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# Stability Analysis for Linear Systems with Time-Varying and Time-Invariant Stochastic Parameters [1]

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**Abstract:** This paper presents a method to guarantee stability of linear stochastic systems. The systems include both time-varying and time-invariant unknown stochastic parameters simultaneously. For analyzing the stability, such a system is represented by an expanded system that contains only the time-invariant stochastic parameter. This expansion excludes the time-varying parameter from the system, which simplifies the stability analysis. Existing methods on robust stability theory can be thus employed to ensure stability of the expanded system. Guaranteeing stability of the expanded system is a necessary and/or sufficient condition for that of the original system. Consequently, the stability of the original system is evaluated by using linear matrix inequalities.

*Keywords:* Robust stability, Stability analysis, Stochastic systems, Time-invariant stochastic parameters, Time-varying stochastic parameters, Uncertain linear systems

## 1. INTRODUCTION

There are various types of uncertain systems in the real world, which should be controlled. An example is an automated vehicle that interacts with manually-operated vehicles with uncertain dynamics (Nishi et al., 2019). A semi-autonomous vehicle shares steering control with a human driver (Saleh et al., 2013). It is important to guarantee stability of such systems under the uncertainty. This paper focuses on the stability of uncertain linear systems for ensuring the safe control.

The uncertainty in systems is described as stochastic parameters that are efficient for stability analysis of the systems. In the field of stochastic control theory, there are two main types of such parameters: *time-invariant* and *time-varying* stochastic parameters. Figure 1 shows examples of state trajectories of systems with such parameters. The time-invariant stochastic parameters are given as random variables that are constant in time (Fisher and Bhattacharya, 2009; Ito et al., 2016). The time-invariant property is applicable to represent variations in system parameters such as manufacturing variations. Meanwhile, time-varying stochastic parameters are randomly changed with the time (Koning, 1982; Ito et al., 2019). Such parameters are suitable for representing the noise effect on the system parameters rather than the static variation.

For describing linear systems including both time-varying and time-invariant stochastic parameters simultaneously, it is promising to regard system matrices as *random polytopes* (Hosoe et al., 2018). The random polytopes consist of vertices that are random matrices (defined in Section 2). However, there are limitations on stability analysis of linear systems with the random polytopes while they can represent various types of uncertain stochastic dynamics. In the ground breaking work (Hosoe et al., 2018), the stability of such systems is not guaranteed in an exact sense due to approximations. The exact stability has been analyzed if the set of the time-varying stochastic parameter is restricted to be not infinite but finite (Hosoe and Hagiwara, 2015). Various methods for the stability have been developed using Kronecker products (Hibey, 1996; Ogura and Martin, 2013), linear matrix inequalities (LMIs) (de Oliveira et al., 1999; Oliveira and Peres, 2005; Zhang et al., 2010), and boundary mapping (İlhan Mutlu et al., 2018; Voßwinkel et al., 2019). These methods focus on systems with either the time-varying or time-invariant parameter, but not both. To the best of our knowledge, while stability of other systems with both the parameters have been analyzed, e.g., (Tabarraie et al., 2016; Gershon and Shaked, 2018), the random polytopes (that we focus on) are different from such systems and/or less restrictive.

To overcome the limitations described above, this paper proposes a method to guarantee stability of linear stochastic systems. The proposed method satisfies the following three requirements simultaneously. First, the target systems consist of the random polytopes representing both the time-varying and time-invariant stochastic parameters. Second, the stability is analyzed in an exact sense without approximations. Third, the set of stochastic parameters need not be finite. The stability of such a system is ensured

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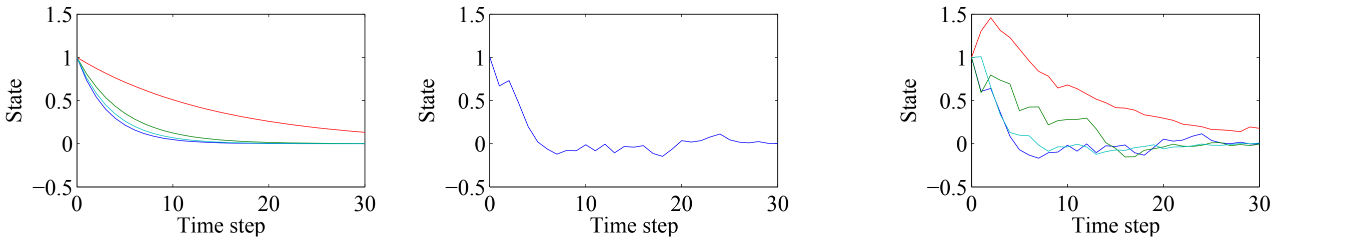


(a) With time-invariant parameters (b) With time-varying parameters (c) With time-invariant and time-varying parameters

Fig. 1. State trajectories of systems with time-invariant and/or time-varying stochastic parameters. Multiple lines in (a) and (c) indicate the trajectories with different values of the time-invariant parameters. Non-smooth (noisy) trajectories in (b) and (c) are invoked due to the time-varying parameters.

using an expanded system with only the time-invariant stochastic parameter. It is shown that guaranteeing stability of the expanded system is a necessary and/or sufficient condition for that of the original stochastic system. Consequently, via this result, a stability condition of the original system reduces to well-known LMIs.

The remainder of this paper is organized as follows. Section 2 describes the problem setting in this paper. Section 3 solves the main problem stated in Section 2. Section 4 demonstrates the proposed method in a numerical simulation. Section 5 concludes this paper and describes future work.

This paper uses the following notation.

