Multiple Multiplicative Actuator Fault Detectability Analysis Based on Invariant Sets for Discrete-time LPV Systems [*]

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Abstract: This paper proposes a generalized *minimum detectable fault* (MDF) computation method based on the set-separation condition between the healthy and faulty residual sets for discrete-time *linear parameter varying* (LPV) systems with bounded inputs and uncertainties. First, we equivalently transform the multiple multiplicative actuator faults into the form of multiple additive actuator faults, which is beneficial to simplify the problem. Then, by considering the 1-norm of the fault vector, we define the generalized MDF in the case of multiple additive actuator faults, which can be computed via solving a simple *linear programming* (LP) problem. Moreover, an analysis of the effect of the input vector on the magnitude of the generalized MDF is made. Since the proposed generalized MDF computation method is robust by considering the bounds of inputs and uncertainties, robust *fault detection* (FD) can be guaranteed whenever the sum of the magnitudes of all occurred faults is larger than the magnitude of the generalized MDF. At the end of this paper, a numerical example is used to illustrate the effectiveness of the proposed method.

Keywords: Invariant Sets, Multiple Multiplicative Actuator Faults, Generalized Minimum Detectable Faults, LPV Systems.

1. INTRODUCTION

Due to the demand of increasing system performance as well as its safety and reliability in modern industries, FD has attracted more and more attention from the scientific community over the past years. Fault occurrence affects the operation of the system to perform in its expected way. To put it simply, FD determines whether the system operates in faulty situations. The set-invariance approach is an important model-based FD method, which concentrates on testing consistency between the measured residuals in real time and the residual set generated in the healthy situation considering unknown but bounded uncertainties (Xu et al., 2014, 2015). Under normal circumstances, when the system reaches the steady state, the measured residual trajectory finally converges to the healthy residual set. Once the measured residuals are outside the healthy residual set, it is indicated that model uncertainties can not explain the residuals alone and faults must have occurred in the system (Montes de Oca et al., 2012). In this case, the measured residual trajectory ultimately converges to the faulty residual set at steady state. Consequently, provided that the healthy and faulty residual sets are separated, FD can be guaranteed to perform in the steady stage.

Under the circumstances, the primary task of invariant set-based FD is to construct the healthy and faulty invariant residual sets. Generally speaking, the robust positively invariant (RPI) set is defined as a bounded region in state space that the system state can be confined inside regardless of the effect of bounded uncertainties. Furthermore, the minimal robust positively invariant (mRPI) set is a unique and compact RPI set contained in any closed RPI set (Ghasemi and Afzalian, 2018). On one hand, a few methods with respect to the construction of the RPI sets for linear time-invariant (LTI) systems have been studied by researchers in recent years. In Kofman et al. (2007), a systematic method to obtain the RPI sets for perturbed LTI systems was proposed. In Olaru et al. (2010), an iterative method was presented which utilized the recursive iteration of the approximation of the system dynamics to approximate the mRPI set. On the other hand, Tan et al. (2019) proposed a novel and practical mRPI set construction method to characterize the healthy and faulty residual sets of perturbed discrete-time LPV systems.

In addition, since the invariant set-based FD performance is highly affected by system uncertainties, the characterization of the MDF of the invariant set-based FD methods is significant for us to know the limits of performance of the FD methods. In Pourasghar et al. (2016) and Kodakkadan

et al. (2017), the characterization of the minimum magnitude of faults affecting an LTI system in the presence of uncertainties was presented using both the observer-based and the set-invariance approaches. However, they only separated the projections of the healthy and faulty residual sets over at least one coordinate to obtain certain conditions for guaranteed MDF, which is a little conservative due to the over-approximation by using the interval hull of the invariant sets instead of the invariant sets. Furthermore, Tan et al. (2019) proposed an effective MDF computation method based on the set-separation condition between the healthy and faulty residual sets for LPV systems, but it merely considered the case of single additive actuator fault. According to Meseguer et al. (2010), the MDF ('*triggering limit*') corresponds to a fault that brings a residual to its threshold, provided that no other faults and nuisance inputs are present. Similarly, the case that multiple faults bring a residual to its threshold should be taken into account. On the basis of Tan et al. (2019), without loss of generality, we propose the generalized definition of the MDF in the multiple multiplicative actuator-faults situation in this paper.

2. SYSTEM DESCRIPTION

2.1 System model

Consider the following discrete-time LPV system affected by multiplicative actuator faults:

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)Gu_k + Ew_k, \tag{1a}$$
$$y_k = C(\theta_k)x_k + F\eta_k, \tag{1b}$$

where $k \in \mathbb{N}$ is the discrete time index. $A(\theta_k) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta_k) \in \mathbb{R}^{n_x \times n_u}$, and $C(\theta_k) \in \mathbb{R}^{n_y \times n_x}$ are respectively the system, input, and output matrices dependent on a varying scheduling vector $\theta_k \in \mathbb{R}^{n_\theta}$. $x_k \in \mathbb{R}^{n_x}$ and $y_k \in \mathbb{R}^{n_y}$ denote the system state and output vectors at time instant k, respectively. $u_k \in \mathbb{R}^{n_u}$ represents the control input vector. The unknown input vector $w_k \in \mathbb{R}^{n_w}$ (including process disturbances, modeling errors, etc.) is contained in a known compact convex set $\mathbf{W} = \{w \in \mathbb{R}^{n_w} | H_w w \leq b_w\}$ containing the origin. Similarly, the measurement noise vector $\eta_k \in \mathbb{R}^{n_\eta}$ also belongs to a given compact convex set $\mathbf{V} = \{\eta \in \mathbb{R}^{n_\eta} | H_\eta \eta \leq b_\eta\}$ containing the origin. $E \in \mathbb{R}^{n_x \times n_w}$ and $F \in \mathbb{R}^{n_y \times n_\eta}$ are constant distribution matrices. The diagonal matrix $G \in \mathbb{R}^{n_u \times n_u}$ models the multiplicative actuator faults. In fact, the matrix G represents changes induced by faults in the matrix $B(\theta_k)$, where $G = diag([G_1, \ldots, G_{n_u}])$ with $0 \leq G_i < 1$ when the i-th actuator is faulty. Additionally, G is the identity matrix I when the system (1) is healthy.

