Untersuchung linearer Systeme 2. Ordnung in der Zustandsebene

Betrachtung anhand der Jordan-Normalform:

$$\underline{\xi} = A\,\underline{\xi} \xrightarrow[\underline{y} = X^{-1}\underline{\xi}]{\text{Transf.}} \dot{\underline{y}} = J\underline{y}$$

$\lambda_1 \neq \lambda_2$: $\left\{\begin{array}{l}\dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2\end{array}\right.$

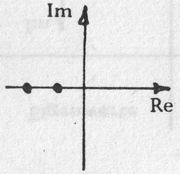
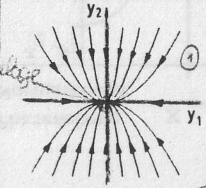
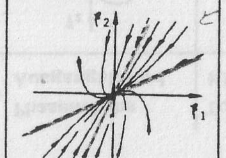
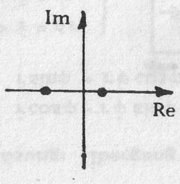
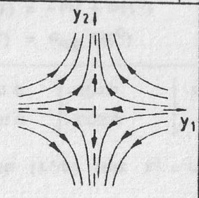
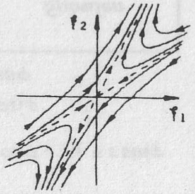
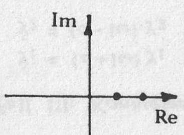
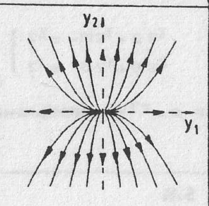
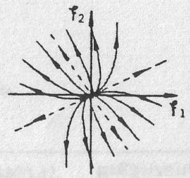
$\lambda_1 = \lambda_2 = \lambda$: $\left\{\begin{array}{l}\dot{y}_1 = \lambda\, y_1 + \delta\, y_2 \\ \dot{y}_2 = \lambda\, y_2\end{array}\right.$, $\delta = 0$ oder 1

Die Gestalt der Phasenkurven hängt von den Eigenwerten λ_1 und λ_2 ab.

Fall I: *Einfache reelle Eigenwerte* $\lambda_1 = \sigma_1$, $\lambda_2 = \sigma_2$

$$\left.\begin{array}{l}\dot{y}_1 = \sigma_1 y_1 \\ \dot{y}_2 = \sigma_2 y_2\end{array}\right\} \frac{dy_2}{dy_1} = \frac{\sigma_2}{\sigma_1}\frac{y_2}{y_1} \rightarrow \int \frac{dy_2}{y_2} = \frac{\sigma_2}{\sigma_1}\int\frac{dy_1}{y_1} + \ln C \rightarrow \ln y_2 = \frac{\sigma_2}{\sigma_1}\ln y_1 + \ln C$$

$$\boxed{\rightarrow \text{Phasenkurven:} \quad y_2 = C\,y_1^{\frac{\sigma_2}{\sigma_1}} \;;\; \frac{\sigma_2}{\sigma_1} > 0\text{: Parabeln} \;;\; \frac{\sigma_2}{\sigma_1} < 0\text{: Hyperbeln}.}$$

Eigenwerte	Gleichungen	Phasenebene Normalkoord. $\xrightarrow{X}$	Phasenebene Ausgangskoord.	Topologische Klassifizierung
	$\dot{y}_1 = -\lvert\sigma_1\rvert y_1$ $\dot{y}_2 = -\lvert\sigma_2\rvert y_2$	Parabel? 1 		asymptotisch stabiler Knoten
	$\dot{y}_1 = -\lvert\sigma_1\rvert y_1$ $\dot{y}_2 = \lvert\sigma_2\rvert y_2$			instabiler Sattel
	$\dot{y}_1 = \lvert\sigma_1\rvert y_1$ $\dot{y}_2 = \lvert\sigma_2\rvert y_2$			instabiler Knoten

Verhalten der Kurven bleibt bei der Abbildung von y nach ξ grundsätzlich gleich: Richtungen bleiben gleich, benachbart. Kurven bleiben benachbart, Geraden bleiben Geraden.

Die Rücktransformation $\underline{\xi} = X\underline{y}$ bewirkt eine *Dreh-Streckung* der $y_1 y_2$-Ebene, wobei die topologische Gestalt der Phasenkurven erhalten bleibt.

Fall II: *Doppelter reeller Eigenwert* $\lambda_1 = \lambda_2 = \sigma$

II.1: *Rangabfall* $d = 2 - Rg(\sigma I - J) = 2$ ⇔ Rangabfall wird durch Transformation nicht verändert ⇒ J kann nur die Form $\begin{bmatrix}\sigma & 0\\ 0 & \sigma\end{bmatrix}$ haben

$$\left.\begin{array}{l}\dot{y}_1 = \sigma\, y_1 \\ \dot{y}_2 = \sigma\, y_2\end{array}\right\} J = \Lambda = \begin{bmatrix}\sigma & 0\\ 0 & \sigma\end{bmatrix}$$

