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Adaptive low-power high-gain observers for lower-triangular systems with input-dependent Lipschitz constant

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Abstract: We combine the low-power high-gain observer recently proposed in Astolfi and Marconi (2015) with the updated-gain technique used in Andrieu et al. (2009). The resulting adaptive low-power high-gain observer inherits the advantages of both techniques and can be used to address the state-estimation problem for Lipschitz systems in lower triangular form with nonlinearities having a Lipschitz constant that depends on a known external input.

1. INTRODUCTION

The state-estimation problem for nonlinear systems in lower-triangular form can be systematically addressed by using the so-called high-gain observers, see, e.g., Emel'Yanov et al. (1989), Tornambe (1992), Deza et al. (1992). This observers are characterized by having an output injection terms which scales with increasing power of a (positive) high-gain parameter. This one has to be chosen large enough to dominate the Lipschitz constants of the nonlinear terms. In this case, asymptotic convergence of the estimation error is ensured. When the Lipschitz constants depends on external inputs, adaptive techniques for the on-line tuning of the high-gain parameter have been proposed in Andrieu et al. (2009); Sanfelice and Praly (2011); Alessandri and Rossi (2015). Although the good robustness properties with respect to model perturbations of high-gain observers, their use for the state observation of systems of large state dimension is limited due to important drawbacks: the sensitivity to high-frequency measurement noise, Astolfi et al. (2016); the peaking phenomenon, Astolfi et al. (2018b); Khalil (2017); possible implementation issues to the large powers of the high-gain parameter multiplying the output injection term, Astolfi and Marconi (2015); Khalil (2017). To address these issues, a new class of observers, denoted as low-power high-gain observers, has been recently proposed in Astolfi and Marconi (2015); Astolfi et al. (2018b); Wang et al. (2017). The new technique proposes an interconnected cascade of high-gain observers of dimension two in which the high-gain parameter shows up with powers 1 and 2, regardless the dimension of the system state dimension. The resulting observer dimension, however, is nearly doubled. Other techniques, addressing one or some of those drawbacks, have been proposed in Boizot et al. (2010); Teel (2016); Khalil (2017); Astolfi et al. (2017, 2018a); Cocetti et al. (2018); Tréangle et al. (2019).

The objective of this work is to investigate the use of adaptive gains in the context of low-power high-gain observers proposed by combining the updated-gain technique described in Andrieu et al. (2009) with the cascade structure proposed in Astolfi and Marconi (2015). We provide sufficient conditions for the design of an adaptive law that tunes on-line the high-gain parameter according to the Lipschitz constant of the nonlinear terms to be dominated. Similar to Andrieu et al. (2015), we suppose that such Lipschitz constants depends on known external inputs. Asymptotic convergence of the estimation error is established. Numerical simulations are then presented to show that asymptotic estimation can be achieved with values of the high-gain parameter much lower than a constant-gain structure, for which, very conservative bounds are needed.

Notation $\mathbb{R}$ denotes the real numbers. Given $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, we compactly denote $(x, y) := (x^\top, y^\top)^\top$.

2. PROBLEM STATEMENT

In this paper are considered single-input single-output nonlinear systems in the following lower triangular form

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(u, y), \\ &\vdots \\ \dot{x}_i &= x_{i+1} + \phi_i(u, y, x_2, ..., x_i), \\ &\vdots \\ \dot{x}_n &= \phi_n(u, y, x_2, ..., x_n), \\ y &= x_1, \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the state of the system, $y \in \mathbb{R}$ is the measured output and $u \in \mathbb{R}^m$ is a known input. In this work, the presence of input disturbances or measurement noise are ignored, although all the analysis could be done to cover such case. The nonlinear functions ϕ_i, $i = 1, \ldots, n$, satisfy the following Lipschitz condition.

Assumption 1. There exists a continuous function $\Omega : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ such that the following is satisfied

$$|\phi_i(u, y, \hat{x}_2, ..., \hat{x}_i) - \phi_i(u, y, x_2, ..., x_i)| \leq \Omega(u) \sum_{j=2}^{i} |\hat{x}_j - x_j| \tag{2}$$

for all $i = 1 \ldots, n$, and all $(\hat{x}, x, y, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$.

In other words, the Lipschitz constant of ϕ_i depends on the value of u. We recall that under Assumption 1, standard high-gain observer design Khalil and Praly (2014) can be applied only if u is living in some *known* compact set $U \subset \mathbb{R}^m$. When the knowledge or an estimate of U is not available, standard techniques fails to work because domination arguments cannot be applied, in other words, a lower bound for the high-gain parameter cannot be established because the Lipschitz constant is unknown. In such case, adaptive techniques need therefore be employed. Therefore, in the rest of the paper, we will suppose that u lives in a bounded but *unknown* compact set U for all $t \geq 0$. This scenario can be also of interest to improve online the performances of the observer when the estimate of U is too "rough", namely to select the lowest possible high-gain parameter ensuring estimate convergence. The aim of this work is to combine the low-power high-gain observer approach Astolfi and Marconi (2015) with the updated-gain approach employed in Andrieu et al. (2009).