It is assumed that the n_θ-dimensional scheduling vector θ_k is a convex combination of given vertices, which generates a convex set $\mathbf{\Theta} = \mathbf{Conv}\{\theta^1, \theta^2, \ldots, \theta^\mathcal{N}\}$. Therefore, a linear affine function $\Phi(\theta_k)$ of θ_k can be written as the convex combination of vertex matrices: $\Phi(\theta_k) = \sum_{i=1}^{\mathcal{N}} \alpha_i \Phi(\theta^i)$, where the weighting coefficients α_i satisfy the constraints $\sum_{i=1}^{\mathcal{N}} \alpha_i = 1$, $0 \leq \alpha_i \leq 1$ and the function $\Phi(\cdot)$ can represent $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$.

2.2 Design of FD observer

Consider the following Luenberger-structure FD observer:

$$\hat{x}_{k+1} = A(\theta_k)\hat{x}_k + B(\theta_k)u_k + L(y_k - \hat{y}_k), \tag{2a}$$
$$\hat{y}_k = C(\theta_k)\hat{x}_k, \tag{2b}$$

where $\hat{x}_k$ and $\hat{y}_k$ are the estimated state and output vectors of the system (1), respectively. $L \in \mathbb{R}^{n_x \times n_y}$ is the gain matrix of the observer (2). In the healthy situation (i.e., G=I), the state-estimation error e_k is defined as $e_k = x_k - \hat{x}_k$. In addition, the dynamics of the state-estimation error e_k in the healthy situation can be obtained as

$$e_{k+1} = (A(\theta_k) - LC(\theta_k))e_k + Ew_k - LF\eta_k. \tag{3}$$

As analyzed in Tan et al. (2019), w_k and η_k do not affect the bounded input-bounded output (BIBO) stability of the dynamics (3). Therefore, let us directly consider the stability of the nominal system:

$$\bar{e}_{k+1} = (A(\theta_k) - LC(\theta_k))\bar{e}_k, \tag{4}$$

where $\bar{e}_k$ is the state-estimation error without considering the effects of w_k and η_k on the dynamics (3).

Theorem 1. According to Daafouz and Bernussou (2001), the dynamics (4) is poly-quadratically stable if and only if there exist symmetric positive definite matrices P_i, P_j, and matrices Q_i of appropriate dimensions such that

$$\begin{bmatrix} Q_i + Q_i^T - P_i & * \\ (A_i - LC_i)Q_i & P_j \end{bmatrix} \succ 0, \quad \forall i,j = 1, 2, \ldots, \mathcal{N}, \tag{5}$$

where the symbol $*$ denotes the transpose of $(A_i - LC_i)Q_i$. In this case, the time-varying parameter-dependent Lyapunov function is given as $V(\bar{e}_k, \theta_k) = \bar{e}_k^T Z(\theta_k)\bar{e}_k$, with $Z(\theta_k) = \sum_{i=1}^{\mathcal{N}} \alpha_i P_i^{-1}$, $\sum_{i=1}^{\mathcal{N}} \alpha_i = 1$, and $0 \leq \alpha_i \leq 1$.

3. SET-THEORETIC ANALYSIS IN THE HEALTHY SITUATION

If the condition of Theorem 1 is fulfilled, as analyzed in Tan et al. (2019), there exist a family of RPI sets as well as the convex hull of the mRPI set denoted as Ω_∞ for the dynamics (3). For convenience, we also call Ω_∞ the mRPI set of the dynamics (3). As mentioned above, a novel and practical mRPI set construction method for perturbed discrete-time LPV systems was reported in Tan et al. (2019) and we recall it in Theorems 2, 3 and 4.

Theorem 2. (Tan et al., 2019) Under the condition of Theorem 1, consider an arbitrarily given initial convex set $\mathbf{E}_0 \supseteq (1+\delta)\Omega_\infty$, where Ω_∞ is the mRPI set of the dynamics (3) and $\delta > 0$. By computing the follwing iterative equation:

$$\bar{\mathbf{E}}_{k+1} = \mathcal{A}(\mathbf{E}_k) \oplus \mathbf{S}, \tag{6a}$$
$$\mathbf{E}_{k+1} = \mathbf{Conv}\{\bar{\mathbf{E}}_{k+1} \cup \mathbf{E}_k\}, \tag{6b}$$

where $\oplus$ represents the Minkowski sum, $\mathcal{A}(\cdot)$ is a set mapping function and $\mathcal{A}(\mathbf{E}_k) = \mathbf{Conv}\{\cup_{i=1}^{\mathcal{N}}(A_i - LC_i)\mathbf{E}_k\}$, $\mathbf{Conv}(\cdot)$ denotes the convex hull of a set, $\mathbf{S} = E\mathbf{W} \oplus (-LF)\mathbf{V}$. There exists a finite $k^* \in \mathbb{N}$ such that $\mathbf{E}_{k^*+1} = \mathbf{E}_{k^*}$. Moreover, $\mathbf{E}_{k^*}$ is an RPI set for the dynamics (3).