- $\boldsymbol{I}_n$: the $n \times n$ identity matrix
- $[\boldsymbol{x}]_i$: the $i$-th component of a vector $\boldsymbol{x} \in \mathbb{R}^n$
- $[\boldsymbol{X}]_{i,j}$: the component in the $i$-th row and $j$-th column of a matrix $\boldsymbol{X} \in \mathbb{R}^{n\times m}$
- $\mathrm{vec}(\boldsymbol{X}) := [[\boldsymbol{X}]_{1,1}, \dots, [\boldsymbol{X}]_{n,1}, [\boldsymbol{X}]_{1,2}, \dots, [\boldsymbol{X}]_{n,2}, \dots, [\boldsymbol{X}]_{1,m}, \dots, [\boldsymbol{X}]_{n,m}]^\top$: the vectorization of the components of a matrix $\boldsymbol{X} \in \mathbb{R}^{n\times m}$
- $\mathrm{vech}(\boldsymbol{Y}) := [[\boldsymbol{Y}]_{1,1}, \dots, [\boldsymbol{Y}]_{n,1}, [\boldsymbol{Y}]_{2,2}, \dots, [\boldsymbol{Y}]_{n,2}, \dots, [\boldsymbol{Y}]_{j,j}, \dots, [\boldsymbol{Y}]_{n,j}, \dots, [\boldsymbol{Y}]_{n,n}]^\top$: the half-vectorization of the lower triangular components of a square matrix $\boldsymbol{Y} \in \mathbb{R}^{n\times n}$
- $\boldsymbol{X}_\mathrm{a} \otimes \boldsymbol{X}_\mathrm{b} \in \mathbb{R}^{n_\mathrm{a}n_\mathrm{b}\times m_\mathrm{a}m_\mathrm{b}}$: the Kronecker product of matrices $\boldsymbol{X}_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a}\times m_\mathrm{a}}$ and $\boldsymbol{X}_\mathrm{b} \in \mathbb{R}^{n_\mathrm{b}\times m_\mathrm{b}}$, given by
$$\boldsymbol{X}_\mathrm{a} \otimes \boldsymbol{X}_\mathrm{b} = \begin{bmatrix} [\boldsymbol{X}_\mathrm{a}]_{1,1}\boldsymbol{X}_\mathrm{b} & \dots & [\boldsymbol{X}_\mathrm{a}]_{1,m_\mathrm{a}}\boldsymbol{X}_\mathrm{b} \\ \vdots & \ddots & \vdots \\ [\boldsymbol{X}_\mathrm{a}]_{n_\mathrm{a},1}\boldsymbol{X}_\mathrm{b} & \dots & [\boldsymbol{X}_\mathrm{a}]_{n_\mathrm{a},m_\mathrm{a}}\boldsymbol{X}_\mathrm{b} \end{bmatrix}$$
- $\mathrm{E}_{\boldsymbol{\lambda}}[\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda})]$: the expectation $\int_{\mathbb{S}_{\boldsymbol{\lambda}}} \boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda})p(\boldsymbol{\lambda})\mathrm{d}\boldsymbol{\lambda}$ of a function $\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda})$ with respect to a stochastic parameter $\boldsymbol{\lambda} \in \mathbb{S}_{\boldsymbol{\lambda}}$ with a probability density function (PDF) $p(\boldsymbol{\lambda})$, where the above integral is replaced with a sum when $\mathbb{S}_{\boldsymbol{\lambda}}$ is a finite set.
- $\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_t}[\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots)]$: the expectation of a function $\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots)$ with respect to stochastic parameters $(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_t)$ for any $t \in \{0,1,\dots\}$, where $\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_t}[\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots)]$ for $t<0$ denotes $\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots)$.

## 2. PROBLEM SETTING

Let us consider a discrete-time linear stochastic system with the state variable $\boldsymbol{x}_t \in \mathbb{R}^n$ at the discrete time $t \in \{0,1,2,\dots\}$:

$$\boldsymbol{x}_{t+1} = \boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t)\boldsymbol{x}_t, \tag{1}$$

where $\boldsymbol{x}_0 \in \mathbb{R}^n$ is given as a deterministic vector. The system matrix $\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t) \in \mathbb{R}^{n\times n}$ depends on both the time-invariant stochastic parameter $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}} \subset \mathbb{R}^K$ and time-varying stochastic parameter $\boldsymbol{\lambda}_t \in \mathbb{S}_{\boldsymbol{\lambda}} \subseteq \mathbb{R}^L$. The system matrix $\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t)$ is defined as the following random polytope in a manner similar to (Hosoe et al., 2018):

$$\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t) := \sum_{k=1}^{K} [\boldsymbol{\omega}]_k \boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t), \tag{2}$$

where the vertex $\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t) \in \mathbb{R}^{n\times n}$ is a continuous function of the time-varying stochastic parameter $\boldsymbol{\lambda}_t$ if $\mathbb{S}_{\boldsymbol{\lambda}}$ is not finite. Various types of stochastic systems can be represented by the form of (1) with (2). Section 4.1 demonstrates such a representation and another example is found in (Hosoe et al., 2018). Throughout this paper, let us assume the following properties for the stochastic parameters.

*Assumption 1.* (Stochastic parameters).

(i) The values of $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}_t$ for any $t$ are unknown.
(ii) The set $\mathbb{S}_{\boldsymbol{\omega}}$ of all possible $\boldsymbol{\omega}$ is given by
$$\mathbb{S}_{\boldsymbol{\omega}} = \left\{ \boldsymbol{\omega} \in \mathbb{R}^K \middle| \forall k, [\boldsymbol{\omega}]_k \geq 0, \sum_{k=1}^{K} [\boldsymbol{\omega}]_k = 1 \right\}, \tag{3}$$
while the PDF of $\boldsymbol{\omega}$ is unknown.
(iii) The time-varying $\boldsymbol{\lambda}_t$ is independent and identically distributed (i.i.d.) from a PDF $p(\boldsymbol{\lambda})$ on a set $\mathbb{S}_{\boldsymbol{\lambda}}$.
(iv) For any $t$, for any $k$ and $k'$ in $\{1,2,\dots,K\}$, and for any $i$, $j$, $i'$, and $j'$ in $\{1,2,\dots,n\}$, the following expectation with respect to $\boldsymbol{\lambda}_t$ is bounded and known:
$$\mathrm{E}_{\boldsymbol{\lambda}_t}\left[[\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t)]_{i,j}[\boldsymbol{A}^{(k')}(\boldsymbol{\lambda}_t)]_{i',j'}\right], \tag{4}$$
while it is admissible that $p(\boldsymbol{\lambda})$ and $\mathbb{S}_{\boldsymbol{\lambda}}$ are unknown.
(v) If $\mathbb{S}_{\boldsymbol{\lambda}}$ is not finite, $p(\boldsymbol{\lambda})$ is continuous on $\mathbb{S}_{\boldsymbol{\lambda}} = \mathbb{R}^L$.

