, decentralized, linear control systems controlled by appropriate, static, linear control rules.

The paper is structured as follows: Section 2 provides preliminaries and problem formulation. Section 3 discusses practical set invariance and stability notions. Sections 4 and 6 comment briefly on a possible approach for computing local controllers and provide concluding remarks. The proofs for the subsequent results are given in the Appendix.

Basic Nomenclature and Definitions The sets of positive, non–negative integers and reals are denoted by $\mathbb{N}_+$, $\mathbb{N}$ and $\mathbb{R}_+$. Given a positive integer $q \in \mathbb{N}_+$ we denote $\mathbb{N}_{[1:q]} := \{1, 2, \ldots, q\}$. Given any positive integer $q \in \mathbb{N}_+$ and a positive integer $r \in \mathbb{N}_{[1:q]}$ we denote $\mathbb{N}_{(q,r)} := \{1, \ldots, r-1, r+1, \ldots, q\} = \mathbb{N}_{[1:q]} \setminus \{r\}$. A set X is said to be a non–trivial set if it is a proper, non–empty, subset of $\mathbb{R}^n$ and it is not a singleton set. Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \ : \ x \in X, \ y \in Y\}$. Given the sequence of sets $\{X_i \subset \mathbb{R}^n\}_{i=a}^{b}$, $a \in \mathbb{N}$, $b \in \mathbb{N}, b > a$, we denote $\bigoplus_{i=a}^{b} X_i := X_a \oplus \cdots \oplus X_b$. Given a set X and a real matrix M of compatible dimensions (possibly a scalar) we define $MX := \{Mx \ : \ x \in X\}$ and $M^{-1}X := \{x \ : \ Mx \in X\}$. Given a matrix $M \in \mathbb{R}^{n \times n}$, $\rho(M)$ denotes the largest absolute value of its eigenvalues. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half–spaces and a *polytope* is the closed and bounded polyhedron. A set $X \subset \mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $X \subset \mathbb{R}^n$ is *a proper C–set or just PC–set* if it is a C–set and contains the origin in its (non–empty) interior. A collection of sets $\{X_i \subset \mathbb{R}^n \ : \ i \in \mathbb{N}_{[1:q]}\}$

is a PC–collection if each X_i is a PC–set. A set $X \subseteq \mathbb{R}^n$ is a symmetric set (with respect to the origin in $\mathbb{R}^n$) if $X = -X$. The family of all subsets of $\mathbb{R}^n$ is denoted by $2^{\mathbb{R}^n}$. The family of non–empty compact subsets in $\mathbb{R}^n$ is denoted by $\mathrm{Com}(\mathbb{R}^n)$. For $X \in \mathrm{Com}(\mathbb{R}^n)$ and $Y \in \mathrm{Com}(\mathbb{R}^n)$, the *Hausdorff semi–distance* and the *Hausdorff distance* of X and Y are given by:

$$h(L, X, Y) := \min_{\alpha}\{\alpha \ : \ X \subseteq Y \oplus \alpha L,\ \alpha \geq 0\} \text{ and}$$
$$H(L, X, Y) := \max\{h(L, X, Y), h(L, Y, X)\},$$

where L is a given, symmetric, proper C–set in $\mathbb{R}^n$ inducing also the vector–norm $|x|_L := \min_{\mu}\{\mu \ : \ x \in \mu L,\ \mu \geq 0\}$. For typographical convenience, we distinguish row vectors from column vectors only when needed and employ the same symbol for a variable x and its vectorized form in the algebraic expressions.

2 Preliminaries & Problem Description

We consider a set of N discrete–time, time–invariant, linear interconnected control systems given by:

$$\forall i \in \mathbb{N}_{[1:N]},\ x_i^+ = A_i x_i + B_i u_i + \sum_{j \in \mathbb{N}_{(N,i)}} C_{(i,j)} x_j, \tag{1}$$

where $\forall i \in \mathbb{N}_{[1:N]},\ x_i \in \mathbb{R}^{n_i}$ is the current state of the i^{th} subsystem, $u_i \in \mathbb{R}^{m_i}$ is the current control of the i^{th} subsystem, $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$ with $n = \sum_{i \in \mathbb{N}_{[1:N]}} n_i$ is the current state of the overall system, $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^m$ with $m = \sum_{i \in \mathbb{N}_{[1:N]}} m_i$ is the current control of the overall system, for each $i \in \mathbb{N}_{[1:N]},\ A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and for each $i \in \mathbb{N}_{[1:N]}$ and each $j \in \mathbb{N}_{(N,i)},\ C_{(i,j)} \in \mathbb{R}^{n_i \times n_j}$.

Besides the physical interconnections defined by the matrices $C_{(i,j)}$, the subsystem variables $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are subject to hard constraints, namely:

$$\forall i \in \mathbb{N}_{[1:N]},\ x_i \in \mathbb{X}_i \text{ and } u_i \in \mathbb{U}_i, \tag{2}$$

where $\forall i \in \mathbb{N}_{[1:N]},\ \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $\mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ are the state and control constraint sets for the i^{th} subsystem. We invoke the following standard assumption:

Assumption 1 *For each $i \in \mathbb{N}_{[1:N]}$,*

(i) the matrix pairs (A_i, B_i) are controllable, and,

(ii) the sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are proper C–sets in $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$.

We examine practical set invariance and stability notions under two clarifying interpretations; the first considers the case when the local controller has only summarized information about the other subsystems:

Interpretation 1 *For any $i \in \mathbb{N}_{[1:N]}$ and at any time instance $k \in \mathbb{N}$, the current state $x_{(k;i)}$ of the subsystem i and the value of the total sum $\sum_{j \in \mathbb{N}_{(N,i)}} C_{(i,j)} x_{(k;j)}$ is known to the i^{th} decision maker when deciding on the control action $u_{(k;i)}$ for the subsystem i.*

The second interpretation is concerned with the case when the local controller has individual information about the interactions with other subsystems:

Interpretation 2 *For any $i \in \mathbb{N}_{[1:N]}$ and at any time instance $k \in \mathbb{N}$, the current state $x_{(k;i)}$ of the subsystem i and the values of the individual summands $C_{(i,j)} x_{(k;j)},\ j \in \mathbb{N}_{(N,i)}$ are known to the i^{th} decision maker when deciding on the control action $u_{(k;i)}$ for the subsystem i.*

Note that, under Interpretations 1 and 2, the states $x_{(k;j)},\ j \in \mathbb{N}_{(N,i)}$ of the other subsystems (or the values of the individual summands $C_{(i,j)} x_{(k;j)},\ j \in \mathbb{N}_{(N,i)}$ under Interpretation 1) are, excluding special cases, not known to the i^{th} decision maker when deciding on the control action $u_{(k;i)}$.