Theorem 3. (Tan et al., 2019) Given an initial RPI set $\mathbf{E}_{k^*}$ for the dynamics (3), the sequence Ω_k: $\Omega_{k+1} = \mathcal{A}(\Omega_k) \oplus \mathbf{S}$, with $\Omega_0 = \mathbf{E}_{k^*}$, ensures that at each iteration, Ω_k is an RPI set of the dynamics (3) and $\Omega_\infty \subseteq \Omega_{k+1} \subseteq \Omega_k \subseteq \Omega_0$ holds for $k \geq 1$. Furthermore, we have $\Omega_\infty = \lim_{k \to +\infty} \Omega_k = \oplus_{i=0}^{\infty} \mathcal{A}^i(\mathbf{S})$, which is the mRPI set of the dynamics (3).

Furthermore, the initialization of $\mathbf{E}_0$ such that $\mathbf{E}_0 \supseteq (1 + \delta)\Omega_\infty$ can be guaranteed based on Theorem 4.

Theorem 4. (Tan et al., 2019) Suppose that the dynamics (3) is stable, the initial convex set $\mathbf{E}_0 \supseteq (1 + \delta)\Omega_\infty$ can

be given by $\mathbf{E}_0 = \oplus_{i=0}^{p^*-1} \mathcal{A}^i(\mathbf{B}(r)) \oplus \frac{p^*\xi}{1-\xi}\mathbf{B}(r)$, where it is indicated that there exist a scalar $\xi \in (0,1)$, $p^* \in \mathbb{N}$ and a box $\mathbf{B}(r) = \{x \in \mathbb{R}^{n_x} | \|x\|_\infty \leq r\}$ containing $\mathbf{S}$, i.e., $\mathbf{S} \subseteq \mathbf{B}(r)$, such that for any $k \geq p^*$, $\mathcal{A}^k(\mathbf{B}(r)) \subseteq \xi\mathbf{B}(r)$.

4. SET-THEORETIC ANALYSIS IN THE FAULTY SITUATION

4.1 Disturbance-free system with multiple multiplicative actuator faults

Above all, let us consider the behavior of the system (1) with multiple multiplicative actuator faults in the absence of w_k and η_k. Therefore, the analysis is made based on the following disturbance-free system:

$$\tilde{x}_{k+1} = A(\theta_k)\tilde{x}_k + B(\theta_k)Gu_k, \quad (7a)$$
$$\tilde{y}_k = C(\theta_k)\tilde{x}_k, \quad (7b)$$

where $u_k = [u_1, \ldots, u_{n_u}]^T$, $G = diag([G_1, \ldots, G_{n_u}])$ with $0 \leq G_i \leq 1$ for all $i = 1, 2, \ldots, n_u$, and G is not the identity matrix I in the faulty situation. In this case, the state-estimation error corresponding to the Luenberger-structure observer (2) is defined as $\tilde{e}_k = \tilde{x}_k - \hat{x}_k$, whose dynamics can be obtained as

$$\tilde{e}_{k+1} = (A(\theta_k) - LC(\theta_k))\tilde{e}_k + B(\theta_k)(G-\text{I})u_k. \quad (8)$$

In order to construct the faulty residual set, we have to transform (8) into a tractable form. Notice that

$$(G-\text{I})u_k = [(G_1-1)u_1, \ldots, (G_{n_u}-1)u_{n_u}]^T$$
$$= [-f_1u_1, \ldots, -f_{n_u}u_{n_u}]^T, \quad (9)$$

where $f_i = 1 - G_i$ with $0 \leq f_i \leq 1$ for all $i = 1, 2, \ldots, n_u$, u_i is the i-th component of the input vector u_k. Since $B(\theta_k) = [B_1(\theta_k), \ldots, B_{n_u}(\theta_k)]$, we have $B(\theta_k)(G-\text{I})u_k = -\sum_{i=1}^{n_u} f_i u_i B_i(\theta_k)$, where $B_i(\theta_k)$ is the i-th column of the matrix $B(\theta_k)$. Therefore, the dynamics (8) is equivalent to the following form:

$$\tilde{e}_{k+1} = (A(\theta_k) - LC(\theta_k))\tilde{e}_k - \sum_{i=1}^{n_u} f_i u_i B_i(\theta_k). \quad (10)$$

Actually, (10) represents the dynamcis of the state-estiamtion error $\tilde{e}_k$ with multiple additive actuator faults. Especially, the dynamcis of the state-estimation error with single additive actuator fault can be derived as

$$\tilde{e}_{k+1}^i = (A(\theta_k) - LC(\theta_k))\tilde{e}_k^i - f_i u_i B_i(\theta_k), \quad (11)$$

where it is assumed that the i-th actuator is faulty. Obviously, according to the superposition principle, we have $\tilde{e}_k = \sum_{i=1}^{n_u} \tilde{e}_k^i$. In order to construct the faulty residual set in the multiple additive actuator-faults situation, we firstly consider the dynamics (11) in the single additive actuator-fault situation. As mentioned above, we directly consider the convex hull of the mRPI set of the dynamics (11). Firstly, we should deal with the product term $u_i B_i(\theta_k)$. It is assumed that the component u_i of the input vector u_k is contained in an interval $\mathbf{U}_i = \{\mu \in \mathbb{R} | u_i^{min} \leq \mu \leq u_i^{max}\}$. Similarly, the column vector $B_i(\theta_k)$ of the matrix $B(\theta_k)$ is contained in a convex set $\mathbf{B}_i(\mathbf{\Theta}) = \{x \in \mathbb{R}^{n_x} | H_{B_i}x \leq b_{B_i}\}$. Then we can obtain a convex set that contains the product term $u_i B_i(\theta_k)$ as the following:

$$u_i B_i(\theta_k) \in \mathbf{B}_i^{u_i}(\mathbf{\Theta}) = \mathbf{Conv}\{(u_i^{min}\mathbf{B}_i(\mathbf{\Theta})) \cup (u_i^{max}\mathbf{B}_i(\mathbf{\Theta}))\}. \quad (12)$$

Suppose that the dynamics (11) is stable, based on the results in Theorems 2, 3 and 4, the mRPI set of the dynamics (11) can be obtained as $f_i\tilde{\mathcal{E}}_i$, where $\tilde{\mathcal{E}}_i = \oplus_{j=0}^\infty \mathcal{A}^j(-\mathbf{B}_i^{u_i}(\mathbf{\Theta}))$ denotes the mRPI set of the dynamics (11) in the case of $f_i = 1$.

