Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Stackelberg Mean-Field-Type Games with Polynomial Cost ⋆

Zahrate El Oula Frihi * **Julian Barreiro-Gomez** **,***
Salah Eddine Choutri **,***,**** **Boualem Djehiche** ****
Hamidou Tembine **,***

* *Lab. of Probability and Statistics (LaPS), Department of Mathematics, Badji-Mokhtar University, B.P.12, 23000, Annaba, Algeria (e-mail: zahrate.frihi@univ-annaba.org)*
** *Learning & Game Theory Laboratory, New York University Abu Dhabi, Saadiyat Campus PO Box 129188, UAE (e-mail: {jbarreiro,tembine}@nyu.edu).*
*** *Research Center on Stability, Instability and Turbulence, New York University Abu Dhabi, Saadiyat Campus PO Box 129188, UAE*
**** *Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden (e-mail: boualem@math.kth.se)*

Abstract: This article presents a class of Stackelberg mean-field-type games with multiple leaders and multiple followers. The decision-makers act in sequential order with informational differences. The state dynamics is driven by jump-diffusion processes and the cost function is non-quadratic and has a polynomial structure. The structures of Stackelberg strategies and costs of the leaders and followers are given in a semi-explicit way in state-and- mean-field-type feedback form. A sufficiency condition is provided using an infinite dimensional partial integro-differential system. The methodology is extended to multi-level hierarchical systems. It is shown that not only the set of decision-makers per level matters but also the number of hierarchical levels plays a key role in the global performance of the system. We also identify specific range of parameters for which the Nash equilibrium coincides with the hierarchical solution independently of the number of layers and the order of play.

Keywords: Mean-field-type games, Stackelberg solutions, hierarchical game design, semi-explicit solutions.

1. INTRODUCTION

One of the central questions in game theory is understanding the difficulties that parties have in reaching a solution when decision-makers act in a sequential order with different information structure. Informational differences provide an appealing explanation for the solution concept. A distinction between the order of play was introduced in von Stackelberg (1934) within the context of game theory. In this original work, he distinguished between the first mover, called the primary decision-maker (the leader), and the second mover, called the secondary decision-maker (the follower). The idea is to investigate the best response and the reaction of the decision-makers when the follower can observe the move of the leader and subsequently acts.

Mean-field-type game theory studies a class of games in which the payoffs and or state dynamics depend not only on the state-action pairs but also the distribution of them. This class of games offers several features:

- a single decision-maker can have a strong impact on the mean-field terms,
- the expected payoffs are not necessarily linear with respect to the state distribution,
- the number of decision-makers is not necessarily infinite.

Games with non-linearly distribution-dependent quantity-of-interest are very attractive in terms of applications because the non-linear dependence of the payoff functions in terms of state distribution allow us to capture risk measures which are functionals of variance, inverse quantile, and or higher moments.

Only few works consider hierarchical structures in mean-field related games. Open-loop Stackelberg solutions are addressed in linear-quadratic setting in Lin et al. (2019); Du and Wu (2019). In the context of large populations, mean-field Stackelberg games are investigated in (Moon and Basar, 2015; Bensoussan et al., 2015, 2017; Averboukh, 2018).

Our contribution can be summarized as follows. This work examines a class of hierarchical mean-field-type games with multiple leaders and multiple followers. Based on infinite dimensional partial integrodifferential equations

⋆ We gratefully acknowledge support from U.S. Air Force Office of Scientific Research under grant number FA9550-17-1-0259. The authors are members of the NYUAD Research Center on Stability, Instability and Turbulence

(PIDEs), we provide semi-explicit solutions of a class of master systems with hierarchical structure and non-quadratic cost, which are not covered by the earlier works.

The rest of this article is structured as follows. We present the model setup in Section 2. Section 3 investigates the Nash equilibrium (no leader). Section 4 presents Stackelberg solution. Finally, concluding remarks are presented in Section 5.

2. THE SETUP

The time horizon of the interaction is $[t_0, t_1], \quad t_0 < t_1$. There are $I \geq 1$ agents. The set of agents is denoted by $\mathcal{I} = \{1, 2, \ldots, I\}$. Agent $i \in \mathcal{I}$ has a control action $u_i \in U_i = \mathbb{R}$. The state x is driven by Drift-Jump-Diffusion of mean-field type given by

$$dx = bdt + \sigma dB + \int_{\Theta} \mu(.,\theta)\tilde{N}(dt, d\theta), \quad x(t_0) \sim m(t_0, .),$$

where

Drift: b,
Diffusion: Brownian motion B,
Jump: $\tilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)dt$,

$$b, \sigma, \mu(.,\theta), : \ [t_0, t_1] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \prod_{j=1}^{I} U_j \to \mathbb{R}.$$

The performance functional of agent i is

$$L_i(u, m_0) = h_i(x(t_1), m(t_1)) + \int_{t_0}^{t_1} l_i(t, x, u, m)dt,$$

where $m(t, dy) = \mathbb{P}_{x(t)}(dy)$ is the probability measure of the state $x(t)$ at time t.

We assume the following information structure:

- perfect state measurement, i.e., all the decision-makers observe the state x
- perfect knowledge of the model i.e., the model above is known by all decision-makers.

In addition, each decision-maker is assumed to have a computability capability such that is able to compute an aggregative term, denoted by m, from the model. Let $\mathcal{U}_i$ be the set of control strategies of decision-maker i that are progressively measurable with the respect to the filtration generated by the unions of events in $\{B, N\}$.