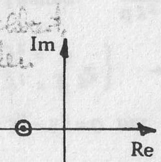
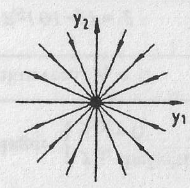
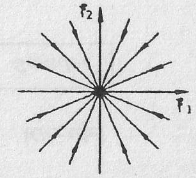
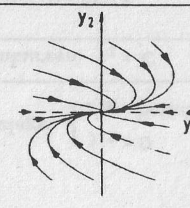
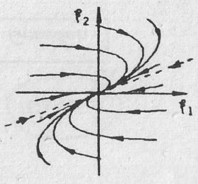
Lösung: $\left.\begin{array}{l}y_1(t) = C_1 e^{\sigma t} \\ y_2(t) = C_2 e^{\sigma t}\end{array}\right\}$ $\boxed{\rightarrow \text{Phasenkurven:} \quad y_2 = C\,y_1 \quad \textit{Ursprungsgeraden}}$

II.2: *Rangabfall* $d = 2 - Rg(\sigma I - J) = 1$

$$\left.\begin{array}{l}\dot{y}_1 = \sigma y_1 + y_2 \\ \dot{y}_2 = \sigma y_2\end{array}\right\} J = \begin{bmatrix}\sigma & 1\\ 0 & \sigma\end{bmatrix}$$

Lösung: $\left.\begin{array}{l}y_1(t) = C_1 e^{\sigma t} + C_2\, t\, e^{\sigma t} \\ y_2(t) = C_2 e^{\sigma t} \qquad \rightarrow t = \frac{1}{\sigma}\ln\frac{y_2}{C_2}\end{array}\right\} \rightarrow y_1 = C_1\frac{y_2}{C_2} + C_2\left(\frac{1}{\sigma}\ln\frac{y_2}{C_2}\right)\frac{y_2}{C_2}$

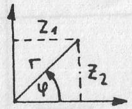
$$\boxed{\rightarrow \text{Phasenkurven:} \quad y_1 = C\,y_2 + \left(\frac{1}{\sigma}\ln\frac{y_2}{C_2}\right) y_2}$$

Eigenwerte	Gleichungen	Phasenebene Normalkoord. $\xrightarrow{X}$	Phasenebene Ausgangskoord.	Topologische Klassifizierung
	II.1: $\dot{y}_1 = -\lvert\sigma\rvert y_1$ $\dot{y}_2 = -\lvert\sigma\rvert y_2$			asymptotisch stabiler entarteter Knoten
	II.2: $\dot{y}_1 = -\lvert\sigma\rvert y_1 + y_2$ $\dot{y}_2 = -\lvert\sigma\rvert y_2$			

Bei positivem doppeltem Eigenwert $\lambda_1 = \lambda_2 = \sigma$ Umkehrung des Durchlaufsinns:
→ instabiler entarteter Knoten

Fall III: *Konjugiert komplexes Eigenwertpaar* $\lambda_{1,2} = \sigma \pm i\,\omega$

$$\left.\begin{array}{l}\dot y_1 = (\sigma + i\omega)\,y_1\\ \dot y_2 = (\sigma - i\omega)\,y_2\end{array}\right\} \xrightarrow[\begin{array}{l} y_1 = \frac12 z_1 - \frac{i}{2} z_2\\ y_2 = \frac12 z_1 + \frac{i}{2} z_2\end{array}]{\text{Transformation}^{*)}} \left\{\begin{array}{l}\dot z_1 = \sigma z_1 + \omega z_2\\ \dot z_2 = -\omega z_1 + \sigma z_2\end{array}\right.$$

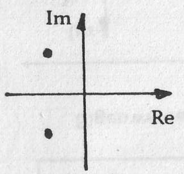
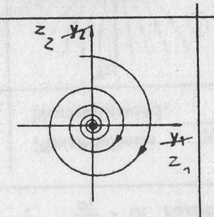
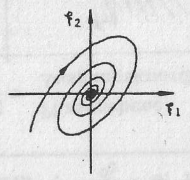
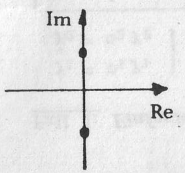
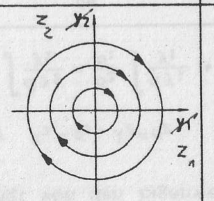
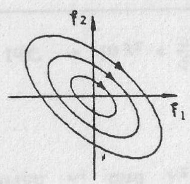
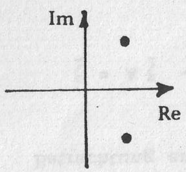
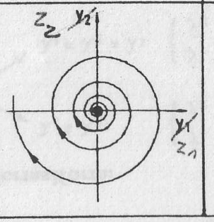
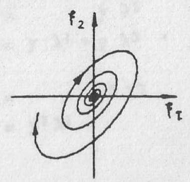