3. MAIN RESULT

In order to present the construction of the observer, we first need to define, as in Astolfi and Marconi (2015), the following matrices. In particular, let (A, B, C) a triplet in prime form of dimension 2, that is

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := (1 \ \ 0),$$

and let the matrices F_i, Q_i, N, be defined as

$$F_i := \begin{pmatrix} -k_{i1} & 1 \\ -k_{i2} & 0 \end{pmatrix}, \ Q_i := \begin{pmatrix} 0 & k_{i1} \\ 0 & k_{i2} \end{pmatrix}, \ N := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

where $k_{i1}, k_{i2} > 0$ are some coefficients to be selected. Finally, let the matrices M, D of dimension $2n-2$ be

$$M := \begin{pmatrix} F_1 & N & 0 & & \cdots & \cdots & 0 \\ Q_2 & F_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & F_i & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{n-2} & F_{n-2} & N \\ 0 & \cdots & \cdots & \cdots & 0 & Q_{n-1} & F_{n-1} \end{pmatrix}, \tag{3}$$

$$D := \mathrm{diag}(1, 2, 2, ..., n-1, n-1, n). \tag{4}$$

As shown in (Astolfi and Marconi, 2015, Lemma 1), it is possible to select the coefficients $k_{i1}, k_{i2} > 0$ so that the matrix M defined in (3) is Hurwitz. Moreover, it is also possible to arbitrarily select its eigenvalues. However, for the purpose of this work, we need some extra properties stated by the following lemma.

Lemma 1. Consider matrices M, D defined in (3), (4). *There exist coefficients $k_{i1}, k_{i2} > 0$ for $i = 1, ..., n-1$, $\mu, \bar{p}, \underline{p}, \bar{\alpha}, \underline{\alpha} > 0$ and a symmetric positive definite matrix P such that*

$$\begin{aligned} \underline{p} I \ &\leq \ P \ \leq \ \bar{p} I, \\ PM + M^\top P \ &\leq \ -\mu P, \\ \underline{\alpha} P \ \leq \ PD + DP \ &\leq \ \bar{\alpha} P. \end{aligned} \tag{5}$$

Proof. The proof is deferred to Section 4.2 where a constructive procedure for designing k_{i1}, k_{i2} is presented. Nevertheless, one can always follow the procedure presented in (Astolfi and Marconi, 2015, Lemma 1) to assign the eigenvalues of M, and then verify, *a posteriori*, the existence of a P satisfying (5), see Section 5. □

The structure of the proposed low-power high-gain observer with updated gain has therefore the following form

$$\begin{aligned} \dot{\xi}_i \ &= \ A\xi_i + N\xi_{i+1} + \Phi_1(u, y, \hat{x}) + \Lambda(L) K_i (y - C\xi_1), \\ &\vdots \\ \dot{\xi}_i \ &= \ A\xi_i + N\xi_{i+1} + \Phi_i(u, y, \hat{x}) \\ &\qquad\qquad + \Lambda(L) K_i (B^\top \xi_{i-1} - C\xi_i), \\ &\vdots \\ \dot{\xi}_{n-1} \ &= \ A\xi_{n-1} + \Phi_{n-1}(u, y, \hat{x}) \\ &\qquad\qquad + \Lambda(L) K_{n-1} (B^\top \xi_{n-2} - C\xi_{n-1}), \\ \hat{x} \ &= \ \Gamma \xi \end{aligned} \tag{6a}$$

where $\xi \in \mathbb{R}^{2n-2}$ is the state of the observer, $\hat{x}$ is its state-estimate with $\Gamma := \mathrm{blkdiag}\ (C, \ \ldots, \ C, I_2) \in \mathbb{R}^{n \times (2n-2)}$, the matrices A, B, C, N that have been defined above, $K_i := (k_{i1} \ k_{i2})^\top$, $\Lambda(L) := \mathrm{diag}(L, \ L^2)$, and the functions Φ_i are defined as

$$\Phi_i(u, y, \hat{x}) := \begin{pmatrix} \phi_i(u, y, \hat{x}_2, ..., \hat{x}_i) \\ \phi_{i+i}(u, y, \hat{x}_2, ..., \hat{x}_i) \end{pmatrix}$$

with the functions ϕ_i defined in system (1). Finally, $L > 0$ denotes the high-gain parameter which is updated according to the following differential equation

$$\dot{L} = L\big(\lambda_1(\lambda_2 - L) + \lambda_3 \Omega(u)\big) \tag{6b}$$

with $\lambda_1, \lambda_2, \lambda_3 > 0$ some parameters to be properly chosen, and Ω the functions defined in Assumption 1. We can state now the main result of this work concerning the convergence of observer (6).

Theorem 1. Consider system (1) *and observer* (6). *Suppose u is a (locally integrable) bounded signal for all $t \geq 0$. Let the coefficients k_{i1}, k_{i2}, $i = 1, \ldots, n-1$ be chosen according to Lemma 1, and let $\lambda_1, \lambda_2, \lambda_3 > 0$ be selected so that $\lambda_1 < \frac{\mu}{\bar{\alpha}}$, $\lambda_2 \geq 1$, and $\lambda_3 \geq 2\bar{p}(\underline{p}\underline{\alpha})^{-1}\varrho$ with $\varrho = \sqrt{2}(2n-3)$. Then, for any initial condition $(x(0), \xi(0)) \in \mathbb{R}^n \times \mathbb{R}^{2n-2}$, $L(0) \geq \lambda_2$ any corresponding solution defined for all $t \geq 0$ satisfy*

$$\lim_{t \to \infty} |x(t) - \hat{x}(t)| = 0.$$

Proof. The proof is deferred to Section 4.1. □

Some qualitative comments of the result of Theorem 1 are given now.

About Assumption 1. If the state $x(t)$ of system (1) is supposed to evolve in a known compact set $X \subset \mathbb{R}^n$ for all $t \geq 0$, then Assumption 1 can be relaxed by asking inequality (2) to hold for all $(\hat{x}, x, y, u) \in X \times X \times Y \times \mathbb{R}^m$ where Y is the projection of X on the first coordinate. Then, as commonly done in high-gain observer design approaches, see, e.g., Astolfi and Marconi (2015) or Khalil and Praly (2014), one can implement the observer (6a) by using a saturated version of ϕ_i on X. Note however that in practice, the compact set X usually depends on U.

