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Loewner Functions for Linear Time-Varying Systems with Applications to Model Reduction ⋆

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Abstract: We introduce a method for model reduction of linear time-varying (LTV) systems by extending the Loewner framework developed for linear time-invariant (LTI) systems. This extension is accomplished by utilizing a state-space interpretation of the Loewner matrices previously developed by the authors. New time-varying Loewner functions are defined, and a Loewner equivalent model is produced using these functions.

Keywords: Model reduction, time-varying systems, state-space realization, interpolation.

1. INTRODUCTION

The problem of model reduction involves finding a simplified model of a dynamical system while preserving desired properties (*e.g.* stability, steady state behaviour). Several approaches for model reduction have been developed. These approaches include moment matching, see Antoulas et al. (1990), Gallivan et al. (2004), Astolfi (2010), Scarciotti and Astolfi (2015), Schulze et al. (2016), Scarciotti and Astolfi (2017), balanced truncation, see Scherpen (1993), Scherpen (1996), Gray and Mesko (1997), Gugercin and Antoulas (2004), and Hankel operators, see Kung and Lin (1981), Glover (1984), Safonov et al. (1990), Fujimoto and Scherpen (2001), Dewilde and Van der Veen (1998). Some of these methods have been applied to the model reduction of linear time-varying (LTV) systems. The systems considered range from systems with simple time-variations, such as the moving load problem, in which only the input matrix varies and/or the moving sensor problem, in which only the output matrix varies, see Fischer et al. (2015), Stykel and Vasilyev (2016), to more complicated plants, in which all system matrices may vary with time. One of the approaches for more complicated time-variations is balancing for LTV systems, see Varga (2000), Lall and Beck (2003), Sandberg and Rantzer (2004), Lang et al. (2016).

An important tool which has been used in the development of reduced-order models and in the solution of the generalized realization problem for LTI systems is the Loewner matrix, see Mayo and Antoulas (2007), which is closely related to the Hankel matrix, see Fiedler (1984) and Belevitch (1970). This matrix was used for the first time to solve rational interpolation problems in Antoulas and Anderson (1986). The Loewner matrix has a special structure and can be factored into two other important matrices: the tangential generalized controllability matrix and the tangential generalized observability matrix. These matrices together can be used to construct linear frequency domain or state-space models as in Antoulas et al. (2014). In light of the time-domain definition of moments in Astolfi (2010), Schulze et al. (2016) has shown that the Loewner framework in Mayo and Antoulas (2007) can be interpreted as a special case of a two-sided moment-matching procedure. Simard and Astolfi (2019) has introduced new objects, the left- and right-Loewner matrices, which allow a new state-space (time-domain) interpretation of the Loewner matrices as the input and output gains of a suitably transformed conceptual experimental setup. Note that the Loewner matrices have typically been interpreted using frequency domain tools. This new interpretation opens up an avenue for more sophisticated usage of the ideas and tools conveyed by the Loewner matrices for LTI systems.

In this paper we utilize this new state-space interpretation to generalize the Loewner method for model reduction from LTI systems to LTV systems. This is accomplished by introducing time-varying generalizations of the Loewner matrices, called Loewner functions, which are then used to construct a model which can produce the exact same left- and right-Loewner functions.

The structure of the paper is as follows. In Sections 2 and 3 we introduce the notation used and we recall preliminary results regarding Loewner matrices and the construction of an interpolating system. In Section 4 we define time-varying generalizations of the Loewner matrices. In Section 5 a conceptual experimental setup

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is introduced, and we perform a transformation exposing the "Loewner functions" of the system. In Section 6 the concept of "Loewner equivalence" is defined, and then a model is constructed (using the original system's Loewner functions) which "interpolates" the Loewner functions produced by the original experimental setup. Finally, in Section 7 we provide a demonstrative model reduction example.

2. PRELIMINARIES

We use standard notation. The set of real numbers is denoted by $\mathbb{R}$. The set of complex numbers is denoted by $\mathbb{C}$. The set of vectors with complex entries with n rows is denoted by $\mathbb{C}^n$. The set of matrices with complex entries with n rows and m columns is denoted by $\mathbb{C}^{n\times m}$. With some abuse of notation $\mathcal{L}_\infty(-\infty,\infty)$ denotes the set of matrix functions with domain $\mathbb{R}$ which are bounded in the infinity norm. The spectrum of a square matrix A is denoted $\sigma(A)$.