2.1 Games with polynomial cost

We investigate the mean-field-type game with the following data: $t_0 = 0$, $t_1 = T > 0$,

$$\begin{aligned} l_i(t, x, u, m) &= q_i \frac{(x - \bar{x})^{2k_i}}{2k_i} + r_i \frac{(u_i - \bar{u}_i)^{2k_i}}{2k_i} \\ &+ c_i (x - \bar{x})^{2k_i - 1}(u_i - \bar{u}_i) \\ &+ \sum_{j \in \mathcal{I}\setminus\{i\}} \epsilon_{ij}(x - \bar{x})^{2(k_i - 1)}(u_i - \bar{u}_i)(u_j - \bar{u}_j) \\ &+ \bar{q}_i \frac{\bar{x}^{2\bar{k}_i}}{2\bar{k}_i} + \bar{r}_i \frac{\bar{u}_i^{2\bar{k}_i}}{2\bar{k}_i} + \bar{c}_i \bar{x}^{2\bar{k}_i - 1}\bar{u}_i \\ &+ \sum_{j \neq i} \bar{\epsilon}_{ij} \bar{x}^{2(\bar{k}_i - 1)} \bar{u}_i \bar{u}_j, \end{aligned} \tag{1a}$$

$$h_i(x, m) = q_{iT} \frac{(x_T - \bar{x}_T)^{2k_i}}{2k_i} + \bar{q}_{iT} \frac{\bar{x}_T^{2\bar{k}_i}}{2\bar{k}_i}, \tag{1b}$$

$$\begin{aligned} b(t, x, u, m) &= b_1(x - \bar{x}) + \bar{b}_1 \bar{x} \\ &+ \sum_{j \in \mathcal{I}} \left[b_{2j}(u_j - \bar{u}_j) + \bar{b}_{2j} \bar{u}_j \right], \end{aligned} \tag{1c}$$

$$\sigma(t, x, u, m) = (x - \bar{x})\tilde{\sigma}, \tag{1d}$$

$$\mu(t, x, u, m, \theta) = (x - \bar{x})\tilde{\mu}(., \theta), \tag{1e}$$

$$\bar{x}(t) = \int y m(t, dy), \tag{1f}$$

$$\bar{u}(t) = \int u(t, y, m) m(t, dy), \tag{1g}$$

where $k_i \geq 1$, $\bar{k}_i \geq 1$, are natural numbers, and the coefficients are time dependent. The coefficient functions $q_i, r_i, \bar{q}_i, \bar{r}_i$, are nonnegative functions.

3. NASH MEAN-FIELD-TYPE EQUILIBRIUM

The risk-neutral mean-field-type game is given by

$$(\mathcal{I}, U_i, \mathcal{U}_i, \mathbb{E}[L_i])_{i \in \mathcal{I}}.$$

A risk-neutral Mean-Field-Type Nash Equilibrium is a solution of the following fixed-point problem:

$$\begin{aligned} &i \in \mathcal{I}, \\ &\mathbb{E}[L_i(u^*, m_0)] \\ &\quad = \inf_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_I^*, m_0)]. \end{aligned}$$

Let $\hat{V}_i(t, m)$ be the optimal cost-to-go from m at time $t \in (t_0, t_1)$ given the strategies of the others, i.e.,

$$\begin{aligned} \hat{V}_i(t, m) &= \inf_{u_i} \mathbb{E}[h_i(x(t_1), m(t_1)) \\ &\quad + \int_t^{t_1} l_i(t, x, u, m)dt' | m(t) = m]. \end{aligned}$$

Let $\hat{V}_{i,m}$ be the Gâteaux derivative of $\hat{V}_i(t, .)$ with the respect to the measure m. Introduce the integrand Hamiltonian as

$$\begin{aligned} &H_i(x, m, (\hat{V}_{j,m}, \hat{V}_{j,xm}, \hat{V}_{j,xxm})_j) \\ &\quad = \inf_{u_i \in U_i} \Big\{ l_i + b\, \hat{V}_{i,xm} + \frac{\sigma^2}{2} \hat{V}_{i,xxm} \\ &\quad + \int_{\Theta} [\hat{V}_{i,m}(t_-, x + \mu) - \hat{V}_{i,m} - \mu \hat{V}_{i,xm}] \nu(d\theta) \Big\}. \end{aligned}$$

Denote the jump operator J as

$$J[\phi_i] := \int_{\Theta} [\phi_{i,m}(t_-, x + \mu) - \phi_{i,m} - \mu \phi_{i,xm}] \nu(d\theta),$$

and let J^* be the adjoint operator of J:

$$\langle J[\phi], m \rangle = \langle \phi, J^*[m] \rangle.$$

A sufficiency condition for a risk-neutral Nash equilibrium system is given by the following (backward-forward) partial integro-differential system

$$i \in \mathcal{I}, \tag{2a}$$

$$\begin{aligned} 0 &= \hat{V}_{i,t}(t, m) \\ &+ \int_x H_i(x, m, (\hat{V}_{j,m}, \hat{V}_{j,xm}, \hat{V}_{j,xxm})_{j \in \mathcal{I}}) m(t, dx), \end{aligned} \tag{2b}$$

$$\hat{V}_i(t_1, m) = \int m(t_1, dy) h_i(y, m(t_1)), \tag{2c}$$

$$m_t = -(mb)_x + \frac{1}{2}(m\sigma^2)_{xx} + J^*[m], \tag{2d}$$

$$m(0) = m_0. \tag{2e}$$

We refer the reader to Bensoussan et al. (2020) for a derivation of this equilibrium system. The system (2) is an infinite PIDE system in m and it provides the Nash equilibrium values of the mean-field-type game. Notice that the PIDE system, satisfied by $\{\hat{V}_{i,m}\}_{i\in\mathcal{I}}$, is not the **equilibrium value** $\{\hat{V}_i\}_{i\in\mathcal{I}}$ in (2). We use (2) to find risk-neutral Nash mean-field-type equilibrium.