For any $i \in \mathbb{N}_{[1:N]}$ we set, for notational compactness, $C_{(i,i)} = I \in \mathbb{R}^{n_i \times n_i}$. For any $i \in \mathbb{N}_{[1:N]}$, let $c_i : \mathbb{R}^n \to \mathbb{R}^{2n_i}$ be given by:

$$c_i(x) = (C_{(i,i)} x_i, \sum_{j \in \mathbb{N}_{(N,i)}} C_{(i,j)} x_j) \tag{3}$$

and, similarly, let $d_i : \mathbb{R}^n \to \mathbb{R}^{Nn_i}$ be given by:

$$d_i(x) = (C_{(i,1)} x_1, \ldots, C_{(i,i)} x_i, , \ldots, C_{(i,N)} x_N). \tag{4}$$

In addition to the invoked interpretations, we are concerned with the utilization of the static linear control rules for control synthesis. In particular, for any $i \in \mathbb{N}_{[1:N]}$ under Interpretation 1, the i^{th} decision maker can deploy, at time $k \in \mathbb{N}$, the linear control rules:

$$u_{(k;i)}(c_i(x_k)) = K_i x_{(k;i)} + L_i \sum_{j \in \mathbb{N}_{(N,i)}} C_{(i,j)} x_{(k;j)}, \tag{5}$$

and, likewise, under Interpretation 2, the linear control rules:

$$u_{(k;i)}(d_i(x_k)) = K_i x_{(k;i)} + \sum_{j \in \mathbb{N}_{(N,i)}} L_{(i,j)} C_{(i,j)} x_{(k;j)}, \tag{6}$$

where, for all $i \in \mathbb{N}_{[1:N]}$, $K_i \in \mathbb{R}^{n_i \times n_i}$, $L_i \in \mathbb{R}^{n_i \times n_i}$ and, for all $j \in \mathbb{N}_{(N,i)}$, $L_{(i,j)} \in \mathbb{R}^{n_i \times n_i}$.

The prime aim of our investigation is concerned with the practical set invariance and stability notions for the set of N discrete–time autonomous systems specified in (1) induced by the linear control structures specified in (5) and (6) and taking the form, $\forall i \in \mathbb{N}_{[1:N]}$,:

$$x_i^+ = A_{(i,i)} x_i + \sum_{j \in \mathbb{N}_{(N,i)}} A_{(i,j)} x_j, \tag{7}$$

where, $\forall i \in \mathbb{N}_{[1:N]}$, $A_{(i,i)} := A_i + B_i K_i$, and, under Interpretation 1, $\forall j \in \mathbb{N}_{(N,i)}$, $A_{(i,j)} := (I + B_i L_i) C_{(i,j)}$, while under Interpretation 2, $\forall j \in \mathbb{N}_{(N,i)}$, $A_{(i,j)} := (I + B_i L_{(i,j)}) C_{(i,j)}$. With these definitions, we utilize the form (7) for the analysis throughout this note. We recall the classical definition in set invariance [3, 7, 8]:

Definition 1 *A set Ω is a positively invariant set for the system $x^+ = Ax$ and constraint set $\mathbb{X}$ if and only if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ it holds that $Ax \in \Omega$ (i.e. $A\Omega \subseteq \Omega \subseteq \mathbb{X}$).*

The most relaxed set invariance and stability notions can be obtained by utilizing Definition 1 and considering the augmented form of the system in (7), namely:

$$x^+ = Ax, \tag{8}$$

where $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is the matrix composed from the matrices $A_{(i,j)}$ specified as in (7). The state and control constraints (2) are taken into account by introducing the constraint set $\mathbb{X}$ on the variables of the system (8). The constraint set $\mathbb{X}$ takes the form

$$\begin{aligned} \mathbb{X} := \{x \ : x \in \mathbb{X}_1 \times \mathbb{X}_2 \times \ldots \times \mathbb{X}_N \text{ and } \forall i \in \mathbb{N}_{[1,N]}, \\ K_i x_i + \sum_{j \in \mathbb{N}_{(N,i)}} K_{(i,j)} x_j \in \mathbb{U}_i\}, \end{aligned} \tag{9}$$

where under Interpretation 1 and when the linear control rules (5) are employed $K_{(i,j)} := L_i C_{(i,j)}$ while under Interpretation 2 and when the linear control rules (6) are employed $K_{(i,j)} := L_{(i,j)} C_{(i,j)}$. Within this setting strict stability of the system (8) reduces to the requirement that $\rho(A) < 1$. In principle, standard set invariance methods can be utilized for the computation of the maximal positively invariant set, say Ω_∞, for the system (8) and constraint set (9). However, such a setting does not permit independent operation of the subsystems (7). Furthermore, the corresponding set invariance computations have to be carried out in $\mathbb{R}^n$ which in the case of large scale systems induces inevitably serious computational obstacles.

The requirement for the independent operation of the set of N subsystems in (7) leads naturally to the induced, independent, set–dynamics given, for all $i \in \mathbb{N}_{[1:N]}$ and all $X = (X_1, X_2, \ldots, X_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$,

$$\begin{aligned} &X_i^+ = F_i(X), \text{ with} \\ &F_i(X) := A_{(i,i)} X_i \oplus \bigoplus_{j \in \mathbb{N}_{(N,i)}} A_{(i,j)} X_j \end{aligned} \tag{10}$$

Before proceeding, let also, for all $i \in \mathbb{N}_{[1:N]}$ and all $X = (X_1, X_2, \ldots, X_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$,

$$U_i(X) := K_i X_i \oplus \bigoplus_{j \in \mathbb{N}_{(N,i)}} K_{(i,j)} X_j \tag{11}$$

A natural and attractive alternative is to look for a positively invariant set of a particular form, namely to aim for the characterization and computation of an invariant set Ω taking the form $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_N$ where, for all $i \in \mathbb{N}_{[1:N]}$, $\Omega_i \subset \mathbb{R}^{n_i}$. Within this setting, a possible notion of set invariance is given by:

Definition 2 *A collection of sets* $\mathbf{\Omega} := \{\Omega_i \ : \ i \in \mathbb{N}_{[1:N]}\}$ *is an invariant collection of sets for the system* (7) *and constraint sets* (2) *if and only if, for all* $i \in \mathbb{N}_{[1:N]}$*,:*

$$\Omega_i \subseteq \mathbb{X}_i, \ U_i((\Omega_1, \Omega_2, \ldots, \Omega_N)) \subseteq \mathbb{U}_i, \ \text{and,} \tag{12a}$$
$$F_i((\Omega_1, \Omega_2, \ldots, \Omega_N)) \subseteq \Omega_i \tag{12b}$$

This alternative definition is computationally attractive as the overall invariant set $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_N$ can be constructed from the collection of sets $\mathbf{\Omega} := \{\Omega_i \ : \ i \in \mathbb{N}_{[1:N]}\}$ and each of sets Ω_i can, in principle, be computed locally. Nevertheless, the requirements in (12) complicate significantly the questions of the existence as well as the detection of a suitable collection of sets $\mathbf{\Omega} := \{\Omega_i \ : \ i \in \mathbb{N}_{[1:N]}\}$. Furthermore, even though this notion seems to be natural within the decentralized framework, it is in fact naive and overly conservative because the condition in (12b) is rather strong and, in fact, too much to ask for.