Lösung: Übergang auf Polarkoordinaten (r,φ) mit $z_1 = r\cos\varphi$, $z_2 = r\sin\varphi$

$$\begin{array}{lll|l|l} \dot r\cos\varphi - r\dot\varphi\sin\varphi & = & \sigma r\cos\varphi + \omega r\sin\varphi & \cdot\cos\varphi & \cdot -\sin\varphi\\ \dot r\sin\varphi + r\dot\varphi\cos\varphi & = & -\omega r\cos\varphi + \sigma r\sin\varphi & \cdot\sin\varphi & \cdot\cos\varphi \end{array}$$

$$\left.\begin{array}{l}\rightarrow \dot r = \sigma r\\ \rightarrow \dot\varphi = -\omega\end{array}\right\}$$

$\boxed{\rightarrow \text{Phasenkurven: } \left\{\begin{array}{l} r(t) = e^{\sigma t}\, r(t_0)\\ \varphi(t) = -\omega t + \varphi(t_0)\end{array}\right. \quad \textit{Spiralen}}$

Eigenwerte	Gleichungen	Phasenebene Normalkoord. $\xrightarrow{X}$	Phasenebene Ausgangskoord.	Topologische Klassifizierung
	$\dot r = -\lvert\sigma\rvert r$ $\dot\varphi = -\omega$			asymptotisch stabiler Strudel
	$\dot r = 0$ $\dot\varphi = -\omega$			grenzstabiler Wirbel
	$\dot r = \lvert\sigma\rvert r$ $\dot\varphi = -\omega$			instabiler Strudel

Fall IV: Mindesters ein reeller Nulleigenwert

IV.1: $\lambda_1 = \sigma > 0$ oder < 0, $\lambda_2 = 0$

$$\left.\begin{array}{l}\dot y_1 = \sigma\, y_1\\ \dot y_2 = 0\end{array}\right\} \rightarrow \text{nichtisolierte Ruhelage} \left\{\begin{array}{l} y_{1R} = 0\\ y_{2R} \text{ beliebig}\end{array}\right. \quad \text{(Gerade)}$$

Lösung: $\dfrac{dy_2}{dy_1} = 0$ $\boxed{\rightarrow \text{Phasenkurven: } y_2 = C \quad \textit{Geraden}}$

IV.2: $\lambda_1 = \lambda_2 = 0$ *mit Rangabfall* $d = 2 - Rg(0I - J) = 1$

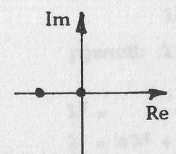
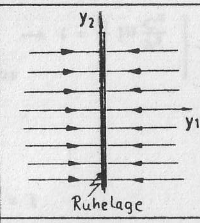
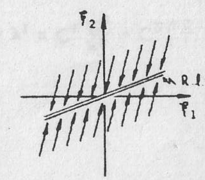
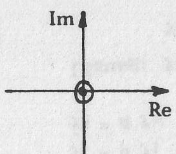
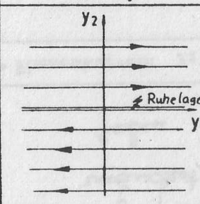
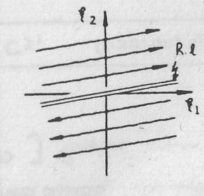
$$\left.\begin{array}{l}\dot y_1 = y_2\\ \dot y_2 = 0\end{array}\right\} \rightarrow \text{nichtisolierte Ruhelage:} \left\{\begin{array}{l} y_{1R} \text{ beliebig}\\ y_{2R} = 0\end{array}\right. \quad \text{(Gerade)}$$

Lösung: $\dfrac{dy_2}{dy_1} = 0$ $\boxed{\rightarrow \text{Phasenkurven: } y_2 = C \quad \textit{Geraden}}$

IV.3: $\lambda_1 = \lambda_2 = 0$ *mit Rangabfall* $d = 2 - Rg(0I - J) = 2$

$$\left.\begin{array}{l}\dot y_1 = 0\\ \dot y_2 = 0\end{array}\right\} \rightarrow \text{nichtisolierte Ruhelage:} \left\{\begin{array}{l} y_{1R} \text{ beliebig}\\ y_{2R} \text{ beliebig}\end{array}\right. \quad \text{(gesamte Zustandsebene)}$$

keine Bewegung möglich

Eigenwerte	Gleichungen	Phasenebene Normalkoord. $\xrightarrow{X}$	Phasenebene Ausgangskoord.	Topologische Klassifizierung
 IV.1	$\dot y_1 = -\lvert\sigma\rvert y_1$ $\dot y_2 = 0$			stabile nichtisolierte Ruhelage ($\sigma > 0$ instabil)
 IV.2	$\dot y_1 = y_2$ $\dot y_2 = 0$			instabile nichtisolierte Ruhelage

Folgerungen für das Verhalten des nichtlinearen Systems in der Umgebung der Ruhelagen

linearisiertes System		nichtlineares System
$Re(\lambda_i) \neq 0$: Knoten, Sattel, Strudel	$\longrightarrow$	ebenfalls Knoten, Sattel, Strudel
$Re(\lambda_i) = 0$: Wirbel, nichtisol. Ruhelagen.	$\longrightarrow$	nichtlineare Terme (Reihenglieder höherer Ordnung bzw. nichtlineare Dgln. der Trajektorien) sind zur Untersuchung heranzuziehen.