The paper Simard and Astolfi (2019) (see also Mayo and Antoulas (2007)) considers a plant described by equations of the form [1]

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Cx, \tag{2}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, and matrices A, B, C of appropriate dimensions. To pose a two-sided tangential interpolation problem one must introduce left tangential data and right tangential data. The right tangential data can be described in matrix form as

$$\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_\rho] \in \mathbb{C}^{\rho\times\rho},$$
$$R = [r_1\ \ldots\ r_\rho] \in \mathbb{C}^{m\times\rho},$$
$$W = [w_1\ \ldots\ w_\rho] \in \mathbb{C}^{p\times\rho},$$

and the left tangential data can be described in matrix form as

$$M = \mathrm{diag}[\mu_1, \ldots, \mu_v] \in \mathbb{C}^{v\times v},$$
$$L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_v \end{bmatrix} \in \mathbb{C}^{v\times p}, \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_v \end{bmatrix} \in \mathbb{C}^{v\times m}.$$

The following assumptions hold for the LTI system (1)-(2), and the tangential data.

Assumption 1. The triple (A, B, C) is a minimal realization of the system (1)-(2).

Assumption 2. The matrices A, Λ, and M have no common eigenvalues, that is

$$\sigma(A) \cap \sigma(\Lambda) = \emptyset,\ \sigma(A) \cap \sigma(M) = \emptyset,\ \sigma(M) \cap \sigma(\Lambda) = \emptyset.$$

Given the transfer matrix $H(s) = C(sI - A)^{-1}B$, the tangential data pertaining to system (1)-(2) are such that

$$w_i = H(\lambda_i)r_i, \quad i = 1, \ldots, \rho, \tag{3}$$

$$v_j = \ell_j H(\mu_j), \quad j = 1, \ldots, v. \tag{4}$$

Given the left and right tangential data, the goal of the two-sided tangential interpolation problem is to construct an interpolating system with transfer function $\tilde{H}(s)$ for which conditions analogous to (3)-(4) hold. A tool that can be used for constructing such an interpolant is the Loewner matrix. The Loewner matrix, $\mathbb{L}$, is defined as the unique (by Assumption 2) solution to the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR,$$

and the shifted Loewner matrix, $\sigma\mathbb{L}$, is defined as the unique (by Assumption 2) solution to the Sylvester equation

$$\sigma\mathbb{L}\Lambda - M\sigma\mathbb{L} = LW\Lambda - MVR.$$

Because the transfer matrix $H(s)$ generates the data in the Loewner matrix, the shifted Loewner matrix is the Loewner matrix associated with the transfer matrix $sH(s)$. These matrices can also be expressed as

$$\mathbb{L} = -YX, \quad \sigma\mathbb{L} = -YAX,$$

where Y and X are the unique solutions to the Sylvester equations

$$YA + LC = MY, \quad AX + BR = X\Lambda.$$

The matrix Y is referred to as the tangential generalized observability matrix, and the matrix X is referred to as the tangential generalized controllability matrix. Furthermore, we have that $V = YB$ and $W = CX$. A very useful construct for the purposes of this paper developed in Simard and Astolfi (2019) is that of the left- and right-Loewner matrices, defined as the unique (by Assumption 2) solutions to

$$M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda = VR, \quad \mathbb{L}^r\Lambda - M\mathbb{L}^r = LW,$$

and the shifted left- and shifted right-Loewner matrices defined as the unique (by Assumption 2) solutions to

$$M\sigma\mathbb{L}^\ell - \sigma\mathbb{L}^\ell\Lambda = MVR, \quad \sigma\mathbb{L}^r\Lambda - M\sigma\mathbb{L}^r = LW\Lambda.$$

This results in

$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r, \quad \sigma\mathbb{L} = \sigma\mathbb{L}^\ell + \sigma\mathbb{L}^r,$$

and

$$\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell, \quad \sigma\mathbb{L}^r = \mathbb{L}^r\Lambda.$$