Note that Assumption 1 (iv) is not strong because we need not the full information of $p(\boldsymbol{\lambda})$ but only the expectations. To analyze stability of the system (1), this paper introduces the following stability notions. Since $\mathbb{S}_{\boldsymbol{\omega}}$ is assumed to be a bounded set, we focus on stability for all $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ robustly.

*Definition 1.* (Robust mean-square stability). The system (1) is said to be robustly mean-square (MS) stable if the following relation holds:

$$\forall \boldsymbol{x}_0 \in \mathbb{R}^n,\ \forall \boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}},\ \lim_{t\to\infty} \mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2] = 0. \tag{5}$$

*Definition 2.* (Exponential robust mean-square stability). The system (1) is said to be exponentially robustly MS stable if there exist $\rho \in (0,1)$ and $\alpha \in (0,\infty)$ such that the following relation holds:

$$\forall \boldsymbol{x}_0 \in \mathbb{R}^n,\ \forall \boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}},\ \forall t \in \{0,1,\dots\},$$
$$\sqrt{\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2]} \le \alpha \|\boldsymbol{x}_0\| \rho^t. \quad (6)$$

*Remark 1.* The system (1) is robustly MS stable if it is exponentially robustly MS stable because the following relation holds:

$$\lim_{t\to\infty} \mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2] \le \lim_{t\to\infty} \alpha^2 \|\boldsymbol{x}_0\|^2 \rho^{2t} = 0. \quad (7)$$

Under these assumptions and definitions, the main problem to be solved in this paper is stated as follows.

*Main problem.* Find necessary and/or sufficient conditions that the system (1) is (exponentially) robustly MS stable.

## 3. PROPOSED METHOD

In this section, we solve the main problem stated in the previous section. Section 3.1 gives an idea to solve it. Section 3.2 presents the solutions, which are the main results. To derive explicit conditions for guaranteeing the stability of the system (1), the main results are applied to existing stability analysis in Section 3.3.

### 3.1 Idea and overview

The key idea for solving the main problem is to expand the system (1) such that the time-varying stochastic parameter $\boldsymbol{\lambda}_t$ is excluded from the system. Such an expanded system involves only the time-invariant stochastic parameter $\boldsymbol{\omega}$ while the original system (1) includes both the time-invariant and time-varying parameters. This exclusion simplifies the stability analysis for stochastic systems.

First, we introduce an important operator to derive the expanded system.

*Definition 3.* (Compression operator $\mathcal{C}$). For any square matrix $\boldsymbol{H} \in \mathbb{R}^{n^2\times n^2}$, the compression operator $\mathcal{C}$ : $\mathbb{R}^{n^2\times n^2} \to \mathbb{R}^{(n(n+1)/2)\times(n(n+1)/2)}$ is defined as follows:

$$\mathcal{C}(\boldsymbol{H}) := \mathcal{L}\boldsymbol{H}\mathcal{D}, \quad (8)$$

where the duplication matrix $\mathcal{D} \in \mathbb{R}^{n^2\times(n(n+1)/2)}$ and elimination matrix $\mathcal{L} \in \mathbb{R}^{(n(n+1)/2)\times n^2}$ are defined such that $\mathcal{D}\mathrm{vech}(\boldsymbol{S}) = \mathrm{vec}(\boldsymbol{S})$ holds for any symmetric matrix $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ and $\mathcal{L}\mathrm{vec}(\boldsymbol{M}) = \mathrm{vech}(\boldsymbol{M})$ holds for any square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, respectively.

*Remark 2.* The duplication matrix $\mathcal{D}$ and elimination matrix $\mathcal{L}$ are uniquely determined for each $n$ (Magnus and Neudecker, 1980). Their explicit definitions are found in (Magnus and Neudecker, 1980). As an example, these matrices with $n = 2$ are given as follows:

$$\mathcal{D} = \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}, \quad (9)$$

$$\mathcal{L} = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{bmatrix}. \quad (10)$$

Using the compression operator $\mathcal{C}$, let us introduce the expanded system:

$$\widetilde{\boldsymbol{x}}_{t+1} = \mathcal{C}(\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega}))\widetilde{\boldsymbol{x}}_t, \quad (11)$$

where $\widetilde{\boldsymbol{x}}_t \in \mathbb{R}^{(n(n+1)/2)}$ is the expanded state at the time $t$. The matrix $\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega}) \in \mathbb{R}^{n^2\times n^2}$ with an expanded time-invariant stochastic parameter $\boldsymbol{\Omega} \in \mathbb{R}^{K\times K}$ is defined as follows:

$$\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega}) := \sum_{k=1}^{K}\sum_{k'=1}^{K} [\boldsymbol{\Omega}]_{k,k'} \widetilde{\boldsymbol{A}}^{(k,k')}, \quad (12)$$

$$\widetilde{\boldsymbol{A}}^{(k,k')} := \mathrm{E}_{\boldsymbol{\lambda}_0}[\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_0) \otimes \boldsymbol{A}^{(k')}(\boldsymbol{\lambda}_0)] = \mathrm{E}_{\boldsymbol{\lambda}_t}[\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t) \otimes \boldsymbol{A}^{(k')}(\boldsymbol{\lambda}_t)]. \quad (13)$$

Note that $\widetilde{\boldsymbol{A}}^{(k,k')} \in \mathbb{R}^{n^2\times n^2}$ is deterministic and known because all its components are known by Assumption 1 (iv). In (13), $\widetilde{\boldsymbol{A}}^{(k,k')}$ is constant in $t$ because $\boldsymbol{\lambda}_t$ is i.i.d. by Assumption 1 (iii). Based on these definitions, we obtain a key connection between the expanded system (11) and the original system (1).