Proposition 1. A risk-neutral mean-field-type Nash equilibrium is given in a semi-explicit way as follows:

$$u_i^{ne} = -\eta_i\left(x - \int ym(dy)\right) - \bar{\eta}_i\int ym(dy), \quad (3a)$$

$$0 = -r_i\eta_i^{2k_i-1} - \sum_{j\neq i}\epsilon_{ij}\eta_j + b_{2i}\alpha_i + c_i, \quad (3b)$$

$$0 = -\bar{r}_i\bar{\eta}_i^{2\bar{k}_i-1} - \sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_j + \bar{b}_{2i}\bar{\alpha}_i + \bar{c}_i, \quad (3c)$$

$$\hat{V}_i(t,m) = \alpha_i\int_x \frac{(x-\int ym(dy))^{2k_i}}{2k_i}m(dx) + \bar{\alpha}_i\frac{(\int ym(dy))^{2\bar{k}_i}}{2\bar{k}_i}, \quad (3d)$$

$$0 = \dot{\alpha}_i + q_i + r_i\eta_i^{2k_i} - 2k_ic_i\eta_i + 2k_i\sum_{j\neq i}\epsilon_{ij}\eta_i\eta_j + 2k_i\alpha_i[b_1 - \sum_{j\in\mathcal{I}}b_{2j}\eta_j] + 2k_i(2k_i-1)\alpha_i\frac{1}{2}\tilde{\sigma}^2 + \alpha_i\int_\Theta[(1+\tilde{\mu})^{2k_i} - 1 - 2k_i\tilde{\mu}]\nu(d\theta), \quad (3e)$$

$$\alpha_i(T) = q_{iT}, \quad (3f)$$

$$0 = \dot{\bar{\alpha}}_i + \bar{q}_i + \bar{r}_i\bar{\eta}_i^{2\bar{k}_i} - 2\bar{k}_i\bar{c}_i\bar{\eta}_i + 2\bar{k}_i\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j + 2\bar{k}_i\bar{\alpha}_i[\bar{b}_1 - \sum_j\bar{b}_{2j}\bar{\eta}_j], \quad (3g)$$

$$\bar{\alpha}_i(T) = \bar{q}_{iT}, \quad (3h)$$

for all $i\in\mathcal{I}$ with

$$\int ym(t,dy) = \left[\int ym(0,dy)\right]e^{\int_0^t\left[\bar{b}_1 - \sum_j\bar{b}_{2j}\bar{\eta}_j\right]dt}, \quad (3i)$$

whenever the above coefficient system admits a unique solution. □

Proof. This proof is developed by solving the PIDE system (2) using the following guess functional of agent i as

$$\hat{V}_i(t,m) = \alpha_i(t)\int_x\frac{(x-\int ym(dy))^{2k_i}}{2k_i}m(dx) + \bar{\alpha}_i(t)\frac{(\int ym(dy))^{2\bar{k}_i}}{2\bar{k}_i},$$

where the coefficient functions $\alpha_i, \bar{\alpha}_i$ need to be determined. We ommit details of this proof due to the lack of space. ■

The uniqueness of the coefficient system (3) in η requires a strong condition. For example for $k=1$, the determinant must be non-zero. When the determinant is zero, the resulting control strategies become non-admissible and the costs become infinite. In the next section we investigate the bi-level case with multiple leaders and multiple followers.

4. MULTIPLE LEADERS AND MULTIPLE FOLLOWERS

We consider the description in (1) in a bi-level hierarchical game with two and more leaders, i.e., $|\mathcal{I}_L|\geq 2$, and two and more followers, i.e., $|\mathcal{I}_F|\geq 2$. We restrict our attention to the admissible strategies which are Lipschitz in the state x. Given the strategies of the leaders $(u_i)_{i\in\mathcal{I}_L}\in\prod_{i\in\mathcal{I}_L}\mathcal{U}_i$, a risk-neutral best response strategy of follower j is a strategy that solves $\inf_{\mathcal{U}_j}\ \mathbb{E}L_j$. The set of risk-neutral best responses of j is denoted by $\text{rnBR}_j((u_i)_{i\in\mathcal{I}_L},(u_{j'})_{j'\in\mathcal{I}_F\setminus\{j\}})$.

A mean-field-type risk-neutral Nash equilibrium between the followers given the first movers' strategies $(u_i)_{i\in\mathcal{I}_L}\in\prod_{i\in\mathcal{I}_L}\mathcal{U}_i$, is a strategic profile $(u_j,\ j\in\mathcal{I}_F)$, of all followers such that for every decision-maker $j\in\mathcal{I}_F$,

$$u_j\in\text{rnBR}_j((u_i)_{i\in\mathcal{I}_L};(u_{j'}^{\text{rn}})_{j'\in\mathcal{I}_F\setminus\{j\}}).$$

The followers solve the following Nash game given the strategy of the leaders $(u_i)_{i\in\mathcal{I}_L}$

$j\in\mathcal{I}_F$,

$$0 = \hat{V}_{j,t}(t,m) + \int_x H_j^r(x,m,(\hat{V}_{j',m},\hat{V}_{j',xm},\hat{V}_{j',xxm})_{j'\in\mathcal{I}_F}|(u_i)_{i\in\mathcal{I}_L})m(t,dx), \quad (4a)$$

$$\hat{V}_j(t_1,m) = \int m(t_1,dy)h_j(y,m(t_1)), \quad (4b)$$

$$H_j^r = \inf_{u_j\in U_j}\left\{l_j + b\ \hat{V}_{j,xm} + \frac{\sigma^2}{2}\hat{V}_{j,xxm} + J[\hat{V}_{j,m}]|(u_i)_{i\in\mathcal{I}_L}\right\}, \quad (4c)$$

$$m_t = -(mb)_x + \frac{1}{2}(m\sigma^2)_{xx} + J^*[m],\ m(0)=m_0. \quad (4d)$$

Then the leader agents solve the following PIDE system:

$i\in\mathcal{I}_L$:

$$0 = \hat{V}_{i,t}(t,m) + \int_x H_i^r(x,m,(\hat{V}_{i',m},\hat{V}_{i',xm},\hat{V}_{i,xxm})_{i'\in\mathcal{I}_L\cup\mathcal{I}_F})m(t,dx), \quad (5a)$$

$$\hat{V}_i(t_1,m) = \int m(t_1,dy)h_i(y,m(t_1)), \quad (5b)$$

$$H_i^r = \inf_{u_i\in U_i}\left\{l_i + b\ \hat{V}_{i,xm} + \frac{\sigma^2}{2}\hat{V}_{i,xxm} + J[\hat{V}_{i,m}] \quad (5c)\right.$$

$$\left.|\{u_j^*(.,(u_i)_{i\in\mathcal{I}_L})\}_{j\in\mathcal{I}_F}\right\}, \quad (5d)$$

$$m_t = -(mb)_x + \frac{1}{2}(m\sigma^2)_{xx} + J^*[m],\ m(0)=m_0. \quad (5e)$$

A minimizer of the integrand Hamiltonian H_i^r, denoted by $u_i^{ss} = u_i^{ss}(t,x,m,(\hat{V}_{i',m},\hat{V}_{i',xm},\hat{V}_{i',xxm})_{i'\in\mathcal{I}_L\cup\mathcal{I}_F})$, provides a candidate Stackelberg strategy of the leader i. A mean-field-type risk-neutral Stackelberg solution among multiple leaders and multiple followers is a strategy $(u_i^{ss})_{i\in\mathcal{I}_L},(u_j^{ss})_{j\in\mathcal{I}_F}$, of all decision-makers such that

$i\in\mathcal{I}_L$,

$$u_i^{ss}\in\arg\min_{u_i\in\mathcal{U}_i}\left\{\mathbb{E}L_i(x,u_i,(u_{i'}^{ss})_{i\in\mathcal{I}_L\setminus\{i\}},(u_j^{ss})_{j\in\mathcal{I}_F}) : u_j^{ss}\in\text{rnBR}_j((u_i^{ss})_{i\in\mathcal{I}_L};(u_{j'}^{ss})_{j'\in\mathcal{I}_F\setminus\{j\}}\right\},$$

and for every follower

$$j \in \mathcal{I}_F,\ u_j^{ss} \in \mathrm{rnBR}_j((u_i^{ss})_{i\in\mathcal{I}_L}; (u_{j'}^{ss})_{j'\in\mathcal{I}_F\setminus\{j\}}).$$

Proposition 2. The risk-neutral Stackelberg mean-field-type solution with multiple leaders and multiple followers is given in a semi-explicit way as follows:

$$u_j^{ss} = -\eta_j \left(x - \int ym(dy)\right) - \bar{\eta}_j \int ym(dy), j \in \mathcal{I}_F, \quad \text{(6a)}$$

$$\begin{aligned}
& j \in \mathcal{I}_F, \\
& 0 = -r_j \eta_j^{2k_j-1} - \sum_{j'\in\mathcal{I}_F\setminus\{j\}} \epsilon_{jj'}\eta_{j'} - \sum_{i\in\mathcal{I}_L} \epsilon_{ji}\eta_i + b_{2j}\alpha_j + c_j, \\
& 0 = -\bar{r}_j \bar{\eta}_j^{2\bar{k}_j-1} - \sum_{j'\in\mathcal{I}_F\setminus\{j\}} \bar{\epsilon}_{jj'}\bar{\eta}_{j'} - \sum_{i\in\mathcal{I}_L} \bar{\epsilon}_{ji}\bar{\eta}_i \\
& \quad + \bar{b}_{2j}\bar{\alpha}_j + \bar{c}_j,
\end{aligned}$$

$$\begin{aligned}
& i \in \mathcal{I}_L, \\
& 0 = -r_i \eta_i^{2k_i-1} - \sum_{i'\in\mathcal{I}_L\setminus\{i\}} \epsilon_{ii'}\eta_{i'} - \sum_{j\in\mathcal{I}_F} \epsilon_{ij}\eta_j + b_{2i}\alpha_i \\
& \quad + \sum_{j\in\mathcal{I}_F} \epsilon_{ij}\eta_i \frac{\epsilon_{ji}}{(2k_j-1)r_j\eta_j^{2k_j-2}} \\
& \quad - \sum_{j\in\mathcal{I}_F} b_{2j} \frac{\epsilon_{ji}}{(2k_j-1)r_j\eta_j^{2k_j-2}}\alpha_i + c_i, \\
& 0 = -\bar{r}_i \bar{\eta}_i^{2\bar{k}_i-1} - \sum_{i'\in\mathcal{I}_L\setminus\{i\}} \bar{\epsilon}_{ii'}\bar{\eta}_{i'} - \sum_{j\in\mathcal{I}_F} \bar{\epsilon}_{ij}\bar{\eta}_j + \bar{b}_{2i}\bar{\alpha}_i \\
& \quad + \sum_{j\in\mathcal{I}_F} \bar{\epsilon}_{ij}\bar{\eta}_i \frac{\bar{\epsilon}_{ji}}{(2\bar{k}_j-1)\bar{r}_j\bar{\eta}_j^{2\bar{k}_j-2}} \\
& \quad - \sum_{j\in\mathcal{I}_F} \bar{b}_{2j} \frac{\bar{\epsilon}_{ji}}{(2\bar{k}_j-1)\bar{r}_j\bar{\eta}_j^{2\bar{k}_j-2}}\bar{\alpha}_i + \bar{c}_i,
\end{aligned}$$