These objects lend themselves to a state-space interpretation of the Loewner matrices, which have typically been interpreted as frequency domain objects. To see this, consider two systems constructed from the left and right tangential data pertaining to (1)-(2):

$$\dot{\zeta}_r = \Lambda\zeta_r + \Delta, \quad v = R\zeta_r,$$

and

$$\dot{\zeta}_\ell = M\zeta_\ell + L\chi, \quad \eta = \zeta_\ell,$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$. Consider the interconnected system resulting from the interconnection equations $v = u$ and $\chi = y$. This has the state-space representation

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ BR & A & 0 \\ 0 & LC & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{5}$$

$$\eta = [0\ 0\ I] \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}. \tag{6}$$

Consider now the coordinates transformation

$$\begin{bmatrix} \zeta_r \\ z_c \\ z_\ell \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ \mathbb{L}^\ell & Y & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix},$$

[1] All signals are assumed to be complex valued for ease of presentation. Therefore all matrices and functions can be complex valued.

yielding the new state-space representation

$$\begin{bmatrix}\dot{\zeta}_r\\ \dot{z}_c\\ \dot{z}_\ell\end{bmatrix} = \begin{bmatrix}\Lambda & 0 & 0\\ 0 & A & 0\\ 0 & 0 & M\end{bmatrix}\begin{bmatrix}\zeta_r\\ z_c\\ z_\ell\end{bmatrix} + \begin{bmatrix}I\\ -X\\ \mathbb{L}^\ell\end{bmatrix}\Delta,$$

$$\eta = [\mathbb{L}^r \;\; -Y \;\; I]\begin{bmatrix}\zeta_r\\ z_c\\ z_\ell\end{bmatrix}.$$

Thus, the Loewner matrices can be interpreted as the input and output "gains" of three systems running in parallel and producing the exact same output as (5)-(6).

Following Mayo and Antoulas (2007), given $\mathbb{L}$, $\sigma\mathbb{L}$, V, and W associated to (5)-(6), an interpolating system of the form (1)-(2) which produces identical Loewner matrices can be constructed. This interpolant is given by

$$\dot{r} = \mathbb{L}^{-1}\sigma\mathbb{L}r - \mathbb{L}^{-1}Vu_r, \quad y_r = Wr,$$

with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$.

3. PROBLEM FORMULATION

In this paper we consider linear time-varying systems of the form

$$\dot{x} = A(t)x + B(t)u, \tag{7}$$

$$y = C(t)x, \tag{8}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, and time-varying matrices $A(t)$, $B(t)$, $C(t)$ of appropriate dimensions. The following assumption holds throughout this paper.

Assumption 3. The triple $(A(t), B(t), C(t))$ describes a minimal realization for every $t \in \mathbb{R}$.

The goal of this work is to extend the interpolation methods based on the use of Loewner matrices for LTI systems to LTV systems by introducing the notion of "Loewner functions", which generalize the notion of Loewner matrices. This goal is accomplished by utilizing the state-space interpretation of the Loewner matrices developed in Simard and Astolfi (2019).

4. THE TIME-VARYING LOEWNER OBJECTS

Before presenting the main results we define time-varying generalizations of the tangential generalized controllability and observability matrices, and of the Loewner matrices. Consider matrix functions of time $\Lambda : \mathbb{R} \to \mathbb{C}^{\rho\times\rho}$, $M : \mathbb{R} \to \mathbb{C}^{v\times v}$, $R : \mathbb{R} \to \mathbb{C}^{m\times\rho}$, and $L : \mathbb{R} \to \mathbb{C}^{v\times p}$. Then the tangential generalized controllability function, $X(\cdot)$, and the tangential generalized observability function, $Y(\cdot)$, for the system (7)-(8) are defined as the unique (by Behr et al. (2018)) solutions to the Sylvester ODEs

$$\frac{dX}{dt} = A(t)X(t) - X(t)\Lambda(t) + B(t)R(t), \; X(t_0) = X_0, \tag{9}$$

and

$$\frac{dY}{dt} = M(t)Y(t) - Y(t)A(t) - L(t)C(t), \; Y(t_0) = Y_0, \tag{10}$$

where X_0 and Y_0 are chosen to be any initial conditions such that $X(\cdot) \in \mathcal{L}_\infty(-\infty,\infty)$ and $Y(\cdot) \in \mathcal{L}_\infty(-\infty,\infty)$. Additionally, for all t we define the functions

$$V(t) := Y(t)B(t), \quad W(t) := C(t)X(t),$$

as time-varying generalizations of the matrices V and W. The time-varying Loewner function is then defined for all t as