Alternative design. Recall that $y = x_1$. Hence, in order to improve the sensitivity to measurement noise, one may want to implement the observer (6a) by using $\hat{x}_1$ instead of y in the functions Φ_i. However, this is possible only by strengthening Assumption 1 as follows

$$|\phi_i(u, \hat{x}_1, ..., \hat{x}_i) - \phi_i(u, x_1, ..., x_i)| \leq \Omega(u) \sum_{j=1}^{i} |\hat{x}_j - x_j|$$

for all $i = 1, \ldots, n$ and all $(\hat{x}, x, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \times \mathbb{R}^m$. In such case, the result of Theorem 1 still holds.

Asymptotic convergence. Theorem 1 establishes the asymptotic convergence of the estimation error thus ensuring that system (6) is an asymptotic observer for plant (1). With respect to standard results on high-gain observers, see Khalil and Praly (2014) or Astolfi and Marconi (2015), exponential convergence, uniform with respect to the initial estimation error, is not established since we are not assuming u in (1) to evolve in known compact sets. It would be possible to give more precise statements by reinforcing Assumption 1.

Peaking phenomenon. Since we do not assume to know a compact set where the state x of system (1) is evolving, we cannot follow the idea in Astolfi et al. (2018b), in which the use of saturation functions is exploited to deal with the peaking phenomenon. Nevertheless, peaking phenomenon can be attenuated by tuning the parameters of the dynamics of L, as shown in the simulations of Section 5.

Design of the parameters. The parameters $\lambda_1, \lambda_2, \lambda_3$ define the dynamics of L, and in particular, its speed of convergence, its sensitivity to u and the DC-gain with respect to the constant u. Theorem 1 provides sufficient conditions for the choice of the parameters for the convergence of the observer (6) which may be very conservative. In practice, we have the following phenomena.

- The ratio $\frac{\lambda_3}{\lambda_1}$ modulates the values of L: larger ratios correspond to larger values of L.
- For a given constant ratio $\frac{\lambda_3}{\lambda_1}$, larger values of λ_1, λ_3, provide a faster response of L to variations of the input u.
- For given constant λ_1, λ_3, smaller values of λ_2 decrease the steady-state value of L in case of constant input u.

In conclusion, the design of $\lambda_1, \lambda_2, \lambda_3$ depends on the type of response we desire on the estimation error and such choice modulates the peaking phenomenon, the rate of convergence, the sensitivity to variations of u and the sensitivity to measurement noise in steady-state. More comments are given in Section 5.

4. PROOFS

4.1 Proof of Theorem 1

Consider the differential equation (6b) governing the dynamics of L in which all λ_i's are positive. First of all, if $L = \lambda_2$, then $\dot{L} = L\lambda_3\Omega(u) \geq 0$. Since $L(0) \geq \lambda_2$, this implies that for all t on the time existence of the solutions, $L(t) \geq \lambda_2$. On another hand, if u is bounded then $\Omega(u)$ is bounded as well. Let $B_\Omega := \sup_{t\in[0,\infty)} \Omega(u)$ and $\bar{L} := \frac{\lambda_3}{\lambda_1} B_\Omega + \lambda_2$. If $L \geq \bar{L}$, then $\dot{L} \leq 0$. This implies that $L(t)$ is defined for all $t \geq 0$ and we conclude that $L(t) \in [\lambda_2, \max\{L(0), \bar{L}\}]$ for any u bounded and any $L(0) \geq \lambda_2$.

Now let us define the following change of coordinates

$$\xi_i \mapsto \varepsilon_i := \left(\frac{x_i - \xi_{i1}}{L^i}, \frac{x_{i+1} - \xi_{i2}}{L^{i+1}}\right) \qquad \forall\, i = 1, \ldots, n-1,$$

$\varepsilon := (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \mathbb{R}^{2n-2}$. The ε-dynamics is then given by (computations are omitted for space reasons)

$$\dot{\varepsilon} = LM\varepsilon - \frac{\dot{L}}{L} D\varepsilon + \Psi(L)\Delta\Phi(u, y, x, \hat{x}) \tag{7}$$

where M, D are defined in (3), (4), $\Psi(L) \in \mathbb{R}^{2n-2\times 2n-2}$ is defined as

$$\Psi(L) := \text{diag}\left(\frac{1}{L}, \frac{1}{L^2}, \frac{1}{L^2}, ..., \frac{1}{L^{n-1}}, \frac{1}{L^{n-1}}, \frac{1}{L^n}\right),$$

and $\Delta\Phi = (\Delta\Phi_1, \ldots, \Delta\Phi_{n-1})$, with $\Delta\Phi_i = (\Delta\Phi_{i1}, \Delta\Phi_{i2})$ and $\Delta\Phi_i(u, y, x, \hat{x}) := \Phi_i(u, y, x) - \Phi_i(u, y, \hat{x})$, for all $i = 1, \ldots, n-1$, where we omitted the arguments for compactness. In order to obtain a bound for $\Psi(L)\Delta\Phi$, first recall that by definition of $\hat{x}_i, e_i, \varepsilon$ we have, for all $L \geq 1$:

$$|x_i - \hat{x}_i| = |x_i - \xi_{i1}| = |e_{i1}| = L^i|\varepsilon_{i1}| \leq L^i|\varepsilon_i|$$