and

$$\begin{aligned}
\hat{V}_i(0,m) &= \alpha_i(0) \int_x \frac{(x - \int ym_0(dy))^{2k_i}}{2k_i} m_0(dx) \\
&\quad + \bar{\alpha}_i(0) \frac{(\int ym_0(dy))^{2\bar{k}_i}}{2\bar{k}_i}, \qquad \text{(6b)}
\end{aligned}$$

$$\begin{aligned}
0 &= \dot{\alpha}_i + q_i + r_i\eta_i^{2k_i} - 2k_i c_i \eta_i + 2k_i \sum_{i'\in\mathcal{I}_L\setminus\{i\}} \epsilon_{ii'}\eta_i\eta_{i'} \\
&\quad + 2k_i \sum_{j\in\mathcal{I}_F} \epsilon_{ij}\eta_i\eta_j \\
&\quad + 2k_i \left[b_1 - \sum_{i'\in\mathcal{I}_L} b_{2i'}\eta_{i'} - \sum_{j\in\mathcal{I}_F} b_{2j}\eta_j \right] \alpha_i \\
&\quad + 2k_i(2k_i-1)\alpha_i \frac{1}{2}\tilde{\sigma}^2 \\
&\quad + \alpha_i \int_\Theta [(1+\tilde{\mu})^{2k_i} - 1 - 2k_i\tilde{\mu}]\nu(d\theta), \qquad \text{(6c)}
\end{aligned}$$

$$\alpha_i(T) = q_{iT}, \qquad \text{(6d)}$$

$$\begin{aligned}
0 &= \dot{\bar{\alpha}}_i + \bar{q}_i + \bar{r}_i \bar{\eta}_i^{2\bar{k}_i} - 2\bar{k}_i \bar{c}_i \bar{\eta}_i \\
&\quad + 2\bar{k}_i \sum_{i'\in\mathcal{I}_L\setminus\{i\}} \bar{\epsilon}_{ii'}\bar{\eta}_i\bar{\eta}_{i'} + 2\bar{k}_i \sum_{j\in\mathcal{I}_F} \bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j \\
&\quad + 2\bar{k}_i \{\bar{b}_1 - \sum_{i'\in\mathcal{I}_L} \bar{b}_{2i'}\bar{\eta}_{i'} - \sum_{j\in\mathcal{I}_F} \bar{b}_{2j}\bar{\eta}_j\}\bar{\alpha}_i, \qquad \text{(6e)}
\end{aligned}$$

$$\bar{\alpha}_i(T) = \bar{q}_{iT}, \qquad \text{(6f)}$$

with

$$\int ym(t,dy) = \left[\int ym(0,dy)\right] e^{\int_0^t [\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j]dt'}, \qquad \text{(6g)}$$

whenever the above coefficient system admits a unique solution. □

Proof. This proof is omitted for brevity. It is developed by following same technique as in Proposition 1. ■

Clearly, the mean-field-type Nash equilibrium (3) differs from the Stackelberg solution (6) when the ϵ_{ij} are non-zero.

4.1 No control-coupling within classes

It follows from (6) that for $\epsilon_{jj'} = 0 = \bar{\epsilon}_{jj'}$ for $(j, j') \in \mathcal{I}_F^2$, the term η_j is explicitly given by

$$\eta_j = \left\{ \frac{-\sum_{i\in\mathcal{I}_L} \epsilon_{ji}\eta_i + b_{2j}\alpha_j + c_j}{r_j} \right\}^{\frac{1}{2k_j-1}},$$

$$\bar{\eta}_j = \left\{ \frac{-\sum_{i\in\mathcal{I}_L} \bar{\epsilon}_{ji}\bar{\eta}_i + \bar{b}_{2j}\bar{\alpha}_j + \bar{c}_j}{\bar{r}_j} \right\}^{\frac{1}{2\bar{k}_j-1}}.$$

No Leader. All Followers: In this case there is no leader. All agents are followers. This case is similar to the model proposed in the Nash game above. The solution is given in (3).

One Leader and Multiple Followers : There is one leader in $\mathcal{I}_L$ and the remaining agents in $\mathcal{I}_F$ are followers. $\mathcal{I} = \mathcal{I}_L \cup \mathcal{I}_F$. We assume that the leader agent $1 \in \mathcal{I}_L$ uses a state-and-mean-field type feedback strategy $u_1(t, x, m)$ and each of the follower agent $j \in \mathcal{I}_F$ finds state-and-mean-field type feedback strategy $u_j(t, x, m, u_1)$ given u_1. The followers solve a Nash game given the strategy of the leader u_1.

Multiple Leaders and One Follower: Since there is only one follower the reaction set of the follower will be computed given the strategies of the leaders.

All leaders and no follower: In this case there is no follower. All agents are leaders. In terms of information structure, this case is similar to the model proposed in the Nash game above. The solution is given in (3).