*Theorem 1.* (Expanded system). For any initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$ and any time-invariant stochastic parameter $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$, suppose the following properties:

$$\widetilde{\boldsymbol{x}}_0 = \mathrm{vech}(\boldsymbol{x}_0\boldsymbol{x}_0^\top), \quad (14)$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}\boldsymbol{\omega}^\top. \quad (15)$$

The following relation between the state and expanded state holds for all $t \in \{0,1,2,\dots\}$:

$$\widetilde{\boldsymbol{x}}_t = \mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[\mathrm{vech}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)]. \quad (16)$$

**Proof.** The proof is described in Appendix A.

*Remark 3.* Theorem 1 indicates that the expanded state $\widetilde{\boldsymbol{x}}_t$ in (11) is equivalent to the second moment of the state $\boldsymbol{x}_t$ in (1) with respect to $(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1})$. Therefore, the expanded system (11) can be used to evaluate the second moment with respect to the time-varying stochastic parameters despite the fact that the expanded system is the time-invariant stochastic system. Although expanded systems similar to (11) have been used for stability analysis of time-varying stochastic systems (Hibey, 1996; Ogura and Martin, 2013), the time-invariant stochastic parameters $\boldsymbol{\omega}$ are not involved.

*Remark 4.* The compression operator $\mathcal{C}$ plays an important role for developing the expanded system (11). Although a straightforward representation of the second moment of the state $\boldsymbol{x}_t$ is $\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[\mathrm{vec}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)]$ (Hibey, 1996; Ogura and Martin, 2013), this includes the duplicated components. Indeed, $[\mathrm{vec}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)]_{i,j} = [\mathrm{vec}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)]_{j,i}$ holds for any $i$ and $j$. By virtue of the compression operator $\mathcal{C}$ in (11), the second moment is given by $\mathrm{vech}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)$ in (16) that omits the duplicated components. The representation of $\mathrm{vech}(\boldsymbol{x}_t\boldsymbol{x}_t^\top)$ is computationally efficient because the dimension of $\mathrm{vech}(\boldsymbol{x}_t\boldsymbol{x}_t^\top) \in \mathbb{R}^{(n(n+1)/2)}$ is smaller than that of $\mathrm{vec}(\boldsymbol{x}_t\boldsymbol{x}_t^\top) \in \mathbb{R}^{n^2}$ for $n \ge 2$.

### 3.2 Main results

This subsection presents the main results in this paper. We derive necessary and/or sufficient conditions for guaranteeing the (exponential) robust MS stability of the original system (1). The necessary/sufficient conditions are that the expanded system (11) is (exponentially) robustly stable under some assumptions. It is notable that the necessity/sufficiency with respect to the stability conditions depends on the set $\mathbb{S}_{\boldsymbol{\Omega}}$ of the expanded time-invariant stochastic parameter $\boldsymbol{\Omega}$. In the following, we introduce two candidates of $\mathbb{S}_{\boldsymbol{\Omega}}$ associated with necessary/sufficient conditions for ensuring the stability of the original system.

First, let us define the (exponential) robust stability of the expanded system (11).

*Definition 4.* (Robust stability). The expanded system (11) is said to be robustly stable if the following relation holds:

$$\forall \widetilde{\boldsymbol{x}}_0 \in \mathbb{R}^{(n(n+1)/2)},\ \forall \boldsymbol{\Omega} \in \mathbb{S}_{\boldsymbol{\Omega}},\quad \lim_{t\to\infty} \|\widetilde{\boldsymbol{x}}_t\|^2 = 0. \tag{17}$$

*Definition 5.* (Exponential robust stability). The expanded system (11) is said to be exponentially robustly stable if there exist $\widetilde{\rho} \in (0,1)$ and $\widetilde{\alpha} \in (0,\infty)$ such that the following relation holds:

$$\forall \widetilde{\boldsymbol{x}}_0 \in \mathbb{R}^{(n(n+1)/2)},\ \forall \boldsymbol{\Omega} \in \mathbb{S}_{\boldsymbol{\Omega}}, \forall t \in \{0,1,\dots\},\quad \|\widetilde{\boldsymbol{x}}_t\| \le \widetilde{\alpha}\|\widetilde{\boldsymbol{x}}_0\|\widetilde{\rho}^t. \tag{18}$$

Next, let us define the two candidates of $\mathbb{S}_{\boldsymbol{\Omega}}$:

$$\mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}} := \left\{ \boldsymbol{\Omega} \in \mathbb{R}^{K\times K} \middle| \forall k,k', [\boldsymbol{\Omega}]_{k,k'} \ge 0, \sum_{k=1}^{K}\sum_{k'=1}^{K} [\boldsymbol{\Omega}]_{k,k'} = 1 \right\}, \tag{19}$$

$$\mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}} := \left\{ \boldsymbol{\Omega} \in \mathbb{R}^{K\times K} \middle| \boldsymbol{\Omega} = \boldsymbol{\omega}\boldsymbol{\omega}^\top,\ \boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}} \right\}. \tag{20}$$

Using these candidates, we derive two necessary/sufficient conditions for the robust MS stability of the original system (1) in the cases that $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$ and $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}}$.

*Theorem 2.* (Sufficient condition). Suppose that $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$. The original system (1) is robustly MS stable if the expanded system (11) is robustly stable.

**Proof.** The proof is described in Appendix B.

*Theorem 3.* (Necessary and sufficient condition). Suppose that $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}}$. The original system (1) is robustly MS stable if and only if the expanded system (11) is robustly stable.

**Proof.** The proof is described in Appendix C.

*Remark 5.* If $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$ holds in Theorem 2, the expanded system (11) reduces to a polytopic linear system. There are existing methods analyzing the robust stability of such a system, e.g., (de Oliveira et al., 1999; Oliveira and Peres, 2005). Section 3.3 demonstrates the combining of Theorem 2 with an existing method. This combination presents a sufficient condition for the stability of the original system (1) in an explicit form.

*Remark 6.* Theorem 3 derives the necessary and sufficient condition for the stability of the original system (1). The expanded system (11) reduces to a non-polytopic uncertain system in the case of $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}}$. While it is difficult to analyze stability of this non-polytopic system without conservativeness, such an analysis may lead to efficient results for the stability of the original system. This problem is included in future work.