$$\mathbb{L}(t) := -Y(t)X(t). \tag{11}$$

We now define the left Loewner function, $\mathbb{L}^\ell(t)$, and the right Loewner function, $\mathbb{L}^r(t)$, as the unique (by Behr et al. (2018)) solutions to the ODEs

$$\frac{d\mathbb{L}^\ell}{dt} = M(t)\mathbb{L}^\ell(t) - \mathbb{L}^\ell(t)\Lambda(t) - V(t)R(t), \; \mathbb{L}^\ell(t_0) = \mathbb{L}^\ell_0, \tag{12}$$

and

$$\frac{d\mathbb{L}^r}{dt} = M(t)\mathbb{L}^r(t) - \mathbb{L}^r(t)\Lambda(t) + L(t)W(t), \; \mathbb{L}^r(t_0) = \mathbb{L}^r_0, \tag{13}$$

where $\mathbb{L}^\ell_0$ and $\mathbb{L}^r_0$ are chosen to be any initial conditions such that $\mathbb{L}^\ell_0 + \mathbb{L}^r_0 = -Y_0X_0$. By (9), (10), and (11) we have that

$$\frac{d\mathbb{L}}{dt} = M(t)\mathbb{L}(t) - \mathbb{L}(t)\Lambda(t) + L(t)W(t) - V(t)R(t). \tag{14}$$

Furthermore, by (12), (13), and (14), we have that, for all t,

$$\mathbb{L}(t) = \mathbb{L}^\ell(t) + \mathbb{L}^r(t). \tag{15}$$

Finally, for all t we define the shifted Loewner matrices as

$$\sigma\mathbb{L}^\ell(t) := M(t)\mathbb{L}^\ell(t), \quad \sigma\mathbb{L}^r(t) := \mathbb{L}^r(t)\Lambda(t),$$

$$\sigma\mathbb{L}(t) := \sigma\mathbb{L}^\ell(t) + \sigma\mathbb{L}^r(t). \tag{16}$$

Note that in the case in which $\Lambda(\cdot)$ and $M(\cdot)$ are constant matrices (*i.e.* $\Lambda(t) = \Lambda$, $M(t) = M$, for all t), then we obtain the ODEs

$$\frac{d\sigma\mathbb{L}^\ell}{dt} = M\frac{d\mathbb{L}^\ell}{dt} = M\sigma\mathbb{L}^\ell(t) - \sigma\mathbb{L}^\ell(t)\Lambda - MV(t)R(t),$$

$$\frac{d\sigma\mathbb{L}^r}{dt} = \frac{d\mathbb{L}^r}{dt}\Lambda = M\sigma\mathbb{L}^r(t) - \sigma\mathbb{L}^r(t)\Lambda + L(t)W(t)\Lambda,$$

$$\frac{d\sigma\mathbb{L}}{dt} = M\sigma\mathbb{L}(t) - \sigma\mathbb{L}(t)\Lambda + L(t)W(t)\Lambda - MV(t)R(t),$$

which can be used to determine the shifted Loewner functions independently from the Loewner functions.

5. THE CONCEPTUAL EXPERIMENTAL SETUP

Similarly to Simard and Astolfi (2019), in this section we construct a conceptual experimental setup to develop system theoretic interpretations of the time-varying Loewner functions corresponding to the matrix functions $\Lambda(\cdot)$, $R(\cdot)$, $M(\cdot)$, $L(\cdot)$, and the plant (7)-(8). To accomplish this, we begin by constructing two LTV systems using $\Lambda(\cdot)$, $R(\cdot)$, $M(\cdot)$, and $L(\cdot)$, namely

$$\dot{\zeta}_r = \Lambda(t)\zeta_r + \Delta, \quad v = R(t)\zeta_r, \tag{17}$$

and

$$\dot{\zeta}_\ell = M(t)\zeta_\ell + L(t)\chi, \quad \eta = \zeta_\ell, \tag{18}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$. Consider now the interconnected system consisting of the plant (7)-(8) along with the systems (17)-(18), resulting from the interconnection equations $v = u$ and $\chi = y$. This yields the state-space realization