4.2 Effect of the total number of leaders on the social cost:

We investigate the effect of the number of leaders in the global performance of the system. The total cost at the Stackelberg solution is

$$S(\mathcal{I}_L, m_0) = \sum_{i\in\mathcal{I}_L} \hat{V}_i(0, m_0) + \sum_{j\in\mathcal{I}_F} \hat{V}_j(0, m_0).$$

For $m_0 = \delta_{x_0}$, and $\bar{k}_i = \bar{k} \geq 1$, the total cost is

$$S(\mathcal{I}_L, m_0) = \left(\sum_{i\in\mathcal{I}_L} \bar{\alpha}_i(0) + \sum_{j\in\mathcal{I}_F} \bar{\alpha}_j(0) \right) \frac{x_0^{2\bar{k}}}{2\bar{k}}.$$

Uniform coupling: When all other parameters are identical across the players except their role, $S(\mathcal{I}_L, m_0)$ can be expressed as a function $|\mathcal{I}_L|$. It follows from (6) that

$$\chi := |\mathcal{I}_L|,$$

$$
\begin{aligned}
0 &= -\bar{r}(\bar{\eta}^{fo})^{2\bar{k}-1} - (|\mathcal{I}| - \chi - 1)\bar{\epsilon}\bar{\eta}^{fo} \\
&\quad - \chi\bar{\epsilon}\bar{\eta}^{lead} + \bar{b}_2\bar{\alpha}^{fo} + c, \\
0 &= -\bar{r}(\bar{\eta}^{lead})^{2\bar{k}-1} - (\chi - 1)\bar{\epsilon}\bar{\eta}^{lead} \\
&\quad - (|\mathcal{I}| - \chi)\bar{\epsilon}\bar{\eta}^{fo} + \bar{b}_2\bar{\alpha}^{lead} + \bar{c} \\
&\quad + \frac{\bar{\epsilon}(|\mathcal{I}| - \chi)(\bar{\epsilon}\bar{\eta}^{lead} - \bar{\alpha}^{lead}\bar{b}_2)}{(2\bar{k}-1)\bar{r}(\bar{\eta}^{fo})^{2\bar{k}-2}}, \\
\bar{\alpha}^{lead}(t_0) &= \bar{q}_{t_1} + \int_{t_0}^{t_1} \Big\{ \bar{q} + \bar{r}(\bar{\eta}^{lead})^{2\bar{k}} - 2\bar{k}\bar{c}\bar{\eta}^{lead} \\
&\quad + 2\bar{k}\bar{\epsilon}\bar{\eta}^{lead}[(\chi - 1)\bar{\eta}^{lead} + (|\mathcal{I}| - \chi)\bar{\eta}^{fo}] \\
&\quad + 2\bar{k}\bar{\alpha}^{lead}[\bar{b}_1 - \bar{b}_2\bar{\eta}^{lead}\chi - \bar{b}_2\bar{\eta}^{fo}(|\mathcal{I}| - \chi)] \Big\} dt \\
\bar{\alpha}^{fo}(t_0) &= \bar{q}_{t_1} + \int_{t_0}^{t_1} \Big\{ \bar{q} + \bar{r}(\bar{\eta}^{fo})^{2\bar{k}} - 2\bar{k}\bar{c}\bar{\eta}^{fo} \\
&\quad + 2\bar{k}\bar{\epsilon}\bar{\eta}^{fo}[(|\mathcal{I}| - \chi - 1)\bar{\eta}^{fo} + \chi\bar{\eta}^{lead}] \\
&\quad + 2\bar{k}\bar{\alpha}^{fo}[\bar{b}_1 - \bar{b}_2\bar{\eta}^{lead}\chi - \bar{b}_2\bar{\eta}^{fo}(|\mathcal{I}| - \chi)] \Big\} dt.
\end{aligned}
$$

The optimal number of leaders is given by

$$|\mathcal{I}_L| \in \arg\min_{\chi} [\chi\bar{\alpha}^{lead}(0) + (|\mathcal{I}| - \chi)\bar{\alpha}^{fo}(0)],$$

where $\bar{\alpha}$ depends on χ as well. We observe that the later function is not necessarily monotone in $\chi = |\mathcal{I}_L|$. This means that increasing the number of leaders in the interaction does not necessarily improve the global performance of the system.

5. FULLY HIERARCHICAL GAME

In the previous sections we had only bi-level problems. Here, we make as many levels as the number of decision-makers. There are $|\mathcal{I}|$ hierarchical levels. In each layer i, decision-maker i chooses a control strategy u_i knowing the control strategy of the preceding decision-makers i.e., $\{i-1, \ldots, 1\}$. This becomes a sequential decision-making problem. We use a backward induction method to solve the hierarchical game problem. This means that the decision-making problem at the last layer I, which is the reaction of decision-maker I can be seen as a mean-field-type control problem. This is because at $i-$th level, the strategies $(u_{i'})_{i' \in \{1,\ldots,i-1\}}$ are already known by decision-maker i.

Proposition 3. The risk-neutral $I-$level hierarchical mean-field-type solution is given in a semi-explicit way as follows:

$$u_i^{hs} = -\eta_i \left(x - \int y m(dy) \right) - \bar{\eta}_i \int y m(dy), i \in \mathcal{I}, \quad (7a)$$

$$
\begin{aligned}
\hat{V}_i(0, m) &= \alpha_i(0) \int_x \frac{(x - \int y m_0(dy))^{2k_i}}{2k_i} m_0(dx) \\
&\quad + \bar{\alpha}_i(0) \frac{(\int y m_0(dy))^{2\bar{k}_i}}{2\bar{k}_i}, \quad (7b)
\end{aligned}
$$

with

$$\int y m(t, dy) = \left[\int y m(0, dy) \right] e^{\int_0^t \left[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j \right] dt}, \quad (7c)$$

where coefficient functions are given by

Level 1 :