for all $i = 2, \ldots, n-1$ and $|x_n - \hat{x}_n| = L^n|\varepsilon_{(n-1)2}| \leq L^n|\varepsilon_{n-1}|$. Therefore, by using the Lipschitz condition of Assumption 1, we also have $|\phi_i(u, y, x) - \phi_i(u, y, \hat{x})| \leq \sum_{j=2}^{i} L^j|\varepsilon_j|$ which gives

$$\begin{aligned}
|\Psi(L)\Delta\Phi(u, y, x, \hat{x})| &\leq \sum_{i=1}^{n-1} \Lambda(L)^{-1}L^{-(i-1)}|\Delta\Phi_i| \\
&\leq \sum_{i=1}^{n-1} \Omega(u)\left(L^{-i}\sum_{j=2}^{i} L^j|\varepsilon_j| + L^{-(i+1)}\sum_{j=2}^{i} L^j|\varepsilon_j|\right) \\
&\leq \varrho\,\Omega(u)|\varepsilon|
\end{aligned}$$

for any $(u, y, x, \hat{x}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and any $L > 1$, with ϱ defined in the statement of the theorem. Now, consider the matrix P defined in Lemma 1, define $V = \varepsilon^\top P\varepsilon$ and compute its derivative along solutions of (7). We obtain

$$\dot{V} \leq -\left(L(\mu - \lambda_1\bar{\alpha}) + \lambda_1\lambda_2\underline{\alpha} + \left(\lambda_3\underline{\alpha} - 2\frac{\bar{p}}{\underline{p}}\varrho\right)\Omega(u)\right)\varepsilon^\top P\varepsilon.$$

Therefore, by selecting $\lambda_1, \lambda_2, \lambda_3$ according to the statement of the theorem, we obtain $\dot{V} \leq -\epsilon LV$ for some $\epsilon > 0$. This shows that $|\varepsilon|$ converges exponentially to zero. Furthermore, since L is bounded for all times, we deduce that also $|e|$ converge exponentially to zero. In light of the definition of e and $\hat{x}$ in (6), we conclude the statement of the theorem.

4.2 Proof of Lemma 1

The proof of this result is a direct consequence of the next Lemmas 2 and 3 which are the base case and the inductive step of a mathematical induction. For this, let us introduce the following matrices which will be used next

$$M_1 := F_{n-1}, \;\; M_{i+1} := \begin{pmatrix} F_{n-i-1} & \bar{N}_{n-i} \\ \bar{Q}_{n-i} & M_i \end{pmatrix} \;\; i = 2, \ldots, n-1,$$

with $\bar{N}_{n-i} \in \mathbb{R}^{2\times 2i}, \bar{Q}_{n-i} \in \mathbb{R}^{2i\times 2}$ defined as

$$\bar{N}_{n-i} := (N^\top\, 0\, ...\, 0)^\top, \qquad \bar{Q}_{n-i} := (Q_{n-i}^\top\, 0\, ...\, 0)^\top,$$

and the matrices F_i, Q_i, N defined as in (3). By construction, $M_{n-1} = M$. Finally, let us define

$$D'_i := \text{diag}(n-i, n-i+1), \;\; D_1 := D'_1, \;\; D_{i+1} := \begin{pmatrix} D'_{i+1} & 0 \\ 0 & D_i \end{pmatrix}.$$

By construction, $D_{n-1} = D$, with D defined in (4). We have the first following result.

Lemma 2. Consider the matrices M_1 and D_1. There exist coefficients $k_{n-1,1}$ and $k_{n-1,2}$ and a positive definite symmetric matrix P_1 such that

$$P_1 M_1 + M_1^T P_1 \leq -\epsilon_1 P_1 \tag{8}$$

$$\underline{\alpha}_1 P_1 \leq P_1 D_1 + D_1 P_1 \leq \bar{\alpha}_1 P_1 \tag{9}$$

for some positive constants ϵ_1, $\underline{\alpha}_1$ and $\bar{\alpha}_1$.

Proof. The proof of the first inequality is based on the construction proposed in (Wang et al., 2017, Lemma 2). Consider in particular the system $\dot{\xi}_1 = F_{n-1}\xi_1$ in which $\xi_1 = \mathrm{col}(\xi_{11}, \xi_{12}) \in \mathbb{R}^2$. Let $\Theta(r)$ be a matrix of the following form

$$\Theta(r) := \begin{pmatrix} r & 0 \\ -r & 1 \end{pmatrix}, \qquad \forall\, r \in \mathbb{R}. \tag{10}$$

Then, consider the following change of variables

$$\xi_1 \mapsto \eta_1 := \Theta(r_1)\xi_1, \quad \eta_{11} := r_1\xi_{11}, \quad \eta_{12} := \xi_{12} - r_1\xi_{11},$$

with $r_1 > 0$ to be chosen, and select $k_{n-1,2} = r_1 k_{n-1,1}$. We obtain

$$\begin{aligned} \dot{\eta}_{11} &= -(k_{n-1,1} - r_1)\eta_{11} + r_1\eta_{12} \\ \dot{\eta}_{12} &= -r_1(\eta_{11} + \eta_{12}). \end{aligned}$$

Now choose the Lyapunov function $V_1 = |\eta_1|^2 = \xi_1^T \Theta(r_1)^T \Theta(r_1)\xi_1$, whose time derivative is given by

$$\dot{V}_1 = -2(k_{n-1,1} - r_1)\eta_{11}^2 - 2r_1\eta_{12}^2.$$

By coming back in the ξ_1-coordinates and by using Young's inequality, the above equality can be rewritten as $\dot{V}_1 \leq -\epsilon_1|\xi_1|^2$ for any $r_1 > 0$, and $k_{n-1} > 2r_1$, with $\epsilon_1 = \min(2r_1^2(k_{n-1,1} - 2r_1), r_1)$. Therefore, by selecting

$$P_1 := \Theta(r_1)^\top \Theta(r_1) = \begin{pmatrix} 2r_1^2 & -r_1 \\ -r_1 & 1 \end{pmatrix},$$

the inequality (8) is verified and this completes the first part of the proof.