We now have derived the necessary/sufficient conditions for the robust MS stability of the original system (1). Finally, this subsection presents a sufficient condition for the exponential robust MS stability of the original system.

*Theorem 4.* (Exponential robust MS stability). Suppose that $\mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}} \subseteq \mathbb{S}_{\boldsymbol{\Omega}} \subseteq \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$. If the expanded system (11) is exponentially robustly stable with a given $\widetilde{\rho} \in (0,1)$ and given $\widetilde{\alpha} \in (0,\infty)$, the system (1) is exponentially robustly MS stable with the following relations:

$$\alpha \ge n^{1/4}\sqrt{\widetilde{\alpha}}, \tag{21}$$

$$\rho \ge \sqrt{\widetilde{\rho}}. \tag{22}$$

**Proof.** The proof is described in Appendix D.

### *3.3 Applications of the main results to existing stability analysis*

As shown in Theorems 2 and 4 in the previous subsection, the (exponential) robust MS stability of the original system (1) is reduced to the (exponential) robust stability of the expanded system (11). The stability of the expanded system can be evaluated by the existing method (de Oliveira et al., 1999) because the expanded system includes only the time-invariant stochastic parameter. In the following, we provide an explicit condition for the stability of the original system by combining the results in the previous subsection with the existing method.

*Theorem 5.* (Explicit sufficient condition for the stability). The original system (1) is exponentially robustly MS stable with a given $\rho \in (0,1)$ if there exist positive definite symmetric matrices $\widetilde{\boldsymbol{P}}^{(k,k')} \succ 0 \in \mathbb{R}^{(n(n+1)/2)\times(n(n+1)/2)}$ for $k \in \{1,\dots,K\}$ and $k' \in \{1,\dots,K\}$ and a square matrix $\widetilde{\boldsymbol{G}} \in \mathbb{R}^{(n(n+1)/2)\times(n(n+1)/2)}$ such that

$$\forall\ k \in \{1,\dots,K\}, k' \in \{1,\dots,K\},\quad \begin{bmatrix} \rho^4\widetilde{\boldsymbol{P}}^{(k,k')} & \mathcal{C}(\widetilde{\boldsymbol{A}}^{(k,k')})^\top\widetilde{\boldsymbol{G}}^\top \\ \widetilde{\boldsymbol{G}}\mathcal{C}(\widetilde{\boldsymbol{A}}^{(k,k')}) & \widetilde{\boldsymbol{G}} + \widetilde{\boldsymbol{G}}^\top - \widetilde{\boldsymbol{P}}^{(k,k')} \end{bmatrix} \succeq 0. \tag{23}$$

Also, the original system (1) is robustly MS stable if there exist $\widetilde{\boldsymbol{P}}^{(k,k')} \succ 0$ and $\widetilde{\boldsymbol{G}}$ such that the strict inequalities ($\succ$) of (23) hold for $\rho = 1$.

**Proof.** The proof is described in Appendix E.

The LMIs (23) can be solved for $\widetilde{\boldsymbol{G}}$ and $\widetilde{\boldsymbol{P}}^{(k,k')}$ using numerical solvers (Boyd et al., 1994) (see an example in Appendix F). Now, we have derived the solvable sufficient conditions that the original system (1) is (exponentially) robustly MS stable.

## 4. NUMERICAL EXAMPLE

In this section, we demonstrate the proposed method. Section 4.1 describes a target system with the simulation setting. Section 4.2 shows the simulation results regarding the stability analysis of the target system.

### *4.1 Simulation setting*

Let us consider the linear stochastic system:

$$\boldsymbol{x}_{t+1} = \begin{bmatrix} [\boldsymbol{q}]_1 & [\boldsymbol{\lambda}_t]_1 \\ [\boldsymbol{q}]_2[\boldsymbol{\lambda}_t]_2 & 0.9 \end{bmatrix} \boldsymbol{x}_t, \tag{24}$$

where $\boldsymbol{\lambda}_t$ and $\boldsymbol{q}$ are unknown two-dimensional time-varying and time-invariant stochastic parameters, respectively. The time-invariant parameter $\boldsymbol{q}$ is distributed on the set $[0.8 - r, 0.8 + r] \times [1.0 - r, 1.0 + r]$, where $r$ determines the range of the set. The time-varying parameter $\boldsymbol{\lambda}_t$ obeys the normal distribution with mean $[0,0]^\top$ and covariance

$\sigma^2 \boldsymbol{I}_2$, where $\sigma$ indicates the size of the covariance. It is notable that $\boldsymbol{q}$ can be replaced with a four-dimensional $\boldsymbol{\omega}$ by using the form of the polytopic stochastic system (1) with (2). This system (24) is represented by (1) and (2) with the following matrices:

$$\begin{cases} \boldsymbol{A}^{(1)}(\boldsymbol{\lambda}_t) := \begin{bmatrix} 0.8 - r & [\boldsymbol{\lambda}_t]_1 \\ (1.0 - r)[\boldsymbol{\lambda}_t]_2 & 0.9 \end{bmatrix}, \\ \boldsymbol{A}^{(2)}(\boldsymbol{\lambda}_t) := \begin{bmatrix} 0.8 - r & [\boldsymbol{\lambda}_t]_1 \\ (1.0 + r)[\boldsymbol{\lambda}_t]_2 & 0.9 \end{bmatrix}, \\ \boldsymbol{A}^{(3)}(\boldsymbol{\lambda}_t) := \begin{bmatrix} 0.8 + r & [\boldsymbol{\lambda}_t]_1 \\ (1.0 - r)[\boldsymbol{\lambda}_t]_2 & 0.9 \end{bmatrix}, \\ \boldsymbol{A}^{(4)}(\boldsymbol{\lambda}_t) := \begin{bmatrix} 0.8 + r & [\boldsymbol{\lambda}_t]_1 \\ (1.0 + r)[\boldsymbol{\lambda}_t]_2 & 0.9 \end{bmatrix}. \end{cases} \tag{25}$$

The proposed stability analysis is used for the system (1) with (2) and (25). If this system is (exponentially) robustly MS stable, the stability of the system (24) is also guaranteed.