A more flexible and non–naive notion is possible as offered by the following definition concerned with the properties of a suitable family of sets:

Definition 3 *A family of sets* $\mathcal{X} \subseteq 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$ *is said to be an invariant family of sets for the system* (7) *and constraint sets* (2) *if and only if, for all* $X = (X_1, X_2, \ldots, X_N) \in \mathcal{X}$ *and all* $i \in \mathbb{N}_{[1:N]}$*,:*

$$X_i \subseteq \mathbb{X}_i, \ U_i((X_1, X_2, \ldots, X_N)) \subseteq \mathbb{U}_i, \tag{13a}$$
$$F_i((X_1, X_2, \ldots, X_N)) \subseteq X_i^+, \ \text{and,} \tag{13b}$$
$$X^+ = (X_1^+, X_2^+, \ldots, X_N^+) \in \mathcal{X}. \tag{13c}$$

The invariance notions of Definition 3 are compatible with the induced, independent, set–dynamics (10) and are, in fact, sufficiently general to capture the classical notions specified in Definition 1. Indeed, for any positively invariant set Ω satisfying Definition 1, it is possible to define the corresponding family of invariant sets $\mathcal{X}$ by forming it from all points $x = (x_1, x_2, \ldots, x_N) \in \Omega$ (i.e. by setting, for sets $X \in \mathcal{X}$, $X = (X_1, X_2, \ldots, X_N) := (\{x_1\}, \{x_2\}, \ldots, \{x_N\})$ with $x = (x_1, x_2, \ldots, x_N) \in \Omega$). Definition 3 also reveals an interplay between the quality of the attainable invariant sets and the information exchange amongst the subsystems in (7). However, the exact set invariance problem reduces to the characterization and computation of invariant families of sets. Even though it is possible to analyze such an exact problem, the corresponding notion is computationally intractable and, hence, motivates the introduction and analysis of the practicable set invariance which allows for the trade–off between naive notions offered in Definition 2 and general notions of Definition 3. This trade–off is attained by considering a family of sets $\mathcal{S}(\mathbb{S}, \Theta)$, given by

$$\mathcal{S}(\mathbb{S}, \Theta) := \{(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \ : \ \theta \in \Theta\}, \tag{14}$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}_+^N$, $\Theta \subseteq \mathbb{R}_+^N$ and $\forall i \in \mathbb{N}_{[1:N]}, \ S_i \in 2^{\mathbb{R}^{n_i}}$, and by introducing the following notion of practical set invariance:

Definition 4 *Given a collection of sets* $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$ *with* $(S_1, S_2, \ldots, S_N) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$ *and a set* $\Theta \subseteq \mathbb{R}_+^N$*, the family of sets* $\mathcal{S}(\mathbb{S}, \Theta)$ *specified by* (14) *is said to be an invariant family of sets for the system* (7) *and constraint sets* (2) *if and only if, for all* $i \in \mathbb{N}_{[1:N]}$ *and all* $(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$*,:*

$$\theta_i S_i \subseteq \mathbb{X}_i, \ U_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)) \subseteq \mathbb{U}_i, \tag{15a}$$
$$F_i((\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)) \subseteq \theta_i^+ S_i, \ \text{and,} \tag{15b}$$
$$(\theta_1^+ S_1, \theta_2^+ S_2, \ldots, \theta_N^+ S_N) \in \mathcal{S}(\mathbb{S}, \Theta). \tag{15c}$$

The problem of our interest is motivated by Definition 4:

Problem 1 *Given a collection of sets* $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$*, detect a collection of functions* $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ *with* $\mu_i(\cdot) \ : \ \mathbb{R}_+^N \to \mathbb{R}_+$*, and a set* $\Theta \subseteq \mathbb{R}_+^N$ *ensuring that the family of sets* $\mathcal{S}(\mathbb{S}, \Theta)$ *given by* (14) *is such that for all* $i \in \mathbb{N}_{[1:N]}$ *and all* $(\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$ *conditions* (15) *are satisfied with* $\forall i \in \mathbb{N}_{[1:N]}, \ \theta_i^+ = \mu_i(\theta)$*. Furthermore, examine the stability properties of the system:*

$$\forall i \in \mathbb{N}_{[1:N]}, \ \theta_i^+ = \mu_i(\theta), \tag{16}$$

relative to the set Θ *and relate them to the stability properties of the dynamics specified in* (7)*.*

3 Practical Set Invariance and Stability

Problem 1 is addressed in two steps and its solution is obtained under a natural assumption on the underlying collection of sets $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$:

Assumption 2 *The collection of sets $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$ is a PC–collection of sets.*

Remark 1 *Note that a direct use of the algebra of convex sets [12, 13] yields the fact that, under Assumption 1, the conditions that $\forall i \in \mathbb{N}_{[1:N]},\ \theta_i S_i \subseteq \mathbb{X}_i$ and $\theta_i K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} K_{(i,j)}\theta_j S_j \subseteq \mathbb{U}_i$ are equivalent to the requirements that $\forall i \in \mathbb{N}_{[1:N]},\ \theta_i \operatorname{convh}(S_i) \subseteq \mathbb{X}_i$ and $\theta_i K_i \operatorname{convh}(S_i) \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} K_{(i,j)}\theta_j \operatorname{convh}(S_j) \subseteq \mathbb{U}_i$. Likewise, the conditions that $\forall i \in \mathbb{N}_{[1:N]},\ \theta_i A_{(i,i)} S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)}\theta_j S_j \subseteq \theta_i^+ S_i$ imply the relations $\forall i \in \mathbb{N}_{[1:N]},\ \theta_i A_{(i,i)} \operatorname{convh}(S_i) \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)}\theta_j \operatorname{convh}(S_j) \subseteq \theta_i^+ \operatorname{convh}(S_i)$ (these requirements are, in fact, equivalent when involved sets S_i are convex). Consequently, Assumption 2 is invoked without loss of generality in an appropriate sense.*

The first step is to specify appropriate collection of functions $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$, where $\forall i \in \mathbb{N}_{[1:N]},\ \mu_i(\cdot) : \ \mathbb{R}^N_+ \to \mathbb{R}_+$, ensuring the satisfaction of the condition (15b). The second is the detection of appropriate set $\Theta \subseteq \mathbb{R}^N_+$ leading together with the collection of functions $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ to the solution of the considered problem.

For a given PC–collection of sets $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$, we can detect the collection of exact functions $\{\mu_i^e(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ where $\forall i \in \mathbb{N}_{[1:N]},\ \mu_i^e(\cdot) : \ \mathbb{R}^N_+ \to \mathbb{R}_+$ is given, for all $\theta = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^N_+$, by:

$$\mu_i^e(\theta) := \min_{\mu\geq 0}\{\mu \ : \ \bigoplus_{j\in\mathbb{N}_{[1:N]}} \theta_j A_{(i,j)} S_j \subseteq \mu S_i\}. \tag{17}$$

The relevant properties of the collection of functions $\{\mu_i^e(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ are discussed by:

Proposition 1 *Suppose Assumption 2 holds. Then, for all $\forall i \in \mathbb{N}_{[1:N]}$, the functions $\mu_i^e(\cdot) : \ \mathbb{R}^N_+ \to \mathbb{R}_+$ given by* (17) *are sublinear functions.*

Motivated by computational aspects and the fact that the collection of exact functions $\{\mu_i^e(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ is a collection of sublinear functions we utilize, for analysis and consequent computations, the collection of linear functions $\{\mu_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ given, for all $i \in \mathbb{N}_{[1:N]}$ and all $\theta \in \mathbb{R}^N_+$, by:

$$\begin{aligned}\mu_i(\theta) &:= \sum_{j\in\mathbb{N}_{[1:N]}} \mu_{(i,j)}\theta_j \text{ with } \forall (i,j) \in \mathbb{N}_{[1:N]} \times \mathbb{N}_{[1:N]},\\ \mu_{(i,j)} &:= \min_{\mu\geq 0}\{\mu \ : \ A_{(i,j)} S_j \subseteq \mu S_i\}.\end{aligned} \tag{18}$$

Clearly, for all $i \in \mathbb{N}_{[1:N]}$ and all $\theta \in \mathbb{R}^N_+$, $\mu_i^e(\theta) \leq \mu_i(\theta)$. We proceed and introduce the dynamics of the θ variable:

$$\theta^+ = M\theta, \tag{19}$$

where $M \in \mathbb{R}^{N\times N}_+$ is the matrix composed from the scalars $\mu_{(i,j)} \in \mathbb{R}_+,\ (i,j) \in \mathbb{N}_{[1:N]} \times \mathbb{N}_{[1:N]}$. In order to ensure the satisfaction of the conditions (15a), we invoke the constraints on the θ variable:

$$\begin{aligned}\Theta_0 := \{\theta \in \mathbb{R}^N_+ \ : \ &\forall i \in \mathbb{N}_{[1:N]},\ \theta_i S_i \subseteq \mathbb{X}_i \text{ and}\\ &\theta_i K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j K_{(i,j)} S_j \subseteq \mathbb{U}_i\}.\end{aligned} \tag{20}$$

It is important to note that, under the given assumptions, the resulting set Θ_0 is compact and convex:

Lemma 1 *Suppose Assumptions 1 and 2 hold. Then the set Θ_0 given by* (20) *is a convex, compact and full–dimensional subset of $\mathbb{R}^N_+$ that contains the origin.*

We now invoke an assumption on the set Θ permitting us to establish the practical set invariance property of the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ given by (14).