$$\begin{bmatrix}\dot{\zeta}_r\\ \dot{x}\\ \dot{\zeta}_\ell\end{bmatrix} = \begin{bmatrix}\Lambda(t) & 0 & 0\\ B(t)R(t) & A(t) & 0\\ 0 & L(t)C(t) & M(t)\end{bmatrix}\begin{bmatrix}\zeta_r\\ x\\ \zeta_\ell\end{bmatrix} + \begin{bmatrix}I\\ 0\\ 0\end{bmatrix}\Delta, \quad (19)$$

$$\eta = [0 \quad 0 \quad I]\begin{bmatrix}\zeta_r\\ x\\ \zeta_\ell\end{bmatrix}, \quad (20)$$

with state $\left[\zeta_r^\top,\ x^\top,\ \zeta_\ell^\top\right]^\top$, input Δ, and output η. Similarly to the LTI experimental setup in Simard and Astolfi (2019), the time-varying Loewner objects are somehow encoded in the interconnected system (19)-(20). To show how these functions are related to the interconnected system we select a specific set of coordinates. We expose these functions in Theorem 1.

Theorem 1. Consider the system (19)-(20). The coordinates transformation

$$\begin{bmatrix}\zeta_r\\ z_c\\ z_\ell\end{bmatrix} := \begin{bmatrix}I & 0 & 0\\ -X(t) & I & 0\\ \mathbb{L}^\ell(t) & Y(t) & I\end{bmatrix}\begin{bmatrix}\zeta_r\\ x\\ \zeta_\ell\end{bmatrix}$$

is such that the system in the new coordinates is described by the equations

$$\begin{bmatrix}\dot{\zeta}_r\\ \dot{z}_c\\ \dot{z}_\ell\end{bmatrix} = \begin{bmatrix}\Lambda(t) & 0 & 0\\ 0 & A(t) & 0\\ 0 & 0 & M(t)\end{bmatrix}\begin{bmatrix}\zeta_r\\ z_c\\ z_\ell\end{bmatrix} + \begin{bmatrix}I\\ -X(t)\\ \mathbb{L}^\ell(t)\end{bmatrix}\Delta,$$

$$\eta = [\mathbb{L}^r(t) \quad -Y(t) \quad I]\begin{bmatrix}\zeta_r\\ z_c\\ z_\ell\end{bmatrix}.$$

Proof. The proof follows by direct calculation. Starting with $z_c = x - X(t)\zeta_r$ we have

$$\begin{aligned}\dot{z}_c &= \dot{x} - \frac{dX}{dt}\zeta_r - X(t)\dot{\zeta}_r\\ &= A(t)z_c - X(t)\Delta\\ &\quad + \Big(B(t)R(t) + A(t)X(t) - X(t)\Lambda(t) - \frac{dX}{dt}\Big)\zeta_r,\end{aligned}$$

which, by (9), becomes

$$\dot{z}_c = A(t)z_c - X(t)\Delta.$$

Next we consider $z_\ell = \zeta_\ell + Y(t)x + \mathbb{L}^\ell(t)\zeta_r$. This yields

$$\begin{aligned}\dot{z}_\ell &= \dot{\zeta}_\ell + \frac{dY}{dt}x + Y(t)\dot{x} + \frac{d\mathbb{L}^\ell}{dt}\zeta_r + \mathbb{L}^\ell(t)\dot{\zeta}_r\\ &= M(t)z_\ell + \mathbb{L}^\ell(t)\Delta\\ &\quad + \Big(L(t)C(t) + Y(t)A(t) - M(t)Y(t) + \frac{dY}{dt}\Big)x\\ &\quad + \Big(V(t)R(t) + \mathbb{L}^\ell(t)\Lambda(t) - M(t)\mathbb{L}^\ell(t) + \frac{d\mathbb{L}^\ell}{dt}\Big)\zeta_r,\end{aligned}$$

which, by (10) and (12), yields

$$\dot{z}_\ell = M(t)z_\ell + \mathbb{L}^\ell(t)\Delta.$$

Finally,

$$\begin{aligned}\eta = \zeta_\ell &= z_\ell - Y(t)x - \mathbb{L}^\ell(t)\zeta_r\\ &= \Big(-Y(t)X(t) - \mathbb{L}^\ell(t)\Big)\zeta_r - Y(t)z_c + z_\ell,\end{aligned}$$

which, by (11) and (15), becomes

$$\eta = \mathbb{L}^r(t)\zeta_r - Y(t)z_c + z_\ell,$$

concluding the proof. □

Theorem 1 lends itself to a system theoretic interpretation of the time-varying Loewner functions: the Loewner functions and the observability/controllability functions can be viewed as the input and output "gains" of three LTV systems connected in parallel and such that the resulting input/output behaviour coincides with that of (19)-(20).