Let us now show that the inequality (9) is verified with the matrix P_1 established above. First, we prove the upper bound of (9). For this, note that

$$\bar{\alpha}_1 P_1 - (P_1 D_1 + D_1 P_1) \geq 0$$

is equivalent to

$$\begin{pmatrix} 2(\bar{\alpha}_1 - (2n-2))r_1^2 & -(\bar{\alpha}_1 - (2n-1))r_1 \\ -(\bar{\alpha}_1 - (2n-1))r_1 & \bar{\alpha}_1 - 2n \end{pmatrix} \geq 0.$$

It implies the following conditions (positivity of the diagonal terms and of the Schur's complement):

$$\begin{cases} \bar{\alpha}_1 \geq 2n, \\ 2(\bar{\alpha}_1 - 2n + 2)r_1^2 - \dfrac{(\bar{\alpha}_1 - (2n-1))^2}{\bar{\alpha}_1 - 2n} r_1^2 \geq 0, \end{cases} \tag{11}$$

which can be verified by selecting $\bar{\alpha}_1$ in the set $[2n - 1 + \sqrt{2}; +\infty[$. Then, to show the lower bound of (9), we need to find $\underline{\alpha}_1 > 0$ such that

$$\underline{\alpha}_1 P_1 - (P_1 D_1 + D_1 P_1) \leq 0$$

Previous inequality leads to the conditions

$$\begin{cases} \bar{\alpha}_1 \leq 2n - 2, \\ 2(\bar{\alpha}_1 - 2n + 2)r_1^2 - \dfrac{(\bar{\alpha}_1 - (2n-1))^2}{\bar{\alpha}_1 - 2n} r_1^2 \leq 0. \end{cases} \tag{12}$$

Constant $\underline{\alpha}_1$ that meet the inequality exists and have to be in the set $[2n - 1 - \sqrt{2}; 2n - 2]$. This shows the existence of $\bar{\alpha}_1 > \underline{\alpha}_1 > 0$ satisfying inequality (9). Note that those values are independent of $r_1 > 0$. Consequently, the proof of Lemma 1 is completed. □

We have now the following lemma concerning the matrices M_i, D_i defined at the beginning of this section..

Lemma 3. Assume there exist a symmetric positive definite matrix P_i and positive constants ϵ_i, $\underline{\alpha}_i$ and $\bar{\alpha}_i$ such that $P_i M_i + M_i^T P_i \leq -\epsilon_i I$ and $\underline{\alpha}_i P_i \leq P_i D_i + D_i P_i \leq \bar{\alpha}_i P_i$. Then there exist coefficients $k_{n-i-1,1}$ and $k_{n-i-1,2}$ and a positive definite symmetric matrix P_{i+1} such that

$$P_{i+1} M_{i+1} + M_{i+1}^T P_{i+1} \leq -\epsilon_{i+1} P_{i+1} \tag{13}$$

$$\underline{\alpha}_{i+1} P_{i+1} \leq P_{i+1} D_{i+1} + D_{i+1} P_{i+1} \leq \bar{\alpha}_{i+1} P_{i+1} \tag{14}$$

for some positive constants ϵ_{i+1}, $\underline{\alpha}_{i+1}$ and $\bar{\alpha}_{i+1}$.

Proof. Again, the proof of this lemma follows the same construction of P_{i+1} proposed in (Wang et al., 2017, Lemma 3). Consider in particular system

$$\begin{aligned} \dot{\xi}_{i+1} &= F_{n-i-1}\xi_{i+1} + \bar{N}_{n-i}\chi_i \\ \dot{\chi}_i &= M_i\chi_i + \bar{Q}_{n-i}\xi_{i+1} \end{aligned}$$

where $\xi_{i+1} = (\xi_{i+1,1}, \xi_{i+1,2}) \in \mathbb{R}^2$, $\chi_i = (\xi_1, ..., \xi_i) \in \mathbb{R}^{2i}$. Let us make the following linear change of coordinate

$$\xi_{i+1} \mapsto \eta_{i+1} := \Theta(r_{i+1})\xi_{i+1}$$

with $\Theta(r)$ is defined in (10) and r_{i+1} is a positive constant to be chosen. By taking $k_{n-i-1,2} = r_{i+1}k_{n-i-1,1}$, the system in the new coordinates can be rewritten as

$$\begin{aligned} \dot{\eta}_{i+1,1} &= -(k_{n-i-1,1} - r_{i+1})\eta_{i+1,1} + r_{i+1}\eta_{i+1,2} \\ \dot{\eta}_{i+1,2} &= -r_{i+1}\eta_{i+1,1} - r_{i+1}\eta_{i+1,2} + \bar{N}_{n-i}\chi_i \\ \dot{\chi}_i &= M_i\chi_i + \Gamma_i(\eta_{i+1,1} + \eta_{i+1,2}) \end{aligned}$$

where $\Gamma_i = \mathrm{col}(k_{n-i,1}, k_{n-i,2}, 0, ..., 0)$. Consider now the positive definite function $V_i = \chi_i^T P_i \chi_i$, with P_i given in the statement of the lemma, and compute its derivative

$$\dot{V}_i \leq -\frac{1}{2}\epsilon_i|\chi_i|^2 + \delta_1\xi_{i+1,2}^2$$

for some positive constant δ_1, independent of r_{i+1} and $k_{n-i-1,1}$. Then, consider the positive definite function

$$W_{i+1} = |\eta_{i+1}|^2 = \xi_{i+1}\Theta(r_{i+1})^T\Theta(r_{i+1})\xi_{i+1}$$

whose time derivative is given by

$$\dot{W}_{i+1} \leq -2r_{i+1}^2[k_{n-i-1,1} - r_{i+1}]\xi_{i+1,1}^2 - \tfrac{3}{4}r_{i+1}\xi_{i+1,2}^2 + \frac{1}{r_{i+1}}|\chi_i|^2.$$