$$
\begin{aligned}
0 &= -r_1\eta_1^{2k_1-1} + c_1 - \sum_{j=2}^{I} \epsilon_{1,j}\eta_j \\
&\quad + \sum_{j=2}^{I} \epsilon_{1,j}\eta_i \frac{\epsilon_{ji}}{(2k_j - 1)r_j} \eta_j^{-2(k_j-1)} \\
&\quad + \left\{ b_{2,1} - \sum_{j=2}^{I} b_{2j} \frac{\epsilon_{j1}}{(2k_j - 1)r_j} \eta_j^{-2(k_j-1)} \right\} \alpha_1, \\
0 &= \dot{\alpha}_1 + q_1 + r_1\eta_1^{2k_1} - 2k_1c_1\eta_1 + 2k_1 \sum_{j=2}^{I} \epsilon_{1j}\eta_1\eta_j \\
&\quad + 2k_1\{b_1 - b_{21}\eta_1 - \sum_{j=2}^{I} b_{2j}\eta_j\}\alpha_1 \\
&\quad + 2k_1(2k_1 - 1)\alpha_1 \frac{1}{2}\tilde{\sigma}^2 \\
&\quad + \alpha_1 \int_{\Theta} [(1 + \tilde{\mu})^{2k_1} - 1 - 2k_1\tilde{\mu}]\nu(d\theta), \\
\alpha_1(T) &= q_{1T}, \\
0 &= -\bar{r}_1\bar{\eta}_1^{2\bar{k}_1-1} + \bar{c}_1 - \sum_{j=2}^{I} \bar{\epsilon}_{1,j}\bar{\eta}_j \\
&\quad + \sum_{j=2}^{I} \bar{\epsilon}_{1,j}\bar{\eta}_1 \frac{\bar{\epsilon}_{j1}}{(2\bar{k}_j - 1)\bar{r}_j} \bar{\eta}_j^{-2(\bar{k}_j-1)} \\
&\quad + \left\{ \bar{b}_{21} - \sum_{j=2}^{I} \bar{b}_{2j} \frac{\bar{\epsilon}_{j1}}{(2\bar{k}_j - 1)\bar{r}_j} \bar{\eta}_j^{-2(\bar{k}_j-1)} \right\} \bar{\alpha}_1, \\
0 &= \dot{\bar{\alpha}}_1 + \bar{q}_1 + \bar{r}_1\bar{\eta}_1^{2\bar{k}_1} - 2\bar{k}_1\bar{c}_1\bar{\eta}_1 + 2\bar{k}_1 \sum_{j=2}^{I} \bar{\epsilon}_{1j}\bar{\eta}_i\bar{\eta}_j \\
&\quad + 2\bar{k}_1\{\bar{b}_1 - \bar{b}_{21}\bar{\eta}_1 - \sum_{j=2}^{I} \bar{b}_{2j}\bar{\eta}_j\}\bar{\alpha}_1, \\
\bar{\alpha}_1(T) &= \bar{q}_{1T},
\end{aligned}
$$

Level i :

$$
\begin{aligned}
0 &= -r_i\eta_i^{2k_i-1} + c_i - \sum_{i'=1}^{i-1} \epsilon_{I-1,i'}\eta_{i'} - \sum_{j=i+1}^{I} \epsilon_{i,j}\eta_j \\
&\quad + \sum_{j=i+1}^{I} \epsilon_{i,j}\eta_i \frac{\epsilon_{ji}}{(2k_j - 1)r_j} \eta_j^{-2(k_j-1)} \\
&\quad + \left[b_{2i} - \sum_{j=i+1}^{I} b_{2j} \frac{\epsilon_{ji}}{(2k_j - 1)r_j} \eta_j^{-2(k_j-1)} \right] \alpha_i, \\
0 &= \dot{\alpha}_i + q_i + r_i\eta_i^{2k_i} - 2k_ic_i\eta_i + 2k_i \sum_{i'=1}^{i-1} \epsilon_{ii'}\eta_i\eta_{i'} \\
&\quad + 2k_i \sum_{j=i+1}^{I} \epsilon_{ij}\eta_i\eta_j \\
&\quad + 2k_i\{b_1 - \sum_{i'=1}^{i-1} b_{2i'}\eta_{i'} - b_{2i}\eta_i - \sum_{j=i+1}^{I} b_{2j}\eta_j\}\alpha_i
\end{aligned}
$$

$$
\begin{aligned}
&+ 2k_i(2k_i-1)\alpha_i\frac{1}{2}\tilde{\sigma}^2\\
&+ \alpha_i\int_\Theta [(1+\tilde{\mu})^{2k_i}-1-2k_i\tilde{\mu}]\nu(d\theta),\\
\alpha_i(T) &= q_{iT},\\
0 &= -\bar{r}_i\bar{\eta}_i^{2\bar{k}_i-1}+\bar{c}_i-\sum_{i'=1}^{i-1}\bar{\epsilon}_{I-1,i'}\bar{\eta}_{i'}-\sum_{j=i+1}^{I}\bar{\epsilon}_{i,j}\bar{\eta}_j\\
&+\sum_{j=i+1}^{I}\bar{\epsilon}_{i,j}\bar{\eta}_i\frac{\bar{\epsilon}_{ji}}{(2\bar{k}_j-1)\bar{r}_j}\bar{\eta}_j^{-2(\bar{k}_j-1)}\\
&+\left[\bar{b}_{2i}-\sum_{j=i+1}^{I}\bar{b}_{2j}\frac{\bar{\epsilon}_{ji}}{(2\bar{k}_j-1)\bar{r}_j}\bar{\eta}_j^{-2(\bar{k}_j-1)}\right]\bar{\alpha}_i,\\
0 &= \dot{\bar{\alpha}}_i+\bar{q}_i+\bar{r}_i\bar{\eta}_i^{2\bar{k}_i}-2\bar{k}_i\bar{c}_i\bar{\eta}_i+2\bar{k}_i\sum_{i'=1}^{i-1}\bar{\epsilon}_{ii'}\bar{\eta}_i\bar{\eta}_{i'}\\
&+2\bar{k}_i\sum_{j=i+1}^{I}\bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j\\
&+2\bar{k}_i\left[\bar{b}_1-\sum_{i'=1}^{i-1}\bar{b}_{2i'}\bar{\eta}_{i'}-\bar{b}_{2i}\bar{\eta}_i-\sum_{j=i+1}^{I}\bar{b}_{2j}\bar{\eta}_j\right]\bar{\alpha}_i,\\
\bar{\alpha}_i(T) &= \bar{q}_{iT},
\end{aligned}
$$