Assumption 3 *The set Θ is a convex and compact subset of $\mathbb{R}^N_+$ such that $0 \in \Theta \subseteq \Theta_0$ and $\forall \theta \in \Theta,\ M\theta \in \Theta$, i.e. the set $\Theta \subseteq \mathbb{R}^N_+,\ 0 \in \Theta$ is a convex, compact and positively invariant set for the dynamics* (19) *subject to constraints* (20)*.*

Remark 2 *If Assumptions 1 and 2 hold then Assumption 3 is invoked without loss of generality. In this case, the standard set recursion given by:*

$$\forall k \in \mathbb{N},\ \Theta_{k+1} := M^{-1}\Theta_k \bigcap \Theta_0, \tag{21}$$

where M and Θ_0 are given by (19) *and* (20) *respectively, results in the monotonically non–increasing sequence of convex and compact sets $\{\Theta_k\}_{k\in\mathbb{N}}$ that admits the limit with respect to the Hausdorff distance, say Θ_∞ (which is itself a non–empty convex and compact set). In fact, this limit is given by:*

$$\Theta_\infty = \bigcap_{k\in\mathbb{N}} \Theta_k, \tag{22}$$

and is the maximal positively invariant set for the system (19) *subject to constraints* (20)*.*

The following proposition addresses the issue of practical set invariance notions.

Proposition 2 *Suppose Assumptions 1, 2 and 3 hold. Then the family of sets $\mathcal{S}(\mathbb{S},\Theta)$ given by* (14) *is an invariant family of sets.*

These practical notions, for a given PC–collection of sets $\mathbb{S} = \{S_i \ :\ i \in \mathbb{N}_{[1:N]}\}$ require merely the detection of the collection of linear functions $\{\mu_i(\cdot) \ :\ i \in \mathbb{N}_{[1:N]}\}$ given by (17) and the corresponding positively invariant set Θ satisfying Assumption 3. As indicated in Remark 2, Assumption 3 is invoked without loss of generality, albeit it is possible to encounter the cases when the corresponding maximal positively invariant set Θ_∞ (and hence any positively invariant set) reduces to a trivial singleton set $\{0\}$. Such a possibility is ruled out under an additional and reasonable assumption on the dynamics specified in (19):

Assumption 4 *The matrix M inducing the dynamics in* (19) *is strictly stable, i.e. $\rho(M) < 1$.*

Under this assumption we can ensure that the set Θ and the corresponding family of sets $\mathcal{S}(\mathbb{S},\Theta)$ are non–trivial:

Proposition 3 *Suppose Assumptions 1–4 hold. Then: (i) there exists a non–trivial set Θ satisfying Assumption 3, and, (ii) for any such set Θ, the corresponding family of sets $\mathcal{S}(\mathbb{S},\Theta)$ given by* (14) *is a non–trivial invariant family of sets.*

Remark 3 *A direct modification of the standard results [8, 9] implies that, under Assumptions 1, 2 and 4, the maximal positively invariant set Θ_∞ in* (22) *is finitely determined. Namely, there exists a finite integer k^* such that $\Theta_{k^*} = \Theta_{k^*+1}$, where sets $\Theta_k,\ k \in \mathbb{N}$ are given as in* (21)*, and, in turn, $\Theta_\infty = \Theta_{k^*}$. When constraint sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are, in addition, polytopic then the set Θ_0 in* (20) *is a non–trivial polytope. In this case, the maximal positively invariant set Θ_∞ is also a non–trivial polytope and it can be computed using the standard techniques [8, 11].*

We turn now our attention to the convergence issues. Before proceeding, let $\mathbf{X}(X_0)$ denote, for any $X_0 = (X_{(0;1)}, X_{(0;2)}, \ldots, X_{(0;N)}) \in 2^{\mathbb{R}^{n_1}} \times 2^{\mathbb{R}^{n_2}} \times \ldots \times 2^{\mathbb{R}^{n_N}}$ the sequence $\{X_k = (X_{(k;1)}, X_{(k;2)}, \ldots, X_{(k;N)})\}_{k\in\mathbb{N}}$ generated by (10), i.e. for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$,

$$X_{(k+1;i)} = F_i(X_k), \tag{23}$$

where the maps $F_i(\cdot),\ i \in \mathbb{N}_{[1:N]}$ are given by (10). Similarly, let $\mathbf{Y}(Y_0)$ denote, for any initial condition $Y_0 = (\theta_{(0;1)}S_1, \theta_{(0;2)}S_2, \ldots, \theta_{(0;N)}S_N)$ with $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \ldots, \theta_{(0;N)}) \in \mathbb{R}^N_+$, the sequence of parametrized sets $\{Y_k = (\theta_{(k;1)}S_1, \theta_{(k;2)}S_2, \ldots, \theta_{(k;N)}S_N)\}_{k\in\mathbb{N}}$ with $\theta_k = (\theta_{(k;1)}, \theta_{(k;2)}, \ldots, \theta_{(k;N)}) \in \mathbb{R}^N_+$ generated by (19), i.e. for all $k \in \mathbb{N}$,

$$\theta_{k+1} = M\theta_k. \tag{24}$$

We can now state our third main result, leading towards practical set invariance of the whole interconnected system:

Theorem 1 *Suppose Assumptions 1–4 hold. Consider the family of sets $\mathcal{S}(\mathbb{S},\Theta)$ given by* (14) *and any sequence $\mathbf{Y}(Y_0)$ generated by* (24) *with $Y_0 \in \mathcal{S}(\mathbb{S},\Theta)$. Then, for all $k \in \mathbb{N}$, (i) $Y_k \in \mathcal{S}(\mathbb{S},\Theta)$, (ii) $\sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, Y_{(k;i)}, \{0\}) \leq a^k b \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, Y_{(0;i)}, \{0\})$ for some scalars $a \in [0,1)$ and $b \in (0,\infty)$, and, (iii) $\forall i \in \mathbb{N}_{[1:N]}$, $H(L_i, Y_{(k;i)}, \{0\}) \to 0$ as $k \to \infty$.*

A relevant consequence of Theorem 1 is:

Corollary 1 *Suppose Assumptions 1–4 hold. Consider the family of sets $\mathcal{S}(\mathbb{S},\Theta)$ given by* (14) *and any two sequence $\mathbf{X}(X_0)$ and $\mathbf{Y}(Y_0)$ generated by* (23) *and* (24) *with, for all $i \in \mathbb{N}_{[1:N]}$, $X_{(0;i)} \subseteq Y_{(0;i)}$ for some $Y_0 \in \mathcal{S}(\mathbb{S},\Theta)$. Then, for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}_{[1:N]}$, (i) $X_{(k;i)} \subseteq Y_{(k;i)}$, (ii) $X_{(k;i)} \subseteq \mathbb{X}_i$ and $U_i(X_k) \subseteq \mathbb{U}_i$, where the maps $U_i(\cdot)$ are given as in* (11)*, and, (iii) $h(L_i, X_{(k;i)}, \{0\}) \to 0$ as $k \to \infty$.*

Remark 4 *Theorem 1 and Corollary 1 allow us now to state covergence and stability properties of the interconnected subsystems, i.e. for any actual state trajectory generated by* (7)*, i.e. for all* $k \in \mathbb{N}$ *and all* $i \in \mathbb{N}_{[1:N]}$, $x_{(k+1;i)} = A_{(i,i)}x_{(k;i)} + \sum_{j\in\mathbb{N}_{(N,i)}} A_{(i,j)}x_{(k;j)}$*, where* $x_0 = (x_{(0;1)}, x_{(0;2)}, \dots, x_{(0;N)})$ *with, for all* $i \in \mathbb{N}_{[1:N]}$, $x_{(0;i)} \in \theta_{(0;i)}S_i$ *and* $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \dots, \theta_{(0;N)}) \in \Theta$*, it holds that, for all* $k \in \mathbb{N}$ *all* $i \in \mathbb{N}_{[1:N]}$*,:*

$$\begin{aligned}
&x_{(k;i)} \in \theta_{(k;i)}S_i \subseteq \mathbb{X}_i,\\
&K_i x_{(k;i)} + \sum_{j\in\mathbb{N}_{(N,i)}} K_{(i,j)}x_{(k;j)} \in U_i((\theta_{(k;1)}S_1, \dots, \theta_{(k;N)}S_N)),\\
&\text{and, } U_i((\theta_{(k;1)}S_1, \dots, \theta_{(k;N)}S_N)) \subseteq \mathbb{U}_i,
\end{aligned}$$

where $\{\theta_k\}_{k\in\mathbb{N}}$ *is generated by* (24)*. Furthermore, any actual state sequence* $\{x_k = (x_{(k;1)}, x_{(k;2)}, \dots, x_{(k;N)})\}_{k\in\mathbb{N}}$ *converges exponentially fast, in a stable manner, to* $(0, 0 \dots, 0)$*. In fact, in view of Theorem 1, the origin is an exponentially stable attractor for the dynamics* (7) *subject to constraints* (9) *with the basin of attraction induced by the set* Θ *(and depending on the set* Θ*). More importantly, the individual subsystems do not require the exact knowledge of the initial conditions of the other subsystems but merely that they belong to appropriate sets; in other words the only requirement for the safe and independent operation of the dynamics* (7) *is the condition that for all* $i \in \mathbb{N}_{[1:N]}$, $x_{(0;i)} \in \theta_{(0;i)}S_i$ *for some* $\theta_0 = (\theta_{(0;1)}, \theta_{(0;2)}, \dots, \theta_{(0;N)}) \in \Theta$*.*

4 A Simple Control Synthesis & Brief Computational Remarks

In view of Interpretations 1 and 2 and due to the static and linear structure of the employed control rules, the following prototype *max–min* infinite–horizon control problem, $\mathbb{P}_{\text{max–min}}$, provides an appropriate way to design the linear control rules specified in (5) and (6) as well as to detect the corresponding collection of sets $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$:

$$V^*(x, w) = \min_u V(x, u, w), \text{ with,} \tag{25a}$$
$$V(x, u, w) = \ell(x, u, w) + V^0(Ax + Bu + Dw), \tag{25b}$$
$$u^*(x, w) = \arg\min_u V(x, u, w), \tag{25c}$$
$$V^0(x) = \max_w V^*(x, w), \text{ and,} \tag{25d}$$
$$w^0(x) = \arg\max_w V^*(x, w). \tag{25e}$$

It is well–known [5] that when $\ell(\cdot, \cdot, \cdot)$ is given by:

$$\ell(x, u, w) := x'Qx + u'Ru - \gamma^2 w'w \tag{26}$$

with $Q \in \mathbb{R}^{n\times n}$, $Q = Q' > 0$, $R \in \mathbb{R}^{m\times m}$, $R = R' > 0$ and when (A, B) is stabilizable and $(A, Q^{\frac{1}{2}})$ is detectable, then there exists a finite scalar γ^* such that for all $\gamma \geq \gamma^*$ the relations (25) result in the solvable generalized H_∞ algebraic Riccati equation and admit the solution:

$$V^0(x) = x'Px \tag{27a}$$
$$u^*(x, w) = Kx + Lw, \text{ and, } w^0(x) = Tx, \tag{27b}$$

for suitable matrices $P = P' > 0$, K, L, and T of compatible dimensions. It is also well–known that, under the conditions indicated above,

$$\begin{aligned}
&\min_u \max_w (\ell(x, u, w) + V^0(Ax + Bu + Dw)) = \\
&\max_w \min_u (\ell(x, u, w) + V^0(Ax + Bu + Dw))
\end{aligned} \tag{28}$$

and that the linear control rule:

$$u^0(x) = (K + LT)x \tag{29}$$

guarantees the performance index in (28). Returning to our setting in (1), it is reasonable for the i^{th} controller to consider the uncertain system:

$$x^+ = A_i x_i + B_i u_i + D_i w_i \tag{30}$$

where the disturbance w_i and matrix D_i are specified accordingly to the considered case arising under Interpretation 1 or 2 (i.e. $D_i = I$ and $w_i = \sum_{j\in\mathbb{N}_{(N,i)}} C_{(i,j)}x_j$ in the case when Interpretation 1 is valid and $D_i = (I\ I\ \dots I)$ and $w_i = (C_{(i,1)}x_1, \dots, C_{(i,i-1)}x_{i-1}, C_{(i,i+1)}x_{i+1}, \dots C_{(i,N)}x_N)$ under Interpretation 2). Within this framework, the i^{th} decision maker can construct the linear control rules specified in (5) or (6) by solving the local version of the *max–min* infinite–horizon control problem, $\mathbb{P}_{\text{max–min}}$ specified in (25) and (26) (in which the matrices A, B, D, Q, R and the scalar γ are replaced by A_i, B_i, D_i, Q_i, R_i and γ_i). Under standard assumptions [5] on the local data (A_i, B_i, D_i, Q_i, R_i and γ_i), the solution to the i^{th} local *max–min* infinite–horizon control problem, $\mathbb{P}_{\max-\min}$ yields the collection of the value functions and control rules given, for all $i \in \mathbb{N}_{[1:N]}$, by:

$$V_i^0(x_i) = x_i' P_i x_i \tag{31}$$

and

$$u_i^*(c_i(x)) = K_i x_i + L_i \sum_{j \in \mathbb{N}_{(N,i)}} C_{(i,j)} x_j \tag{32}$$

when Interpretation 1 is valid and

$$u_i^*(d_i(x)) = K_i x_i + \sum_{j \in \mathbb{N}_{(N,i)}} L_{(i,j)} C_{(i,j)} x_j \tag{33}$$

under Interpretation 1, for suitable matrices $P_i = P_i' > 0$, K_i, L_i, and $L_{(i,j)}$ of compatible dimensions (and where $c_i(\cdot)$ and $d_i(\cdot)$ are given as in (3) and (4)).