After that, we consider the dynamics (10) to construct the residual set in the multiple additive actuator-faults situation. Based on the properties of invariant sets, the state-estimation error $\tilde{e}_k^i$ will always be contained in the mRPI set $f_i\tilde{\mathcal{E}}_i$ when the system (7) reaches the steady state (i.e., $\tilde{e}_k^i \in f_i\tilde{\mathcal{E}}_i$). Suppose that the dynamics (11) is stable for all $i = 1, 2, \ldots, n_u$, then we have $\tilde{e}_k^i \in f_i\tilde{\mathcal{E}}_i, \forall i = 1, 2, \ldots, n_u$. Therefore, if we consider the dynamics (10) with multiple additive actuator faults, the corresponding state-estimation error $\tilde{e}_k$ will be contained in a convex set $\tilde{\mathcal{E}}$ which includes the mRPI set of the dynamics (10) as

$$\tilde{e}_k = \sum_{i=1}^{n_u} \tilde{e}_k^i \in \oplus_{i=1}^{n_u} f_i\tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}. \quad (13)$$

4.2 Healthy and faulty residual sets

Combining (3) with (10), we can further derive the dynamics of the state-estimation error e_k^f in the multiple additive actuator-faults situation under the effects of w_k and η_k:

$$e_{k+1}^f = (A(\theta_k) - LC(\theta_k))e_k^f + Ew_k - LF\eta_k - \sum_{i=1}^{n_u} f_i u_i B_i(\theta_k), \quad (14)$$

with $e_k^f = e_k + \tilde{e}_k = e_k + \sum_{i=1}^{n_u} \tilde{e}_k^i$ leading to the invariant set characterization: $\mathcal{E}^f = \mathcal{E} \oplus \tilde{\mathcal{E}} = \mathcal{E} \oplus \{\oplus_{i=1}^{n_u} f_i\tilde{\mathcal{E}}_i\}$, where $\mathcal{E} = \Omega_\infty$ represents the mRPI set of the dynamics (3). Furthermore, we define the following residual vector corresponding to (3) in the healthy situation: $r_k = y_k - \hat{y}_k = C(\theta_k)e_k + F\eta_k$, whose set version is

$$\mathcal{R} = \mathcal{C}(\mathcal{E}) \oplus F\mathbf{V}, \quad (15)$$

where $\mathcal{C}(\mathcal{E}) = \mathbf{Conv}\{\cup_{i=1}^{\mathcal{N}} C_i\mathcal{E}\}$. Similarly, we can derive the residual vector in the multiple additive actuator-faults situation: $r_k^f = C(\theta_k)e_k^f + F\eta_k = C(\theta_k)e_k + F\eta_k + C(\theta_k)\sum_{i=1}^{n_u} \tilde{e}_k^i = r_k + \sum_{i=1}^{n_u} C(\theta_k)\tilde{e}_k^i$, whose set version can be represented as

$$\mathcal{R}^f = \mathcal{R} \oplus \{\oplus_{i=1}^{n_u} \mathcal{C}(f_i\tilde{\mathcal{E}}_i)\} = \mathcal{R} \oplus \{\oplus_{i=1}^{n_u} f_i\mathcal{C}(\tilde{\mathcal{E}}_i)\}, \quad (16)$$

where $\mathcal{R}^f$ is the faulty residual set containing the residual vector r_k^f, which is dependent on all $f_i, i = 1, 2, \ldots, n_u$. For convenience, we define the fault vector $f = [f_1, \ldots, f_{n_u}]^T$ in the multiple actuator-faults situation. According to the detection criterion of the invariant set-based FD, we need to check whether $r_k \in \mathcal{R}$ holds or not in real time. If that relationship does not hold (i.e., $r_k \notin \mathcal{R}$) after a time instant $k-1$ where $r_{k-1} \in \mathcal{R}$, it indicates that the system (1) is faulty at time instant k. Otherwise, we believe that the system (1) still operates in the healthy situation. Once there are some faults occurred in the system (1), according to the properties of invariant sets, it is noted that the residual vector r_k will converge to the faulty residual set $\mathcal{R}^f$. Consequently, provided that there is no intersection between the healthy residual set $\mathcal{R}$ and the faulty residual set $\mathcal{R}^f$, i.e., $\mathcal{R} \cap \mathcal{R}^f = \emptyset$, the effective detection of the faults can be guaranteed in the steady stage.