Level I :

$$
\begin{aligned}
\eta_I &= \left(\frac{-\sum_{j=1}^{I-1}\epsilon_{I,j}\eta_j+b_{2I}\alpha_I+c_I}{r_I}\right)^{\frac{1}{2k_I-1}},\\
0 &= \dot{\alpha}_I+q_I+r_I\eta_I^{2k_I}-2k_Ic_I\eta_I+2k_I\sum_{i'=1}^{I-1}\epsilon_{Ii'}\eta_I\eta_{i'}\\
&+2k_I\{b_1-\sum_{i'=1}^{I-1}b_{2i'}\eta_{i'}-b_{2I}\eta_I\}\alpha_I\\
&+2k_I(2k_I-1)\alpha_I\frac{1}{2}\tilde{\sigma}^2\\
&+\alpha_I\int_\Theta[(1+\tilde{\mu})^{2k_I}-1-2k_I\tilde{\mu}]\nu(d\theta),\\
\alpha_I(T) &= q_{IT},\\
\bar{\eta}_I &= \left(\frac{-\sum_{j=1}^{I-1}\bar{\epsilon}_{I,j}\bar{\eta}_j+\bar{b}_{2I}\bar{\alpha}_I+\bar{c}_I}{\bar{r}_I}\right)^{\frac{1}{2\bar{k}_I-1}},\\
0 &= \dot{\bar{\alpha}}_I+\bar{q}_I+\bar{r}_I\bar{\eta}_I^{2\bar{k}_I}-2\bar{k}_I\bar{c}_I\bar{\eta}_I+2\bar{k}_I\sum_{i'=1}^{I-1}\bar{\epsilon}_{Ii'}\bar{\eta}_I\bar{\eta}_{i'}\\
&+2\bar{k}_I\{\bar{b}_1-\sum_{i'=1}^{i-1}\bar{b}_{2i'}\bar{\eta}_{i'}-\bar{b}_{2I}\bar{\eta}_I\}\bar{\alpha}_I,\\
\bar{\alpha}_I(T) &= \bar{q}_{IT},
\end{aligned}
$$

whenever these equations admit a solution. □

Proof. This proof is omitted for brevity. It is developed by following same technique as in Proposition 1. ■

We observe that

- For $\epsilon_{ij} \neq 0, \bar{\epsilon}_{ij} \neq 0$, the order of the play matters because of the informational difference between the decision-makers at different level of hierarchy in (7). One open question that we leave for future investigation is to find the optimal ordering among all permutations of heterogenous decision-makers.
- When all the ϵ_{ij} and $\bar{\epsilon}_{ij}$ are zero, the Nash equilibrium coincides with the bi-level solution, which also coincides with any level hierarchical solution. The order of the play and the informational difference does not generate an extra advantage for the first mover in this particular case.

CONCLUSIONS

We have studied a hierarchical mean-field-type game structure with state dynamics driven by jump-diffusion processes and with non-quadratic cost functionals. We have computed, in a semi-explicit manner, the Stackelberg equilibrium corresponding to different scenarios. The considered cases comprise a bi-level case with multiple either leaders or followers, and the case involving several levels. Besides, a discussion about the circumstances under which the Nash equilibrium coincides with the Stackelberg solution has been developed.

REFERENCES

Averboukh, A.Y. (2018). Stackelberg solution for first-order mean-field game with a major player. *Izv. Inst. Mat. Inform. Udmurt.*, 52.

Bensoussan, A., Chau, M.H.M., and Yam, S.C.P. (2015). Mean-field stackelberg games: Aggregation of delayed instructions. *SIAM J. Control Optim.*, 53(4), 2237–2266.

Bensoussan, A., Chau, M., Lai, Y., and Yam, S. (2017). Linear-quadratic mean field stackelberg games with state and control delays. *SIAM Journal on Control and Optimization*, 55(4), 2748–2781.

Bensoussan, A., Djehiche, B., Tembine, H., and Yam, S.C.P. (2020). Mean-field-type games with jump and regime switching. *Dynamic Games and Applications https://doi.org/10.1007/s13235-019-00306-2.*

Du, K. and Wu, Z. (2019). Linear-quadratic stackelberg game for mean-field backward stochastic differential system and application. *Mathematical Problems in Engineering*, Article ID 1798585, 17 pages.

Lin, Y., Jiang, X., and Zhang, W. (2019). An open-loop stackelberg strategy for the linear quadratic mean-field stochastic differential game. *IEEE Transactions on Automatic Control*, 64(1), 97–110.

Moon, J. and Basar, T. (2015). Linear-quadratic stochastic differential stackelberg games with a high population of followers. *in Proc. 54th IEEE Conf. Decision Control, Osaka, Japan*, 2270–2275.

von Stackelberg, H. (1934). Marktform und gleichgewicht. *Springer-Verlag, Berlin.*