6. LOEWNER EQUIVALENT MODELS

In this section we develop an LTV system, reminiscent of that presented in Mayo and Antoulas (2007), which interpolates the Loewner functions generated from (7)-(8). However, as we are no longer using frequency domain tools, we must define what we mean by an interpolant when referring to LTV systems, for which conditions (3)-(4) hold little meaning.

Definition 1. (**Loewner Equivalence**). Let Σ and $\overline{\Sigma}$ be two systems admitting:

- left- and right-Loewner functions $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$, and $\bar{\mathbb{L}}^\ell(\cdot)$, $\bar{\mathbb{L}}^r(\cdot)$, respectively,
- tangential generalized observability and controllability matrices $Y(\cdot)$, $X(\cdot)$, and $\bar{Y}(\cdot)$, $\bar{X}(\cdot)$, respectively.

Then Σ and $\overline{\Sigma}$ are called Loewner equivalent at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$ if there exists $Y(t_0)$, $X(t_0)$, $\bar{Y}(t_0)$, and $\bar{X}(t_0)$, with $Y(t_0)X(t_0) = \bar{Y}(t_0)\bar{X}(t_0)$, such that $\mathbb{L}^\ell(t_0) = \bar{\mathbb{L}}^\ell(t_0)$ and $\mathbb{L}^r(t_0) = \bar{\mathbb{L}}^r(t_0)$ imply $\mathbb{L}^\ell(t) = \bar{\mathbb{L}}^\ell(t)$ and $\mathbb{L}^r(t) = \bar{\mathbb{L}}^r(t)$, for all $t \in \mathbb{R}$.

We say that an LTV system interpolates the Loewner functions of another system at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$ when the two systems are Loewner equivalent at the given $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$. That is, for the same $\Lambda(\cdot)$, $R(\cdot)$, $M(\cdot)$, $L(\cdot)$, the interpolating system produces the exact same left- and right-Loewner functions. We can now define what a reduced order model is in the Loewner sense.

Definition 2. (**Reduced Order Model**). Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$ and $v < n$.

Considering Theorem 1 it is easy to see that (provided suitable stability conditions hold) any system which produces the same left-Loewner and right-Loewner functions as (19)-(20) has input/output behaviour that differs only in the transient response (*i.e.* the response related to $A(\cdot)$ and $\bar{A}(\cdot)$).

We now construct an LTV system which produces the same Loewner functions as (19)-(20). This model, which is equivalent to (19)-(20) in the Loewner sense, is constructed using the Loewner functions generated by (19)-(20).

Theorem 2. Consider the connected system (19)-(20) with $\rho = v$ (*i.e.* $\mathbb{L}(\cdot)$ is square). Let $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, and $W(\cdot)$ be the associated Loewner functions, with $\mathbb{L}(t)$ non-singular for all $t \geq t_0$. Define the system

$$\begin{aligned}\overbrace{(\mathbb{L}(t)r)}^{\cdot} - \dot{\mathbb{L}}^r(t)r = \mathbb{L}^r(t)\dot{r} + \overbrace{(\mathbb{L}^\ell(t)r)}^{\cdot} &= \mathbb{L}(t)\dot{r} + \dot{\mathbb{L}}^\ell(t)r\\ &= \sigma\mathbb{L}(t)r - V(t)u_r, \quad (21)\\ y_r &= W(t)r, \quad (22)\end{aligned}$$

with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$. Then the system (21)-(22) is Loewner equivalent to the system (7)-(8) at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$.

Remark 3. The left- and right-Loewner functions are necessary for the definition of the Loewner equivalent model (21)-(22); this contrasts with the LTI case in which the left- and right-Loewner matrices are not directly used in the Loewner equivalent model, see Mayo and Antoulas (2007) and Simard and Astolfi (2019).