5. COMPUTATION OF THE GENERALIZED MDF

In this section, an effective method to compute the generalized MDF by considering the constraint $\mathcal{R} \cap \mathcal{R}^f = \emptyset$ is presented. It is noticed that the mathematical description of the MDF for single additive actuator fault has already been presented in Tan et al. (2019). Without loss of generality, here we further propose the generalized MDF in the multiple additive actuator-faults situation as follows:

$$\min \quad \|f\|_1 \tag{17}$$
$$s.t. \quad \mathcal{R} \cap \mathcal{R}^f = \emptyset; 0 \le f_i \le 1, \ \forall i = 1, 2, \dots, n_u,$$

where $\|f\|_1$ denotes the 1-norm of the fault vector f. Since $\|f\|_1 = \sum_{i=1}^{n_u} |f_i| = \sum_{i=1}^{n_u} f_i$, the optimization problem (17) is equivalent to the following form:

$$\min \quad \sum_{i=1}^{n_u} f_i \tag{18}$$
$$s.t. \quad \mathcal{R} \cap \mathcal{R}^f = \emptyset; 0 \le f_i \le 1, \ \forall i = 1, 2, \dots, n_u.$$

As a matter of fact, the constraints $0 \le f_i \le 1$ ($i = 1, 2, \dots, n_u$) describe a hypercube in n_u-dimensional Euclidean space. Furthermore, the optimal solution of the optimization problem (18) describes a hyperplane denoted as $\sum_{i=1}^{n_u} f_i = \lambda$ with respect to each component f_i of the fault vector f, where λ is the generalized MDF. Notice that $0 \le f_i \le 1$ for all $i = 1, 2, \dots, n_u$, we have $0 \le \sum_{i=1}^{n_u} f_i \le n_u$, where $\sum_{i=1}^{n_u} f_i = 0$ and $\sum_{i=1}^{n_u} f_i = n_u$ describe a vertex of the above hypercube, respectively. And all the available values that satisfy $\sum_{i=1}^{n_u} f_i = \rho$ ($0 < \rho < n_u$) represent a series of parallel hyperplanes contained in that hypercube. The line between the point $\sum_{i=1}^{n_u} f_i = 0$ and the point $\sum_{i=1}^{n_u} f_i = n_u$ is perpendicular to all these parallel hyperplanes. In terms of the definition of the generalized MDF, we hold the opinion that the system (1) is faulty as long as the magnitude of $\sum_{i=1}^{n_u} f_i$ is larger than the magnitude of λ.

In order to illustrate better, we consider a special case $n_u = 3$ in Fig. 1. As shown in Fig. 1, the area between the red point $\sum_{i=1}^{3} f_i = 3$ and the plane $\sum_{i=1}^{3} f_i = \lambda$ indicates the range of guaranteed detectable faults in the steady stage, while the area between the blue point $\sum_{i=1}^{3} f_i = 0$ and the plane $\sum_{i=1}^{3} f_i = \lambda$ indicates that the system (1) is healthy or in faulty situations that can not be guaranteed to detect. To sum up, the generalized MDF describes a boundary where the healthy residual set $\mathcal{R}$ and the faulty residual set $\mathcal{R}^f$ are just separate. Therefore, we aim to figure out the optimization problem (18) hereinafter to obtain the value of the generalized MDF. Unfortunately, it is complex and can not be solved directly.

In the following Theorem 5, the optimization problem (18) is transformed into a simple LP problem to obtain the generalized MDF. Before presenting Theorem 5, we first give a relevant corollary. Kvasnica (2005) presented a computation method of the Minkowski sum of two polytopes which are given in H-representation. In consequence, we further propose the result of the Minkowski sum of multiple polytopes in Corollary 1.

Corollary 1. If n_u known polytopes $\mathcal{P}_i$ ($i = 1, 2, \dots, n_u$) are given in H-representation, i.e., $\mathcal{P}_i = \{x_i \in \mathbb{R}^n | H_i x_i \le b_i\}$ ($i = 1, 2, \dots, n_u$), their Minkowski sum $\mathcal{M} = \overset{n_u}{\underset{i=1}{\oplus}} \mathcal{P}_i$ can be computed by the following projection:

$$\mathcal{M} = \left\{ r \in \mathbb{R}^n \middle| \begin{array}{l} \exists x_i, i = 1, 2, \dots, n_u - 1, \ s.t. \\ \begin{bmatrix} H_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & H_{n_u-1} & 0 \\ -H_{n_u} & \dots & -H_{n_u} & H_{n_u} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_u-1} \\ r \end{bmatrix} \le \begin{bmatrix} b_1 \\ \vdots \\ b_{n_u-1} \\ b_{n_u} \end{bmatrix} \end{array} \right\}.$$

Theorem 5. For the optimization problem (18), the magnitude of the generalized MDF can be obtained by solving the following LP problem:
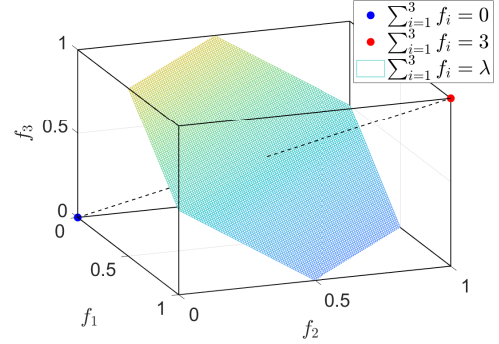

Fig. 1. The blue point represents $\sum_{i=1}^{3} f_i = 0$ while the red point represents $\sum_{i=1}^{3} f_i = 3$. And the plane $\sum_{i=1}^{3} f_i = \lambda$ represents the generalized MDF.

$$\min \quad -\sum_{i=1}^{n_u} f_i$$
$$s.t. \quad \begin{cases} Hx \le b, H_\eta y \le b_\eta, \\ Hx - Hz \le b, H_\eta y - H_\eta t \le b_\eta, \\ \tilde{H}_i x_i \le f_i \tilde{b}_i, \ \forall i = 1, 2, \dots, n_u - 1, \\ -\tilde{H}_{n_u} z - \tilde{H}_{n_u} Ft - \sum_{i=1}^{n_u-1} \tilde{H}_{n_u} x_i \le f_{n_u} \tilde{b}_{n_u}, \\ 0 \le f_i \le 1, \ \forall i = 1, 2, \dots, n_u, \end{cases} \tag{19}$$

where $\mathbf{V} = \{\eta \in \mathbb{R}^{n_\eta} | H_\eta \eta \le b_\eta\}$, $\mathcal{C}(\mathcal{E}) = \{x \in \mathbb{R}^{n_y} | Hx \le b\}$, and $\mathcal{C}(\tilde{\mathcal{E}}_i) = \{x_i \in \mathbb{R}^{n_y} | \tilde{H}_i x_i \le \tilde{b}_i\}$ ($i = 1, 2, \dots, n_u$).