For guaranteeing the system (24), we should clarify the admissible values of the range $r$ of the time-invariant parameter and the standard deviation $\sigma$ of the time-varying parameter. In the next subsection, the proposed method finds pairs of admissible $r$ and $\sigma$ such that the system (24) is (exponentially) robustly MS stable.

### *4.2 Simulation results*

We evaluated pairs of admissible $r$ and $\sigma$ for guaranteeing the (exponential) robust MS stability of the system (24). Recall that the pairs of $r$ and $\sigma$ are admissible if feasible solutions to the LMIs (23) are found. A bisection method estimated the maximum admissible value of $r$ for each $\sigma$ such that the feasible solutions are obtained for $\rho = 0.999$. Here, the LMIs (23) were solved by the MATLAB solver, feasp (Gahinet et al., 1995), as shown in Appendix F. The pairs of admissible $r$ and $\sigma$ were plotted in Fig. 2. The green and blue markers indicate the results in the cases that either the time-varying parameter or the time-invariant parameter is deterministic, i.e., $\boldsymbol{\lambda}_t = [0, 0]^\top$ or $\boldsymbol{q} = [0.8, 1.0]^\top$, respectively. Meanwhile, the proposed method can evaluate the stability of the system including both types of stochastic parameters. Hence, the pairs of such admissible values were obtained, denoted by the red markers. Indeed, Fig. 3 shows that the states were stable for various stochastic parameters included in the admissible ranges. We confirmed that the proposed method is successfully applied to guaranteeing the stability of linear systems including both types of stochastic parameters.

## 5. CONCLUSION

This paper presented a method to guarantee stability of linear systems with both time-varying and time-invariant stochastic parameters. For analyzing the stability, an expanded system was developed that excludes the time-varying parameter from the system. Such exclusion simplifies the stability analysis for the system. It was shown that guaranteeing stability of the expanded system is a necessary/sufficient condition for that of the original system. Based on this result, a sufficient condition for the stability of the original system was derived as LMIs. Future work will focus on design of controllers that stabilize linear systems including both the parameters.

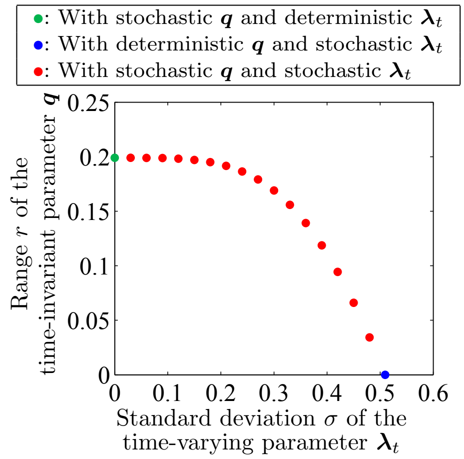


Fig. 2. Results for admissible limits on the time-invariant and time-varying stochastic parameters.

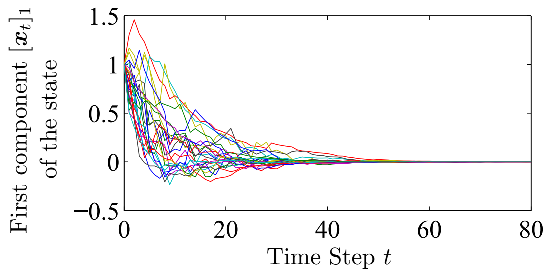


Fig. 3. Control results with admissible values of the time-invariant and time-varying stochastic parameters. Different lines indicate various values of the parameters.

## REFERENCES

Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. SIAM, Philadelphia, PA.

de Oliveira, M.C., Bernussou, J., and Geromel, J.C. (1999). A new discrete-time robust stability condition. *Systems & Control Letters*, 37(4), 261–265.

Fisher, J. and Bhattacharya, R. (2009). Linear quadratic regulation of systems with stochastic parameter uncertainties. *Automatica*, 45(12), 2831–2841.

Gahinet, P., Nemirovski, A., Laub, A.J., and Chilali, M. (1995). *LMI Control Toolbox User's Guide*. The MathWorks, Inc., Natick, MA.

Gershon, E. and Shaked, U. (2018). Stochastic discrete-time systems with delay - robust vertex-dependent $h_\infty$ state-feedback control. In *Proc. of 26th Mediterranean Conf. on Control and Automation*, 499–504.

Hibey, J.L. (1996). Stochastic stability theory for systems containing interval matrices. *IEEE Trans. on Aerospace and Electronic Systems*, 32(4), 1385–1391.

Hosoe, Y. and Hagiwara, T. (2015). Extension of the concept of random polytopes and robust stabilization synthesis. In *Proc. of 2015 European Control Conf.*, 2858–2863.

Hosoe, Y., Hagiwara, T., and Peaucelle, D. (2018). Robust stability analysis and state feedback synthesis for discrete-time systems characterized by random polytopes. *IEEE Trans. on Automatic Control*, 63(2), 556–562.

İlhan Mutlu, Schrödel, F., Mihailescu-Stoica, D., Alaa, K., and Söylemez, M.T. (2018). A case study on determining stability boundaries of parameter uncertain systems. In *Proc. of 26th Mediterranean Conf. on Control and Automation*, 643–648.

Ito, Y., Fujimoto, K., Tadokoro, Y., and Yoshimura, T. (2016). On stochastic optimal control for linear systems with robust stability. In *Proc. of IEEE 55th Conf. Decision and Control*, 5390–5395.

Ito, Y., Fujimoto, K., Tadokoro, Y., and Yoshimura, T. (2019). Risk-sensitive linear control for systems with stochastic parameters. *IEEE Trans. on Automatic Control*, 64(4), 1328–1343.

Koning, W.L.D. (1982). Infinite horizon optimal control of linear discrete time systems with stochastic parameters. *Automatica*, 18(4), 443–453.

Magnus, J.R. and Neudecker, H. (1980). The elimination matrix: Some lemmas and applications. *SIAM Journal on Algebraic Discrete Methods*, 1(4), 422–449.

Nishi, T., Doshi, P., and Prokhorov, D. (2019). Merging in congested freeway traffic using multipolicy decision making and passive actor-critic learning. *IEEE Trans. on Intelligent Vehicles*, 4, 287–297.