The collection of the value functions $\{V_i^0(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ provides a natural choice for the corresponding collection of sets $\mathbb{S} = \{S_i \ : \ i \in \mathbb{N}_{[1:N]}\}$. In particular, the sets S_i, $i \in \mathbb{N}_{[1:N]}$ can be chosen to be the ellipsoidal sets given, for all $i \in \mathbb{N}_{[1:N]}$, by:

$$S_i := \{x_i \ : \ x_i' P_i x_i \leq 1\}. \tag{34}$$

Furthermore, in this case, the matrix M inducing the dynamics in (19) can be easily constructed by evaluating the smallest non–negative scalars $\mu_{(i,j)}$ satisfying

$$\forall i \in \mathbb{N}_{[1:N]}, \ A_{(i,i)}' P_i A_{(i,i)} \leq \mu_{(i,i)}^2 P_i, \tag{35}$$

where $A_{(i,i)} := (A_i + B_i K_i)$, and, for all $i \in \mathbb{N}_{[1:N]}$,

$$\forall j \in \mathbb{N}_{(N,i)}, \ A_{(i,j)}' P_j A_{(i,j)} \leq \mu_{(i,j)}^2 P_i \tag{36}$$

where $A_{(i,j)} := (I + B_i L_i) C_{(i,j)}$ under Interpretation 1 or $A_{(i,j)} := (I + B_i L_{(i,j)}) C_{(i,j)}$ under Interpretation 2. In addition, when the constraint sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are polytopes:

$$\mathbb{X}_i := \{x_i \ : \ \forall l_i \in \mathbb{N}_{[1:q_i]}, \ \phi_{(i,l_i)}' x_i \leq 1\} \text{ and} \tag{37a}$$
$$\mathbb{U}_i := \{u_i \ : \ \forall p_i \in \mathbb{N}_{[1:r_i]}, \ \psi_{(i,p_i)}' u_i \leq 1\}, \tag{37b}$$

with $\forall l_i \in \mathbb{N}_{[1:q_i]}, \ \phi_{(i,l_i)} \in \mathbb{R}^{n_i}$ and $\forall p_i \in \mathbb{N}_{[1:r_i]}, \ \psi_{(i,p_i)} \in \mathbb{R}^{m_i}$, then the set Θ_0 specified in (20) is a polytope:

$$\begin{aligned}\Theta_0 := \{\theta \ : \ &\forall i \in \mathbb{N}_{[1:N]}, \ \forall l_i \in \mathbb{N}_{[1:q_i]}, \ h_{(i,i,l_i)} \theta_i \leq 1, \text{ and,} \\ &\forall p_i \in \mathbb{N}_{[1:r_i]}, \ h_{(i,i,p_i)} \theta_i + \sum_{j \in \mathbb{N}_{(N,i)}} h_{(i,j,p_i)} \theta_j \leq 1\},\end{aligned}$$

with $h_{(i,i,l_i)} := (\phi_{(i,l_i)}' P_i^{-1} \phi_{(i,l_i)})^{\frac{1}{2}}$, $h_{(i,i,p_i)} := (\psi_{(i,p_i)}' K_i P_i^{-1} K_i' \psi_{(i,p_i)})^{\frac{1}{2}}$ and $h_{(i,j,p_i)} := (\psi_{(i,p_i)}' K_{(i,j)} P_j^{-1} K_{(i,j)}' \psi_{(i,p_i)})^{\frac{1}{2}}$. As already mentioned in Remark 3, the standard techniques and tools can be employed for the computation of the sets Θ_k, $k \in \mathbb{N}$ given by (21) as well as the maximal positively invariant set Θ_∞ given by (22). We remark that in this setting, under assumption that $\rho(M) < 1$, the maximal positively invariant set is finitely determined and is a non–trivial polytope.

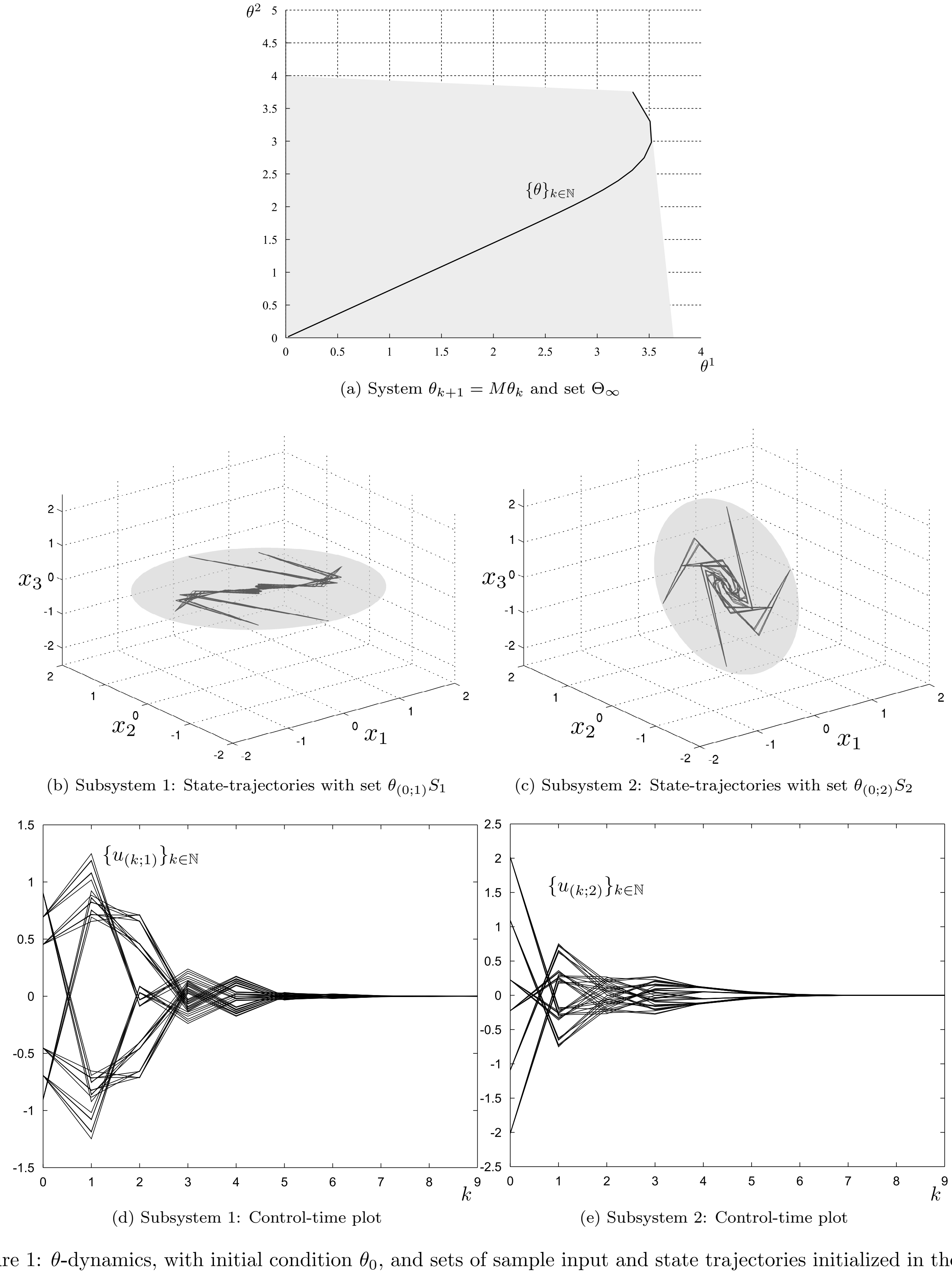
Remark 5 *An alternative way for the design of the local linear controllers (i.e. matrices K_i, L_i, and $L_{(i,j)}$) and the corresponding quadratic functions $\{V_i(\cdot) \ : \ i \in \mathbb{N}_{[1:N]}\}$ (i.e. matrices $P_i = P_i' > 0$) is to utilize the systematic design methods based on the linear matrix inequality which are thoroughly investigated in [15, 16].*