Proof. Consider the dual case of (18), then we obtain the following optimization problem:

$$\max \quad \sum_{i=1}^{n_u} f_i \tag{20}$$
$$s.t. \quad \mathcal{R} \cap \mathcal{R}^f \ne \emptyset; 0 \le f_i \le 1, \ \forall i = 1, 2, \dots, n_u.$$

In fact, for any magnitude of $\sum_{i=1}^{n_u} f_i$ larger than the optimal solution of the problem (20), the constraints in (18) are satisfied, therefore the optimal solution here represents an infimum for the optimization problem (18). Furthermore, the optimization problem (20) is equivalent to the following optimization problem:

$$\min \quad -\sum_{i=1}^{n_u} f_i \tag{21}$$
$$s.t. \quad \mathcal{R} \cap \mathcal{R}^f \ne \emptyset; 0 \le f_i \le 1, \ \forall i = 1, 2, \dots, n_u.$$

Based on (15) and (16), considering the constraint $\mathcal{R} \cap \mathcal{R}^f \ne \emptyset$ in (21), we have

$$\mathcal{R} \cap \mathcal{R}^f \ne \emptyset \Leftrightarrow 0 \in \mathcal{R}^f \oplus (-\mathcal{R}) \Leftrightarrow 0 \in \mathcal{R} \oplus \{\oplus_{i=1}^{n_u} f_i \mathcal{C}(\tilde{\mathcal{E}}_i)\} \oplus (-\mathcal{R})$$
$$\Leftrightarrow 0 \in \mathcal{C}(\mathcal{E}) \oplus (-\mathcal{C}(\mathcal{E})) \oplus F(\mathbf{V} \oplus (-\mathbf{V})) \oplus \{\oplus_{i=1}^{n_u} f_i \mathcal{C}(\tilde{\mathcal{E}}_i)\}. \tag{22}$$

Since the sets $\mathcal{E}$ and $\tilde{\mathcal{E}}_i$ are the mRPI set of the dynamics (3) and (11), respectively, both of them are known polytopes. Consequently, the convex hulls $\mathcal{C}(\mathcal{E})$ and $\mathcal{C}(\tilde{\mathcal{E}}_i)$ ($i = 1, 2, \dots, n_u$) are known polytopes too. For the convenience of illustration, it is assumed that $\mathcal{C}(\mathcal{E}) = \{x \in \mathbb{R}^{n_y} | Hx \le b\}$, $\mathcal{C}(\tilde{\mathcal{E}}_i) = \{x_i \in \mathbb{R}^{n_y} | \tilde{H}_i x_i \le \tilde{b}_i\}$ ($i = 1, 2, \dots, n_u$). After that, in the light of Corollary 1, we have

$$\mathcal{C}(\mathcal{E}) \oplus (-\mathcal{C}(\mathcal{E})) = \left\{ z \in \mathbb{R}^{n_y} \middle| \exists x, \ s.t. \begin{bmatrix} H & 0 \\ H & -H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \le \begin{bmatrix} b \\ b \end{bmatrix} \right\},$$

$$F(\mathbf{V} \oplus (-\mathbf{V})) =$$
$$\left\{ \beta \in \mathbb{R}^{n_y} \middle| \exists y, t, \ s.t. \ \beta = Ft, \begin{bmatrix} H_\eta & 0 \\ H_\eta & -H_\eta \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} \le \begin{bmatrix} b_\eta \\ b_\eta \end{bmatrix} \right\}. \tag{23}$$

In addition, let $\mathcal{S} = \mathcal{C}(\mathcal{E}) \oplus (-\mathcal{C}(\mathcal{E})) \oplus F(\mathbf{V} \oplus (-\mathbf{V})) \oplus \{\oplus_{i=1}^{n_u} f_i \mathcal{C}(\tilde{\mathcal{E}}_i)\}$, which can be computed as

$$\mathcal{S} = \left\{ m \in \mathbb{R}^{n_y} \middle| \exists x, y, z, t, r, x_i, \ s.t. \ m = z + \beta + r, \right.$$

$$\begin{bmatrix} H & 0 \\ H & -H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} b \\ b \end{bmatrix}, \beta = Ft, \begin{bmatrix} H_\eta & 0 \\ H_\eta & -H_\eta \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} \leq \begin{bmatrix} b_\eta \\ b_\eta \end{bmatrix},$$

$$\tilde{H}_i x_i \leq f_i \tilde{b}_i, \forall i = 1, 2, \ldots, n_u - 1,$$

$$\left. \tilde{H}_{n_u} r - \sum_{i=1}^{n_u-1} \tilde{H}_{n_u} x_i \leq f_{n_u} \tilde{b}_{n_u} \right\}$$

$$= \left\{ m \in \mathbb{R}^{n_y} \middle| \exists x, y, z, t, x_i, \ s.t. \right.$$

$$\begin{bmatrix} H & 0 \\ H & -H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} b \\ b \end{bmatrix}, \beta = Ft, \begin{bmatrix} H_\eta & 0 \\ H_\eta & -H_\eta \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} \leq \begin{bmatrix} b_\eta \\ b_\eta \end{bmatrix},$$