Proof. Let $\bar{X}(\cdot)$, $\bar{Y}(\cdot)$, $\bar{V}(\cdot)$, $\bar{W}(\cdot)$, $\bar{\mathbb{L}}^\ell(\cdot)$, $\bar{\mathbb{L}}^r(\cdot)$, $\bar{\mathbb{L}}(\cdot)$, $\sigma\bar{\mathbb{L}}^\ell(\cdot)$, $\sigma\bar{\mathbb{L}}^r(\cdot)$, $\sigma\bar{\mathbb{L}}(\cdot)$ be the time-varying Loewner objects associated with the system (21)-(22) interconnected with the generators (17)-(18) by the equations $v = u_r$ and $\chi = y_r$. Recall that there could be multiple sets of valid Loewner functions for this system, however in this proof we choose a particular set. In order to prove the result we show that, for all t, $\bar{X}(t) = I, \bar{Y}(t) = -\mathbb{L}(t), \bar{W}(t) = W(t), \bar{V}(t) = V(t)$, and finally that $\bar{\mathbb{L}}^\ell(t) = \mathbb{L}^\ell(t)$ and $\bar{\mathbb{L}}^r(t) = \mathbb{L}^r(t)$ are a set of time-varying Loewner objects belonging to $\mathcal{L}_\infty(-\infty, \infty)$. We start by rearranging (21)-(22) so that

$$\dot{r} = \mathbb{L}(t)^{-1}\Big(\sigma\mathbb{L}(t) - \frac{d\mathbb{L}^\ell}{dt}\Big)r - \mathbb{L}(t)^{-1}V(t)u_r, \quad y_r = W(t)r.$$

The generalized controllability function of the interpolating model is the solution to

$$\frac{d\bar{X}}{dt} = \mathbb{L}(t)^{-1}\Big(\sigma\mathbb{L}(t) - \frac{d\mathbb{L}^\ell}{dt}\Big)\bar{X}(t) - \bar{X}(t)\Lambda(t) - \mathbb{L}(t)^{-1}V(t)R(t), \quad \bar{X}(t_0) = I. \tag{23}$$

Subbing (12) and (16) into (23) yields

$$\frac{d\bar{X}}{dt} = \big(\Lambda(t)\bar{X}(t) - \bar{X}(t)\Lambda(t)\big) + \mathbb{L}(t)^{-1}\big(\bar{X}(t) - I\big),$$

which is solved by the $\mathcal{L}_\infty(-\infty, \infty)$ function $\bar{X}(t) = I$ for all t. The generalized observability function of the interpolating model is the solution to

$$\frac{d\bar{Y}}{dt} = M(t)\bar{Y}(t) - \bar{Y}(t)\mathbb{L}(t)^{-1}\Big(\sigma\mathbb{L}(t) - \frac{d\mathbb{L}^\ell}{dt}\Big) - L(t)W(t), \quad \bar{Y}(t_0) = -\mathbb{L}(t_0). \tag{24}$$

Subbing (12) and (16) into (24) yields

$$\frac{d\bar{Y}}{dt} = M(t)\bar{Y}(t) - \bar{Y}(t)\Lambda(t) - \bar{Y}(t)\mathbb{L}(t)^{-1}V(t)R(t) - L(t)W(t),$$

which, by (14), is solved by the $\mathcal{L}_\infty(-\infty, \infty)$ function $\bar{Y}(t) = -\mathbb{L}(t)$ for all t. We therefore have, for all t,

$$\bar{W}(t) = W(t)\bar{X}(t) = W(t),$$
$$\bar{V}(t) = -\bar{Y}(t)\mathbb{L}(t)^{-1}V(t) = V(t),$$
$$\bar{\mathbb{L}}(t) = -\bar{Y}(t)\bar{X}(t) = \mathbb{L}(t).$$

The interpolating model's left-Loewner function is the solution to

$$\frac{d\bar{\mathbb{L}}^\ell}{dt} = M(t)\bar{\mathbb{L}}^\ell(t) - \bar{\mathbb{L}}^\ell(t)\Lambda(t) - \bar{V}(t)R(t), \quad \bar{\mathbb{L}}^\ell(t_0) = \mathbb{L}^\ell(t_0), \tag{25}$$

and the interpolating model's right-Loewner function is the solution to

$$\frac{d\bar{\mathbb{L}}^r}{dt} = M(t)\bar{\mathbb{L}}^r(t) - \bar{\mathbb{L}}^r(t)\Lambda(t) + L(t)\bar{W}(t), \quad \bar{\mathbb{L}}^r(t_0) = \mathbb{L}^r(t_0). \tag{26}$$

It is now easy to see that, by (12) and (25), $\bar{\mathbb{L}}^\ell(t) = \mathbb{L}^\ell(t)$, and, by (13) and (26), $\bar{\mathbb{L}}^r(t) = \mathbb{L}^r(t)$, for all t. □

Theorem 2 can be used to construct reduced order models of the system (7)-(8) in the Loewner sense by simply setting the dimension $\rho < n$.