5 Illustrative Example

We consider a six dimensional system consisting of two interconnected systems:

$$A_1 = \begin{pmatrix} 0 & 0.5 & 1 \\ -0.5 & -1 & 0 \\ 1 & -0.5 & 0.5 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ C_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & 0.5 & 1 \\ -0.5 & 1 & 0 \\ 0 & -0.5 & 0.5 \end{pmatrix}, \ B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ C_{21} = \begin{pmatrix} 0.1 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}.$$

The constraint sets are $\mathbb{X}_1 = \mathbb{X}_2 = \{x \in \mathbb{R}^3 \ : \ |x|_\infty \leq 5\}$ and $\mathbb{U}_1 = \mathbb{U}_2 = \{u \in \mathbb{R} \ : \ |u| \leq 2\}$. The max–min, static, linear control rules are obtained as described in Section 4 with $Q_1 = Q_2 = 1.5I$, $R_1 = R_2 = 1$ and $\gamma_1 = 3.4912$ and $\gamma_2 = 4.5654$. These control rules are described via the matrices $K_1 = (-0.4262 \ 0.6104 \ -0.3906)$, $L_1 = (-0.0891 \ -0.3535 \ -0.6029)$, $K_2 = (-0.9142 \ 0.2425 \ -0.8192)$ and $L_2 = (-0.6872 \ 0.4541 \ -0.2641)$. The collection of sets $\{S_1, S_2\}$ is obtained according to (34) from the corresponding solutions to generalized H_∞ algebraic Riccati equations. The corresponding matrix M obtained from (35)–(36) is strictly stable. The set Θ is chosen to be the maximal positively invariant set Θ_∞ as indicated in Remark 2.



(a) System $\theta_{k+1} = M\theta_k$ and set Θ_∞

(b) Subsystem 1: State-trajectories with set $\theta_{(0;1)}S_1$

(c) Subsystem 2: State-trajectories with set $\theta_{(0;2)}S_2$

(d) Subsystem 1: Control-time plot

(e) Subsystem 2: Control-time plot

Figure 1: θ-dynamics, with initial condition θ_0, and sets of sample input and state trajectories initialized in the sets $\theta_{(0;i)}S_i$, with $i \in \{1,2\}$.

In the top part of Figure 1 we show the set Θ_∞ and the sequence $\{\theta_k\}_{k\in\mathbb{N}}$ for $\theta_0 = (3.3416, 3.7521)$. A set of state and control time plots for a range of initial conditions $x_{(0,1)} \in \theta_{(0,1)}S_1$ and $x_{(0,2)} \in \theta_{(0,2)}S_2$ is also shown in Figure 1. As expected (in view of Theorem 1, Corollary 1 and Remark 4), the variables of both subsystems satisfy constraints and converge exponentially fast to the origin.

6 Concluding Remarks

In this paper we discussed exact and practical set invariance notions for decentralized discrete time systems which are physically interconnected. The exact set invariance notions were formalized by employing invariant families of sets. The practical set invariance notion was achieved by considering a parameterized invariant family of sets. It was pointed out that, under mild assumptions, the introduced practical notions are computationally tractable and provide guarantees for safe, stable and independent operation of the subsystems forming the overall decentralized system.

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Appendix

Proof of Proposition 1:

By construction, $\mu_i^e(\cdot) : \mathbb{R}_+^N \to \mathbb{R}_+$ and $\forall i \in \mathbb{N}_{[1:N]}$, $\mu_i^e(0) = 0$. For any $\lambda \in \mathbb{R}_+$ and any $\theta \in \mathbb{R}_+^N$ we have $\bigoplus_{j\in\mathbb{N}_{[1:N]}} \lambda\theta_j A_{(i,j)} S_j = \lambda(\bigoplus_{j\in\mathbb{N}_{[1:N]}} \theta_j A_{(i,j)} S_j)$ and, hence, $\mu_i^e(\lambda\theta) \leq \lambda\mu_i^e(\theta)$. But, $\mu_i^e(\lambda\theta) < \lambda\mu_i^e(\theta)$ is, in view of (17), not possible without a contradiction on the optimality of $\mu_i^e(\theta)$ and, hence, $\mu_i^e(\lambda\theta) = \lambda\mu_i^e(\theta)$. Likewise, for any $\theta^1 \in \mathbb{R}_+^N$ and $\theta^2 \in \mathbb{R}_+^N$ it holds that $\bigoplus_{j\in\mathbb{N}_{[1:N]}} (\theta_j^1 + \theta_j^2) A_{(i,j)} S_j = (\bigoplus_{j\in\mathbb{N}_{[1:N]}} \theta_j^1 A_{(i,j)} S_j) \oplus (\bigoplus_{j\in\mathbb{N}_{[1:N]}} \theta_j^2 A_{(i,j)} S_j) \subseteq \mu_i^e(\theta^1) S_i \oplus \mu_i^e(\theta^2) S_i = (\mu_i^e(\theta^1) + \mu_i^e(\theta^2)) S_i$ and, hence, $\mu_i^e(\theta^1 + \theta^2) \leq \mu_i^e(\theta^1) + \mu_i^e(\theta^2)$. The functions $\mu_i^e(\cdot) : \mathbb{R}_+^N \to \mathbb{R}_+$, $i \in \mathbb{N}_{[1:N]}$ are sublinear.

Proof of Lemma 1:

Pick any $\theta^1 \in \Theta_0$ and $\theta^2 \in \Theta_0$ and any $\lambda = (\lambda^1, \lambda^2) \in \Lambda$, $\Lambda := \{(\lambda^1, \lambda^2) \in \mathbb{R}_+^2 : \lambda^1 + \lambda^2 = 1\}$. By convexity of $\mathbb{R}_+^N$ it holds that $\theta^\lambda := \lambda^1\theta^1 + \lambda^2\theta^2 \in \mathbb{R}_+^N$. Due to Assumptions 1 and 2, we have that, $\forall i \in \mathbb{N}_{[1:N]}$, $\theta^\lambda S_i = (\lambda^1\theta_i^1 + \lambda^2\theta_i^2) S_i = \lambda^1\theta_i^1 S_i \oplus \lambda^2\theta_i^2 S_i \subseteq \lambda^1\mathbb{X}_i \oplus \lambda^2\mathbb{X}_i = (\lambda^1 + \lambda^2)\mathbb{X}_i = \mathbb{X}_i$. Likewise, $\forall i \in \mathbb{N}_{[1:N]}$, $\theta_i^\lambda K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j^\lambda K_{(i,j)} S_j = \lambda^1(\theta_i^1 K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j^1 K_{(i,j)} S_j) \oplus \lambda^2(\theta_i^2 K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j^2 K_{(i,j)} S_j) \subseteq \lambda^1\mathbb{U}_i \oplus \lambda^2\mathbb{U}_i = (\lambda^1 + \lambda^2)\mathbb{U}_i = \mathbb{U}_i$. Hence, the set Θ_0 is a convex subset of $\mathbb{R}_+^N$. The set Θ_0 is clearly a closed subset of $\mathbb{R}_+^N$. Furthermore, due to Assumptions 1 and 2, the conditions that $\forall i \in N_{[1:N]}$, $\theta_i S_i \subseteq \mathbb{X}_i$ guarantee that Θ_0 is also bounded and, hence, Θ_0 is a compact subset of $\mathbb{R}_+^N$. The fact that $0 \in \Theta_0$ is clear. Let, for all $i \in \mathbb{N}_{[1:N]}$, $\eta_i := \max_{\eta\geq 0}\{\eta : \eta S_i \subseteq \mathbb{X}_i \text{ and } \eta K_i S_i \subseteq \mathbb{U}_i\}$. Clearly, due to Assumptions 1 and 2, it holds that, for all $i \in \mathbb{N}_{[1:N]}$, $0 < \eta_i < \infty$. Let for all $i \in \mathbb{N}_{[1:N]}$, $\bar{\theta}_i := (0, \ldots, 0, \eta_i, 0, \ldots, 0)$ and $\bar{\Theta} := \{0\} \cup \{\bar{\theta}_i : i \in \mathbb{N}_{[1:N]}\}$. By construction, we have that, for all $i \in \mathbb{N}_{[1:N]}$, $\bar{\theta}_i \in \Theta_0$ and that $\text{convh}(\bar{\Theta})$ is a full–dimensional subset of $\mathbb{R}_+^N$ containing the origin. Consequently, since Θ_0 is convex, it follows that $\text{convh}(\bar{\Theta}) \subseteq \Theta_0$ and Θ_0 is a full–dimensional subset of $\mathbb{R}_+^N$. The claimed properties of the set Θ_0 are verified.