$$\tilde{H}_i x_i \leq f_i \tilde{b}_i, \forall i = 1, 2, \ldots, n_u - 1,$$

$$\left. \tilde{H}_{n_u}(m - z - \beta) - \sum_{i=1}^{n_u-1} \tilde{H}_{n_u} x_i \leq f_{n_u} \tilde{b}_{n_u} \right\}$$

$$= \left\{ m \in \mathbb{R}^{n_y} \middle| \exists x, y, z, t, x_i, \ s.t. \right.$$

$$\begin{bmatrix} H & 0 \\ H & -H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} b \\ b \end{bmatrix}, \begin{bmatrix} H_\eta & 0 \\ H_\eta & -H_\eta \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} \leq \begin{bmatrix} b_\eta \\ b_\eta \end{bmatrix},$$

$$\tilde{H}_i x_i \leq f_i \tilde{b}_i, \forall i = 1, 2, \ldots, n_u - 1,$$

$$\left. \tilde{H}_{n_u}(m - z - Ft) - \sum_{i=1}^{n_u-1} \tilde{H}_{n_u} x_i \leq f_{n_u} \tilde{b}_{n_u} \right\}. \quad (24)$$

Since $\mathcal{R} \cap \mathcal{R}^f \neq \emptyset \Leftrightarrow 0 \in \mathcal{S}$ based on (22), the constraint $0 \in \mathcal{S}$ can lead to a series of linear constraints $Hx \leq b$, $H_\eta y \leq b_\eta$, $Hx - Hz \leq b$, $H_\eta y - H_\eta t \leq b_\eta$, $\tilde{H}_i x_i \leq f_i \tilde{b}_i$ $(i = 1, 2, \ldots, n_u - 1)$, $-\tilde{H}_{n_u} z - \tilde{H}_{n_u} Ft - \sum_{i=1}^{n_u-1} \tilde{H}_{n_u} x_i \leq f_{n_u} \tilde{b}_{n_u}$. Finally, we obtain the LP problem (19). ∎

In summary, Theorem 5 presents the method to compute the magnitude of the generalized MDF. No matter how the component u_i $(i = 1, 2, \ldots, n_u)$ of the input vector u_k varies in the interval $\mathbf{U}_i = \{\mu \in \mathbb{R} | u_i^{min} \leq \mu \leq u_i^{max}\}$, FD can always be guaranteed in the steady stage by using the invariant set-based method as long as the sum of the magnitudes of all occurred faults is larger than the magnitude of the generalized MDF. In fact, Theorem 5 describes a more universal circumstance and it includes the case of computing the magnitude of MDF for single additive actuator fault, which is presented in Tan et al. (2019). Moreover, if we consider a special case that $n_u = 1$, we will obtain the same results as Tan et al. (2019). Nevertheless, the results obtained from the optimization problem (19) may be deemed to be conservative since the set of the input vector u_k is considered here. Actually, we can reduce the conservatism of results on the magnitude of the generalized MDF by decreasing the varying range of the input vector u_k. Particularly, if the specific value $\bar{u}_i$ of the component u_i is known to compute the set $\mathbf{B}_i^{\bar{u}_i}(\mathbf{\Theta}) = \bar{u}_i \mathbf{B}_i(\mathbf{\Theta})$ instead of computing the convex hull set $\mathbf{B}_i^{u_i}(\mathbf{\Theta})$ in (12), we can obtain a less conservative magnitude of the generalized MDF. In this case, the magnitude of the generalized MDF is dependent on the value of each component u_i and the whole solving process is similar to the above which is omitted here.

6. NUMERICAL EXAMPLE

We consider the discrete-time LPV system (1) with

$$A(\theta_k) = \begin{bmatrix} 0.85 & 0.2\theta_k(2) \\ 0.1\theta_k(1) & 0.75 \end{bmatrix}, C(\theta_k) = \begin{bmatrix} 0.04\theta_k(2) & 0.3 \\ 0.05 & 0.01\theta_k(1) \end{bmatrix},$$

$$B(\theta_k) = \begin{bmatrix} 0.5678\theta_k(1) & 0.4265\theta_k(1) & 0.2464\theta_k(1) \\ 0.3492\theta_k(2) & 0.7347\theta_k(2) & 0.6386\theta_k(2) \end{bmatrix},$$

$$E = \begin{bmatrix} 0.6324 & 0.2785 \\ 0.0975 & 0.5469 \end{bmatrix}, F = \begin{bmatrix} 0.8147 & 0.1270 \\ 0.9058 & 0.9134 \end{bmatrix},$$

with $\theta_k = [\theta_k(1), \theta_k(2)]^T$. The bounding sets of w_k and η_k are designed as $\mathbf{W} = \{w \in \mathbb{R}^2 | \|w\|_\infty \leq 0.02\}$ and $\mathbf{V} = \{\eta \in \mathbb{R}^2 | \|\eta\|_\infty \leq 0.02\}$. It is assumed that the component u_i $(i = 1, 2, 3)$ of the input vector u_k is contained in the intervals $\mathbf{U}_1 = \{\mu \in \mathbb{R} | 1.5 \leq \mu \leq 1.8\}$, $\mathbf{U}_2 = \{\mu \in \mathbb{R} | 0.8 \leq \mu \leq 1.2\}$, and $\mathbf{U}_3 = \{\mu \in \mathbb{R} | 1.0 \leq \mu \leq 1.6\}$, respectively. The bounding set of the scheduling vector θ_k is given as $\mathbf{\Theta} = \mathbf{Conv}\{[0.5, 0.5]^T, [1, 0.5]^T, [1, 1]^T, [0.5, 1]^T\}$. Furthermore, the gain matrix L of the FD observer (2) is designed as $L = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$. According to Theorem 1, the linear matrix inequalities (5) can be solved and thus we can obtain proper matrices to verify poly-quadratical stability of the nominal system (4) using YALMIP.