Then, consider the Lyapunov function $V_i + W_{i+1}$. By choosing r_{i+1} such that $r_{i+1} = \max\{2\delta_1, \frac{4}{\epsilon_i}\}$ and $k_{n-i-1,1}$ satisfying $k_{n-i-1,1} > 2r_{i+1}$, and by combining together previous inequalities, we obtain

$$\begin{aligned} \dot{V}_i + \dot{W}_{i+1} \leq &-\frac{\epsilon_i}{4}|\chi_i|^2 - \frac{1}{4}r_{i+1}\xi_{i+1,2}^2 \\ &- 2r_{i+1}^2(k_{n-i-1,1} - 2r_{i+1})\xi_{i+1,1}^2. \end{aligned}$$

Therefore, by selecting $\chi_{i+1} = (\xi_{i+1}, \chi_i)$ and

$$P_{i+1} := \mathrm{blkdiag}(P'_{i+1}, P_i), \quad P'_{i+1} := \Theta(r_{i+1})^\top\Theta(r_{i+1}).$$

By differentiating the Lypaunov function

$$V_{i+1,1} = \chi_{i+1}^\top P_{i+1}\chi_{i+1}$$

we finally obtain $\dot{V}_{i+1} \leq -\epsilon_{i+1}|\chi_{i+1}|^2$ in which

$$\epsilon_{i+1} = \min\left\{\frac{\epsilon_i}{4}, 2r_{i+1}^2(k_{n-i-1,1} - 2r_{i+1}), \frac{1}{4}r_{i+1}\right\}.$$

Therefore $P_{i+1}M_{i+1} + M_{i+1}^T P_{i+1} \leq -\epsilon_{i+1}P_{i+1}$, showing inequality (13) and completing the first part of the proof.

Let us now show that the second inequality is verified with the matrix P_{i+1} that has been established. By the statement of the lemma, we know there exist positive constants $\underline{\alpha}_i, \bar{\alpha}_i$ such that $\underline{\alpha}_i P_i \le P_i D_i + D_i P_i \le \bar{\alpha}_i P_i$. Therefore, by using the definition of P_{i+1} given above, we compute (14). Its lower bound is given by

$$\underline{\alpha}_{i+1}\begin{pmatrix} P'_{i+1} & 0 \\ 0 & P_i \end{pmatrix} \le \begin{pmatrix} P'_{i+1}D'_{i+1} + D'_{i+1}P'_{i+1} & 0 \\ 0 & P_i D_i + D_i P_i \end{pmatrix}$$

while the upper bound is computed as

$$\begin{pmatrix} P'_{i+1}D'_{i+1} + D'_{i+1}P'_{i+1} & 0 \\ 0 & P_i D_i + D_i P_i \end{pmatrix} \le \bar{\alpha}_{i+1}\begin{pmatrix} P'_{i+1} & 0 \\ 0 & P_i \end{pmatrix}.$$

Due to the triangular structure of previous expression, we obtain the following two independent conditions equivalent to (14)

$$\begin{cases} \underline{\alpha}'_{i+1}P'_{i+1} \le P'_{i+1}D'_{i+1} + D'_{i+1}P'_{i+1} \le \bar{\alpha}'_{i+1}P'_{i+1} \\ \underline{\alpha}_{i+1}P_i \le P_i D_i + D_i P_i \le \bar{\alpha}_{i+1}P_i \end{cases} \tag{15}$$

The second condition is verified for some $\bar{\alpha}_i, \underline{\alpha}_i$ by assumption. Therefore, we focus on the first. Let us drop the subscript $i+1$ in the notation, and note that the matrix $P'(= P'_{i+1})$ previously defined has the same structure of the matrix P_1 defined in Lemma 2. We can re-use the same arguments to show the existence of $\bar{\alpha}' > \underline{\alpha}' > 0$ satisfying $\underline{\alpha}'P' \le P'D' + D'P' \le \bar{\alpha}'P'$, for $\underline{\alpha}' \in [2n-2i+1-\sqrt{2}; 2n-2i]$ and $\bar{\alpha}' \in [2n-2i+1+\sqrt{2}; +\infty[$ where n is the order of the system. Thus, the two conditions being verified, by selecting $\underline{\alpha}_{i+1} = \min(\underline{\alpha}_i, \underline{\alpha}')$ and $\bar{\alpha}_{i+1} = \max(\bar{\alpha}_i, \bar{\alpha}')$, we show that both conditions in (15) are verified, thus establishing inequality (14) and completing the proof of Lemma 3. □

Finally, by applying iteratively Lemma 3 it is possible to prove Lemma 1 by recalling that $M_{n-1} = M$ and $D_{n-1} = D$.