Proof of Proposition 2:

Let $X \in \mathcal{S}(\mathbb{S}, \Theta)$, then $X = (\theta_1 S_1, \theta_2 S_2, \ldots, \theta_N S_N)$ for some $\theta \in \Theta$. Since $\Theta \subseteq \Theta_0$ it follows that, for all $i \in \mathbb{N}_{[1:N]}$, $\theta_i S_i \subseteq \mathbb{X}_i$ and $\theta_i K_i S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j K_{(i,j)} S_j \subseteq \mathbb{U}_i$. By Assumption 3 and definition of the functions $\mu_i(\cdot) : i \in \mathbb{N}_{[1:N]}$ in (17), we have $\forall i \in \mathbb{N}_{[1:N]}$, $\theta_i A_{(i,i)} S_i \oplus \bigoplus_{j\in\mathbb{N}_{(N,i)}} \theta_j A_{(i,j)} S_j \subseteq \theta_i^+ S_i$ with $\theta_i^+ = \sum_{j\in\mathbb{N}_{[1:N]}} \mu_{(i,j)}\theta_j$ given as in (18). But, since $\theta \in \Theta$, Assumption 3 guarantees that $\theta^+ := M\theta \in \Theta$ and, consequently, $(\theta_1^+ S_1, \theta_2^+ S_2, \ldots, \theta_N^+ S_N) \in \mathcal{S}(\mathbb{S}, \Theta)$. Hence, the family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ is an invariant family of sets.

Proof of Proposition 3:

(i) Assumption 4 implies the existence of a PC–set in $\mathbb{R}^N$, say L, and a scalar $\lambda \in [0, 1)$ such that $ML \subseteq \lambda L$. Since $M\mathbb{R}_+^N \subseteq \mathbb{R}_+^N$ it follows that $\bar{L} := L \cap \mathbb{R}_+^N$ is a convex, compact and full–dimensional subset of $\mathbb{R}_+^N$ that contains the origin and is such that $M\bar{L} \subseteq \bar{L}$. Lemma 1 yields the fact that the set Θ_0 defined in (20) is a convex, compact and full–dimensional subset of $\mathbb{R}_+^N$ that contains the origin. Hence, there exists a positive scalar d such that the set $d\bar{L}$ is contained in Θ_0 (recall the definition of the set $\text{convh}(\bar{\Theta})$ in the proof of Lemma 1 and the fact that $\text{convh}(\bar{\Theta}) \subseteq \Theta_0$). The set $d\bar{L}$ is a non–trivial set and hence $\Theta = d\bar{L}$ verifies the claim. Note also that, since $M\bar{L} \subseteq \bar{L}$ and, in turn, $Md\bar{L} \subseteq d\bar{L}$, it holds that, for all $k \in \mathbb{N}$, $d\bar{L} \subseteq \Theta_\infty \subseteq \Theta_k$ where sets Θ_k and Θ_∞ are given by (21) and (22) and, hence, the set Θ_∞ is also a non–trivial set verifying the claim. (ii) This fact follows immediately from (i).

Proof of Theorem 1:

(i) By construction, since $\mathcal{S}(\mathbb{S}, \Theta)$ is an invariant family of sets, we have that $Y_k \in \mathcal{S}(\mathbb{S}, \Theta)$ implies $Y_{k+1} \in \mathcal{S}(\mathbb{S}, \Theta)$. Since $Y_0 \in \mathcal{S}(\mathbb{S}, \Theta)$ the principle of mathematical induction yields the first fact. (ii) Due to Assumption 2 there exists a pair of scalars $\eta_1 \in (0, \infty)$ and $\eta_2 \in (0, \infty)$ such that, for all $i \in \mathbb{N}_{[1:N]}$, $\eta_1 L_i \subseteq S_i \subseteq \eta_2 L_i$. In turn, for any $\theta = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}_+^N$, $\eta_1\theta_i L_i \subseteq \theta_i S_i \subseteq \eta_2\theta_i L_i$ and $\eta_1 \sum_{i\in\mathbb{N}_{[1:N]}} \theta_i \leq \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, \theta_i S_i, \{0\}) \leq \eta_2 \sum_{i\in\mathbb{N}_{[1:N]}} \theta_i$. Assumption 4 implies the existence of two scalars $\tilde{a} \in [0, 1)$ and $\tilde{b} \in (0, \infty)$ such that, for all $k \in \mathbb{N}$, $|\theta_k|_L \leq \tilde{a}^k\tilde{b}|\theta_0|_L$. Since $\eta_1 \sum_{i\in\mathbb{N}_{[1:N]}} \theta_i \leq \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, \theta_i S_i, \{0\}) \leq \eta_2 \sum_{i\in\mathbb{N}_{[1:N]}} \theta_i$ there exists a pair of scalars $\eta_3 \in (0, \infty)$ and $\eta_4 \in (0, \infty)$ such that $\eta_3|\theta|_L \leq \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, \theta_i S_i, \{0\}) \leq \eta_4|\theta|_L$. In turn, since for all $k \in \mathbb{N}$, $|\theta_k|_L \leq \tilde{a}^k\tilde{b}|\theta_0|_L$ with $\tilde{a} \in [0, 1)$ and $\tilde{b} \in (0, \infty)$, it follows that there exists a pair of scalars $a \in [0, 1)$ and $b \in (0, \infty)$ such that, for all $k \in \mathbb{N}$, $\sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, Y_{(k;i)}, \{0\}) \leq a^k b \sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, Y_{(0;i)}, \{0\})$. (iii) By (ii) we have $\sum_{i\in\mathbb{N}_{[1:N]}} H(L_i, Y_{(k;i)}, \{0\}) \to 0$ as $k \to \infty$ and, hence, $\forall i \in \mathbb{N}_{[1:N]}$, $H(L_i, Y_{(k;i)}, \{0\}) \to 0$ as $k \to \infty$.