Ogura, M. and Martin, C.F. (2013). Generalized joint spectral radius and stability of switching systems. *Linear Algebra and its Applications*, 439(8), 2222–2239.

Oliveira, R.C.L.F. and Peres, P.L. (2005). Stability of polytopes of matrices via affine parameter-dependent lyapunov functions: Asymptotically exact lmi conditions. *Linear Algebra and its Applications*, 405, 209–228.

Saleh, L., Chevrel, P., Claveau, F., Lafay, J.F., and Mars, F. (2013). Shared steering control between a driver and an automation: Stability in the presence of driver behavior uncertainty. *IEEE Trans. on Intelligent Transportation Systems*, 14(2), 974–983.

Tabarraie, M., Niapour, S.M., and Shafai, B. (2016). Robust $l_\infty$-induced filtering and deconvolution of a wide class of linear discrete-time stochastic systems. *Signal Processing*, 122, 213–227.

Voßwinkel, R., Pyta, L., Schrödel, F., İlhan Mutlu, Mihailescu-Stoica, D., and Bajcinca, N. (2019). Performance boundary mapping for continuous and discrete time linear systems. *Automatica*, 107, 272–280.

Zhang, X., Tsiotras, P., and Iwasaki, T. (2010). Lyapunov-based exact stability analysis and synthesis for linear single-parameter dependent systems. *International Journal of Control*, 83(9), 1823–1838.

## Appendix A. PROOF OF THEOREM 1

The statement is shown using mathematical induction. For any $s \in \{0, 1, 2, \dots\}$, supposing that (16) holds for $t = s$, we obtain

$$\begin{aligned}
&\mathrm{vech}(\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\boldsymbol{x}_{s+1}\boldsymbol{x}_{s+1}{}^\top]) \\
&= \mathcal{L}\mathrm{vec}(\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\boldsymbol{x}_{s+1}\boldsymbol{x}_{s+1}{}^\top]) \\
&= \mathcal{L}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\mathrm{vec}(\boldsymbol{x}_{s+1}\boldsymbol{x}_{s+1}{}^\top)] \\
&= \mathcal{L}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\mathrm{vec}(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)\boldsymbol{x}_s\boldsymbol{x}_s{}^\top\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)^\top)] \\
&= \mathcal{L}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)\otimes\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s))\mathrm{vec}(\boldsymbol{x}_s\boldsymbol{x}_s{}^\top)] \\
&= \mathcal{L}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)\otimes\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s))\mathcal{D}\mathrm{vech}(\boldsymbol{x}_s\boldsymbol{x}_s{}^\top)] \\
&= \mathcal{C}(\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)\otimes\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s))]) \\
&\qquad \times \mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\mathrm{vech}(\boldsymbol{x}_s\boldsymbol{x}_s{}^\top)] \\
&= \mathcal{C}(\mathrm{E}_{\boldsymbol{\lambda}_s}[(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s)\otimes\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_s))])\widetilde{\boldsymbol{x}}_s. \qquad \text{(A.1)}
\end{aligned}$$

For any $t$, $\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega})$ is represented as follows

$$\begin{aligned}
&\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega}) \\
&= \sum_{k=1}^{K}\sum_{k'=1}^{K}[\boldsymbol{\omega}\boldsymbol{\omega}^\top]_{k,k'}\mathrm{E}_{\boldsymbol{\lambda}_t}[\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t)\otimes\boldsymbol{A}^{(k')}(\boldsymbol{\lambda}_t)] \\
&= \mathrm{E}_{\boldsymbol{\lambda}_t}\left[\left(\sum_{k=1}^{K}[\boldsymbol{\omega}]_k\boldsymbol{A}^{(k)}(\boldsymbol{\lambda}_t)\right)\otimes\left(\sum_{k'=1}^{K}[\boldsymbol{\omega}]_{k'}\boldsymbol{A}^{(k')}(\boldsymbol{\lambda}_t)\right)\right] \\
&= \mathrm{E}_{\boldsymbol{\lambda}_t}[(\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t)\otimes\boldsymbol{A}(\boldsymbol{\omega},\boldsymbol{\lambda}_t))]. \qquad \text{(A.2)}
\end{aligned}$$

Substituting this property into (A.1) yields

$$\mathrm{vech}(\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_s}[\boldsymbol{x}_{s+1}\boldsymbol{x}_{s+1}{}^\top]) = \mathcal{C}(\widetilde{\boldsymbol{A}}(\boldsymbol{\Omega}))\widetilde{\boldsymbol{x}}_s. \qquad \text{(A.3)}$$

Substituting this equation into (11) with $t = s$ leads to (16) for $t = s + 1$. Since (16) was supposed for $t = 0$, (16) holds for all $t \in \{0, 1, 2, \dots\}$ by the mathematical induction. This completes the proof.

## Appendix B. PROOF OF THEOREM 2

For any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$, the relation of $\boldsymbol{\omega}\boldsymbol{\omega}^\top \in \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$ holds because

$$\forall k, k', \quad [\boldsymbol{\omega}\boldsymbol{\omega}^\top]_{k,k'} \geq 0, \qquad \text{(B.1)}$$

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}[\boldsymbol{\omega}\boldsymbol{\omega}^\top]_{k,k'} = 1. \qquad \text{(B.2)}$$

holds. Thus, for any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ and any $\boldsymbol{x}_0$, there exist $\widetilde{\boldsymbol{x}}_0$ and $\boldsymbol{\Omega}$ such that (14) and (15) hold. For such $\widetilde{\boldsymbol{x}}_0$ and $\boldsymbol{\Omega}$, Theorem 1 gives (16) for all $t \in \{0, 1, 2, \dots\}$. For any $i$, there exists $j$ such that

$$\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i^2] = [\widetilde{\boldsymbol{x}}_t]_j \leq \|\widetilde{\boldsymbol{x}}_t\|, \qquad \text{(B.3)}$$

holds because of (16). If the expanded system (11) is robustly stable, we obtain the following asymptotic convergence for all $i \in \{1, \dots, n\}$:

$$\lim_{t\to\infty}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i^2] \leq \lim_{t\to\infty}\|\widetilde{\boldsymbol{x}}_t\| = 0. \qquad \text{(B.4)}$$

This convergence holds for any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ and any $\boldsymbol{x}_0$. Therefore, the system (1) is robustly MS stable. This completes the proof.