5. ACADEMIC EXAMPLE

In order to illustrate the interest of an observer designed as explained in this paper, we consider the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u\sin(x_2) - x_2. \end{cases} \tag{16}$$

where $u = \bar{u}\sin(t)$, with $\bar{u} > 0$. In the simulation, the initial conditions are selected as $x(0) = (1, 2, 3)$. It is possible to verify that system (16) verifies Assumption 1 with $\Omega(u) := 1 + |u|$. For comparative purposes, we considered a low-power high-gain observer (6a) with fixed gain L (as in Astolfi and Marconi (2015)) and a low-power high-gain observer with dynamic L given by (6b). For both observers, the coefficients k_{ij} are chosen, by following the procedure of (Astolfi and Marconi, 2015, Lemma 1), as $K_1 = (1.5, 1.05)$, $K_2 = (1.5, 0.2632)$, so that the eigenvalues of M in (3) are between $-.5$ and -1. It is possible to verify the the matrix P defined as solution to $PM + M^\top P = -0.1I$ verifies Lemma 1 with $\underline{p} = 0.029$, $\bar{p} = 3.301$, $\mu = 0.3$, $\underline{\alpha} = 1.136$, $\bar{\alpha} = 7.342$. Both observers are initialized in the origin.

First, we recall that a low-power high-gain observer (6a) with fixed L should take a value of L proportional to $\bar{u}$. If $\bar{u}$ is unknown, convergence for a constant given L cannot be always ensured. This is shown in Figure 2, where, for $L = 5$, $\bar{u}$ is selected as $\bar{u} = 5$, $\bar{u} = 15$ and $\bar{u} = 30$: by augmenting its value, convergence of the estimation error $|x - \hat{x}|$ is no more ensured and divergence occurs. Figure 3 shows the evolution of the estimation error $|x(t) - \hat{x}(t)|$ for the proposed low-power high-gain observer (6a) with updated gain (6b), with $\lambda_1 = 0.2$, $\lambda_2 = 1$ and $\lambda_3 = 0.15$, in the same three cases $\bar{u} = 5$, $\bar{u} = 15$ and $\bar{u} = 30$. Although convergences is always guaranteed, larger values of $\bar{u}$ results in a faster convergence rate. The evolution of the gain L in the three different scenario is depicted in Figure 4. Due to the oscillating behaviour of u, we can observe that L has an oscillating behaviour around $L = 3.5$, $L = 8.5$ and $L = 16$ respectively. Note that even though L is increased, the peaking phenomenon is not critically augmented as in the case of standard high-gain observers with fixed L. Finally, in Figure 5 we studied the influence of the parameters λ_i in the dynamics (6b) of L for $\bar{u} = 30$. We can see that a decrease of the ratio $\frac{\lambda_1}{\lambda_3}$ causes an increase of the values of L. Similarly, an increase of λ_2 causes larger values of L. We recall, indeed, that the conditions of Theorem 1 are only sufficient and not necessary.

6. CONCLUSION

In this work we combined the techniques of adaptive gain proposed in Andrieu et al. (2009) with the low-power structure of Astolfi and Marconi (2015). The resulting adaptive observer retains therefore the good properties explored in Astolfi et al. (2018b), that is the implementation of a high-gain parameter with powers up to 2 regardless the dimension of the system state and the good sensitivity properties with respect to measurement noise, and, at the same time, provides a self-tuning strategy with the aim of reducing the value of the implemented high-gain parameter with respect to a constant-gain approach. The adaptive law proposed in this work relies on the assumption that the Lipschitz constants of the nonlinear terms depend on a known external input. Future works will study the active use of such external input to the aim of further improving the performances in presence of measurement noise; the development of an adaptive law to address the case in which the Lipschitz constants of the nonlinear terms depend also on the measured output and the estimated state in addition to the input; the use of different gains with separated dynamics, one for each two-sized block of the observer, to obtain less conservative conditions with respect to the ones in Assumption 1; the use of the proposed adaptive observer in output feedback stabilization contexts (Wang et al. (2015); Teel and Praly (1994); Andrieu et al. (2008); Praly and Jiang (2004)).

REFERENCES

Alessandri, A. and Rossi, A. (2015). Increasing-gain observers for nonlinear systems: Stability and design. *Automatica*, 57, 180–188.

Andrieu, V., Nadri, M., Serres, U., and Vivalda, J.C. (2015). Self-triggered continuous–discrete observer with updated sampling period. *Automatica*, 62, 106–113.

Andrieu, V., Praly, L., and Astolfi, A. (2008). Homogeneous approximation, recursive observer design, and output feedback. *SIAM Journal on Control and Optimization*, 47(4), 1814–1850.

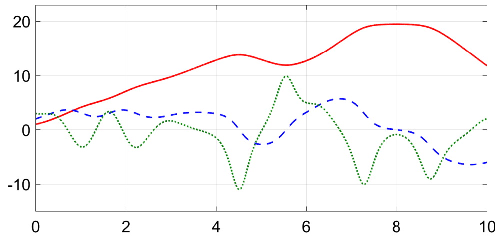

Fig. 1. Evolution of the state x of example (16), with $\bar{u} = 30$. Red line: x_1. Dashed blue line: x_2. Dotted green line: x_3.

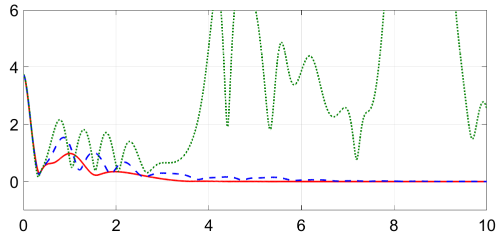

Fig. 2. Estimation error norm $|x(t) - \hat{x}(t)|$ for gain $L = 5$ constant and different values of $\bar{u}$. Red line: $\bar{u} = 5$. Dashed blue line: $\bar{u} = 15$. Dotted green line: $\bar{u} = 30$.