Firstly, we can obtain the mRPI sets $\mathcal{E}$ and $\tilde{\mathcal{E}}_i$ $(i = 1, 2, 3)$ based on Theorems 2, 3 and 4. The construction process of these mRPI sets is ommitted and readers can refer to Tan et al. (2019) for more details. Then, according to Theorem 5, the generalized MDF in the multiple additive actuator-faults situation is obtained as $\lambda = 0.5426$ ($f_1 = 0.5426, f_2 = f_3 = 0$). Thus, for any actuator faults, as long as the sum of all occurred faults is larger than this threhold, the detection of the faults can be guaranteed. We assume the following faults scenario: from $k = 0$ to $k = 40$, the system operates in the healthy situation. From $k = 41$ to $k = 100$, we inject three faults ($f_1 = 0.2426, f_2 = 0.1, f_3 = 0.2$) into the system. For convenience, we directly consider drawing the interval hull (the two blue lines) of the healthy residual set $\mathcal{R}$. The results of on-line FD for the generalized MDF $\lambda = 0.5426$ are shown in Fig. 2. It is observed that from $k = 45$ to $k = 100$, the measured residual r_k in real time is outside $\mathcal{R}$ and the faults are detected. As mentioned above, as long as the sum of the magnitudes of all occurred faults is larger than the magnitude of the generalized MDF, FD can be guaranteed by utilizing our proposed method.

For the reason that the input vector u_k can affect the magnitude of the generalized MDF, we can reduce the conservatism of results on the magnitude of the generalized MDF by decreasing the varying range of the input vector u_k. For the sake of illustration, we consider the case that $n_u = 1$ and utilize the variation of the component u_1 to analyze the effect of the varying range of u_k on the magnitude of the generalized MDF. The magnitudes of the generalized MDF for single actuator fault with respect to different varying ranges of the component u_1 are displayed in Table 1. In addition, for a specific value $\bar{u}_1$ inside the varying range of the component u_1, we can also compute the corresponding magnitude of the generalized MDF.

Table 1. The magnitudes of the generalized MDF w.r.t. different varying ranges of u_1

Varying range of u_1	the generalized MDF
$u_1 \in [1.5, 1.8]$	0.5426
$u_1 \in [1.6, 1.8]$	0.5087
$u_1 \in [1.7, 1.8]$	0.4788
$u_1 \in [1.75, 1.8]$	0.4651

It is worth mentioning that we can obtain the same results as Table 1 by using the MDF computation method for single additive actuator fault presented in Tan et al. (2019), which verify the universality of Theorem 5. To
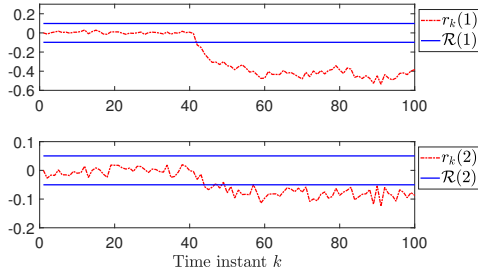
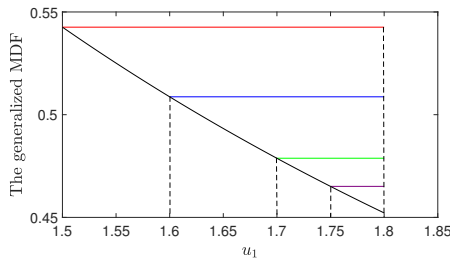Fig. 2. On-line FD for the generalized MDF $\lambda = 0.5426$.

Fig. 3. The different magnitudes of the generalized MDF w.r.t. different varying ranges of u_1.

dispaly and illustrate better, we show the generalized MDF for specific values of u_1 and results of Table 1 in Fig. 3. In Fig. 3, the black line represents the magnitudes of the generalized MDF for different specific values of u_1, which is plotted by computing a magnitude of the generalized MDF with a step increment of 0.003 from 1.5 to 1.8. The purple line represents the magnitude of the generalized MDF when $u_1 \in [1.75, 1.8]$. Similarly, the green line and blue line represent the magnitudes of the generalized MDF when $u_1 \in [1.7, 1.8]$, $u_1 \in [1.6, 1.8]$, respectively. It is observed that in each small interval (i.e., [1.75,1.8], [1.7,1.8], [1.6,1.8]), the black line is always below the purple line, green line, and blue line. Additionally, the purple line and green line are both below the blue line since the blue line has a larger varying range of the component u_1, which exactly conforms to the theoretic analysis that the conservatism of results on the magnitude of the generalized MDF can be reduced by decreasing the varying range of the input vector u_k. The red line represents the magnitude of the generalized MDF when $u_1 \in [1.5, 1.8]$, whose result is the most conservative since all the values of the component u_1 have been considered. Therefore, it can be found that all other lines are below the red line. According to the analysis above, it can be concluded that if we know more infomation on the input vector u_k, we can reduce the conservatism of results on the magnitude of the generalized MDF as expected. Moreover, we can see from the black line in Fig. 3 that the larger the component u_1 is, the smaller the magnitude of the generalized MDF, which is practical and meaningful.

7. CONCLUSION

This paper characterizes the generalized MDF of the invariant set-based robust FD methods for perturbed discrete-time LPV systems affected by multiple multiplicative actuator faults. The main contribution is threefold. First, we obtain the dynamics of the state-estimation error

in the multiple multiplicative actuator-faults situation and transform it into a tractable form. Second, by considering the 1-norm of the fault vector, we define the generalized MDF in the multiple additive actuator-faults situation, which can be computed via solving a simple LP problem. Third, an analysis of the effect of the input vector on the magnitude of the generalized MDF is made. In the future, we are devoted to extend these results to sensor faults.

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