Note that the fact that systems (7)-(8) and (21)-(22) are Loewner equivalent at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$ has an important implication for the steady-state responses of the systems connected with (17)-(18) provided that the steady-state response is well-defined. Because the two systems are Loewner equivalent, by Theorem 1, the outputs of the connected systems differ only in the free responses associated to $A(\cdot)$ and to $\mathbb{L}(\cdot)^{-1}\big(\sigma\mathbb{L}(\cdot) - \frac{d\mathbb{L}^\ell}{dt}\big)$. Furthermore, if both (7)-(8) and (21)-(22) are asymptotically stable systems, then they produce the same steady-state behaviour when interconnected to systems (17)-(18).

7. MODEL REDUCTION EXAMPLE

In this section we use the tools developed in the previous sections in a demonstrative example. Consider a time-varying system of the form (7)-(8) defined by the equations

$$\dot{x} = \begin{bmatrix} -1 & \sin(t) \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0]\, x. \tag{27}$$

We would like to produce a reduced order model of this system in the Loewner sense when the generators (17)-(18) have the matrices $\Lambda(t) = 0$, $R(t) = 1$, $M(t) = -3$, and $L(t) = 1$. By direct calculation, and setting the initial conditions so that the experimental setup is initialized in steady-state, the tangential generalized controllability and observability functions are

$$X(t) = \begin{bmatrix} 1 + \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ 1 \end{bmatrix},$$

$$Y(t) = \begin{bmatrix} -\frac{1}{2} & \frac{\sin(t)}{5} - \frac{\cos(t)}{10} \end{bmatrix},$$

which implies that

$$W(t) = 1 + \frac{\sin(t)}{2} - \frac{\cos(t)}{2},$$
$$V(t) = -\frac{1}{2} + \frac{\sin(t)}{5} - \frac{\cos(t)}{10}.$$

Similarly, the Loewner functions are

$$\mathbb{L}(t) = \frac{1}{2} + \frac{\sin(t)}{20} - \frac{3\cos(t)}{20},$$
$$\mathbb{L}^\ell(t) = \frac{1}{6} + \frac{\cos(t)}{20} - \frac{\sin(t)}{20},$$
$$\mathbb{L}^r(t) = \frac{1}{3} + \frac{\sin(t)}{10} - \frac{\cos(t)}{5},$$

and the shifted Loewner functions are

$$\sigma\mathbb{L}^\ell(t) = -\frac{1}{2} + \frac{3\sin(t)}{20} - \frac{3\cos(t)}{20}, \quad \sigma\mathbb{L}^r(t) = 0,$$
$$\sigma\mathbb{L}(t) = -\frac{1}{2} + \frac{3\sin(t)}{20} - \frac{3\cos(t)}{20}.$$

Using Theorem 2 we have that the system defined by the equations, which are well-defined for all t,

$$\dot{r} = -\frac{\left(1 - \frac{2\sin(t)}{5} + \frac{\cos(t)}{5}\right)}{\left(1 + \frac{\sin(t)}{10} - \frac{3\cos(t)}{10}\right)}\left(r - u_r\right),$$

$$y_r = \left(1 + \frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right)r,$$

is equivalent in the Loewner sense to the system (27) at $(\Lambda(\cdot), R(\cdot), M(\cdot), L(\cdot))$, and is thus a reduced order model in the Loewner sense [2].

8. CONCLUSION

We have used a state-space interpretation of the Loewner matrices for LTI systems, developed in Simard and Astolfi (2019), in order to generalize the methods of model reduction using Loewner matrices to LTV systems. This has been accomplished by introducing Loewner functions, the time-varying generalization of Loewner matrices. We have then shown how to construct an interpolating system using the Loewner functions, and finally we have demonstrated the application of the methods via an example.

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[2] While the dimensionality of the system may not be the best measure of complexity of a model it is, probably, the only quantitative measure available.