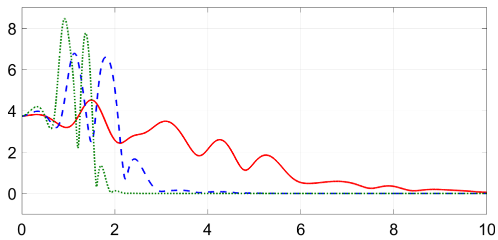

Fig. 3. Estimation error norm $|x(t) - \hat{x}(t)|$ with updated gain (6b) for different values of $\bar{u}$. Red line: $\bar{u} = 5$. Dashed blue line: $\bar{u} = 15$. Dotted green line: $\bar{u} = 30$.

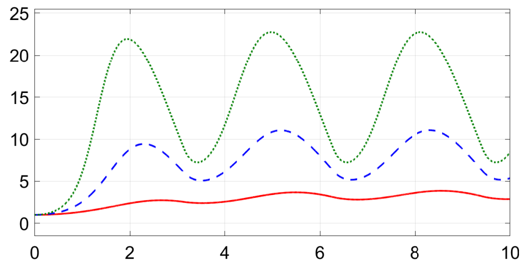

Fig. 4. Evolution of the updated observer gain L with parameters $\lambda_1 = 0.2$, $\lambda_2 = 1$, $\lambda_3 = 0.15$ for different values of $\bar{u}$. Red line: $\bar{u} = 5$. Dashed blue line: $\bar{u} = 15$. Dotted green line: $\bar{u} = 30$.

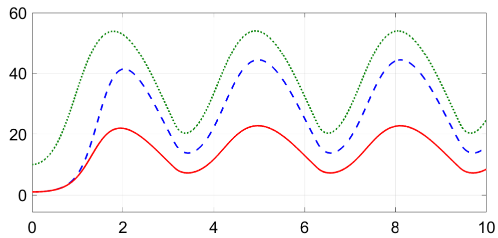

Fig. 5. Evolution of the gain L for $\bar{u} = 30$ and different values of $\lambda_1, \lambda_2, \lambda_3$. Red line: $\lambda_1 = 0.2, \lambda_2 = 1, \lambda_3 = 0.15$. Dashed blue line: $\lambda_1 = 0.1, \lambda_2 = 1, \lambda_3 = 0.15$. Dotted green line: $\lambda_1 = 0.1, \lambda_2 = 10, \lambda_3 = 0.15$.

Andrieu, V., Praly, L., and Astolfi, A. (2009). High gain observers with updated gain and homogeneous correction terms. *Automatica*, 45(2), 422–428.

Astolfi, D., Alessandri, A., and Zaccarian, L. (2017). Stubborn iss redesign for nonlinear high-gain observers. *IFAC-PapersOnLine*, 50(1), 15422–15427.

Astolfi, D., Jungers, M., and Zaccarian, L. (2018a). Output injection filtering redesign in high-gain observers. In *2018 European Control Conference*, 1957–1962.

Astolfi, D. and Marconi, L. (2015). A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11), 3059–3064.

Astolfi, D., Marconi, L., Praly, L., and Teel, A. (2016). Sensitivity to high-frequency measurement noise of nonlinear high-gain observers. *IFAC-PapersOnLine*, 49(18), 862–866.

Astolfi, D., Marconi, L., Praly, L., and Teel, A.R. (2018b). Low-power peaking-free high-gain observers. *Automatica*, 98, 169–179.

Boizot, N., Busvelle, E., and Gauthier, J.P. (2010). An adaptive high-gain observer for nonlinear systems. *Automatica*, 46(9), 1483–1488.

Cocetti, M., Tarbouriech, S., and Zaccarian, L. (2018). High-gain dead-zone observers for linear and nonlinear plants. *IEEE Control Systems Letters*, 3(2), 356–361.

Deza, F., Busvelle, E., Gauthier, J., and Rakotopara, D. (1992). High gain estimation for nonlinear systems. *Systems & control letters*, 18(4), 295–299.

Emel'Yanov, S., Korovin, S., Nikitin, S., and Nikitina, M. (1989). Observers and output differentiators for nonlinear systems. *Soviet Physics Doklady*, 34, 412.

Khalil, H.K. (2017). Cascade high-gain observers in output feedback control. *Automatica*, 80, 110–118.

Khalil, H.K. and Praly, L. (2014). High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 24(6), 993–1015.

Praly, L. and Jiang, Z.P. (2004). Linear output feedback with dynamic high gain for nonlinear systems. *Systems & Control Letters*, 53(2), 107–116.

Sanfelice, R.G. and Praly, L. (2011). On the performance of high-gain observers with gain adaptation under measurement noise. *Automatica*, 47(10), 2165–2176.

Teel, A. and Praly, L. (1994). Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems & Control Letters*, 22(5), 313–325.

Teel, A.R. (2016). Further variants of the Astolfi/Marconi high-gain observer. In *2016 American Control Conference*, 993–998.

Tornambe, A. (1992). High-gain observers for non-linear systems. *International Journal of Systems Science*, 23(9), 1475–1489.

Tréangle, C., Farza, M., and M'Saad, M. (2019). Filtered high gain observer for a class of uncertain nonlinear systems with sampled outputs. *Automatica*, 101, 197–206.

Wang, L., Astolfi, D., Marconi, L., and Su, H. (2017). High-gain observers with limited gain power for systems with observability canonical form. *Automatica*, 75, 16–23.

Wang, L., Astolfi, D., Su, H., Marconi, L., and Isidori, A. (2015). Output stabilization for a class of nonlinear systems via high-gain observer with limited gain power. *IFAC-PapersOnLine*, 48(11), 730–735.