## Appendix C. PROOF OF THEOREM 3

The sufficiency is proved in a manner similar to the proof of Theorem 2. Thus, the system (1) is robustly MS stable if the expanded system (11) is robustly stable. We show the necessity in the following.

For any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ and any $\boldsymbol{x}_0$, let us suppose that (14) and (15) hold to satisfy (16) for all $t \in \{0, 1, 2, \dots\}$ in Theorem

1. For any $i$ and $j$, applying Cauchy-Schwarz inequality to the expectations of $\boldsymbol{x}_t$ yields

$$\begin{aligned}&|\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i[\boldsymbol{x}_t]_j]|\\ &\le (\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i{}^2]\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[[\boldsymbol{x}_t]_j{}^2])^{1/2}. \quad \text{(C.1)}\end{aligned}$$

If the system (1) is robustly MS stable, this property indicates

$$\lim_{t\to\infty}|\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i[\boldsymbol{x}_t]_j]| = 0. \quad \text{(C.2)}$$

Combining this convergence with (16) gives

$$\lim_{t\to\infty}\widetilde{\boldsymbol{x}}_t = 0. \quad \text{(C.3)}$$

Here, for any $\boldsymbol{\Omega} \in \mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}}$, there exists $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ such that (15) holds. Also, for any $\widetilde{\boldsymbol{x}}_0$, there exist initial states $\boldsymbol{x}_0$ of the original system such that a linear combination of the initial states $\boldsymbol{x}_0$ is equal to $\widetilde{\boldsymbol{x}}_0$. Therefore, satisfying (C.3) for any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ and any $\boldsymbol{x}_0$ leads to the robust stability of the expanded system (11) under the condition $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}}$. This completes the proof.

## Appendix D. PROOF OF THEOREM 4

Since $\mathbb{S}_{\boldsymbol{\Omega},\mathrm{b}} \subseteq \mathbb{S}_{\boldsymbol{\Omega}} \subseteq \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$, for any $\boldsymbol{\omega} \in \mathbb{S}_{\boldsymbol{\omega}}$ and any $\boldsymbol{x}_0$, there exist $\widetilde{\boldsymbol{x}}_0$ and $\boldsymbol{\Omega}$ such that (14) and (15) hold. For such $\widetilde{\boldsymbol{x}}_0$ and $\boldsymbol{\Omega}$, Theorem 1 gives (16) for all $t \in \{0, 1, 2, \ldots\}$. Using Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n}[\boldsymbol{v}]_i^2\right)\left(\sum_{i=1}^{n}[\boldsymbol{w}]_i^2\right) \ge \left(\sum_{i=1}^{n}[\boldsymbol{v}]_i[\boldsymbol{w}]_i\right)^2, \quad \text{(D.1)}$$

with the settings of $[\boldsymbol{v}]_i = \mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i^2]$ and $[\boldsymbol{w}]_i = 1/\sqrt{n}$, we obtain

$$\begin{aligned}\|\widetilde{\boldsymbol{x}}_t\|^2 = \sum_{i=1}^{(n(n+1)/2)}[\widetilde{\boldsymbol{x}}_t]_i^2 &\ge \sum_{i=1}^{n}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i^2]^2\sum_{i=1}^{n}\frac{1}{n}\\ &\ge \frac{1}{n}\left(\sum_{i=1}^{n}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[[\boldsymbol{x}_t]_i^2]\right)^2\\ &= \frac{1}{n}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2]^2. \quad \text{(D.2)}\end{aligned}$$

Meanwhile,

$$\begin{aligned}\|\widetilde{\boldsymbol{x}}_0\|^2 = \sum_{i=1}^{(n(n+1)/2)}[\widetilde{\boldsymbol{x}}_0]_i^2 &\le \sum_{i=1}^{n}\sum_{j=1}^{n}[\boldsymbol{x}_0]_i^2[\boldsymbol{x}_0]_j^2\\ &= \left(\sum_{i=1}^{n}[\boldsymbol{x}_0]_i^2\right)\left(\sum_{j=1}^{n}[\boldsymbol{x}_0]_j^2\right)\\ &= \|\boldsymbol{x}_0\|^4. \quad \text{(D.3)}\end{aligned}$$

If the expanded system (11) is exponentially robustly stable, substituting (D.2) and (D.3) into (18) yields

$$\frac{1}{\sqrt{n}}\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2] \le \widetilde{\alpha}\|\boldsymbol{x}_0\|^2\widetilde{\rho}^t. \quad \text{(D.4)}$$

Taking the root of the above equation gives

$$\sqrt{\mathrm{E}_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{t-1}}[\|\boldsymbol{x}_t\|^2]} \le n^{1/4}\sqrt{\widetilde{\alpha}}\|\boldsymbol{x}_0\|(\sqrt{\widetilde{\rho}})^t. \quad \text{(D.5)}$$

Therefore, substituting (21) and (22) yields (6). This completes the proof.

## Appendix E. PROOF OF THEOREM 5

If there exist $\widetilde{\boldsymbol{G}}$ and $\widetilde{\boldsymbol{P}}^{(k,k')} \succ 0$ satisfying (23), the expanded system (11) with $\mathbb{S}_{\boldsymbol{\Omega}} = \mathbb{S}_{\boldsymbol{\Omega},\mathrm{a}}$ is exponentially robust stable with $\widetilde{\rho} = \rho^2$, based on the result in (de Oliveira et al., 1999, Theorem 2) and a manner similar to (Hosoe et al., 2018, Theorems 1 and 2). Therefore, using Theorem 4 shows that the system (1) is exponentially robustly MS stable with the given $\rho$. The robust MS stability is also ensured in a similar manner by combining Theorem 2 with the result in (de Oliveira et al., 1999, Theorem 2). This completes the proof.

## Appendix F. NUMERICAL SOLUTION TO THE LMIS (23)

To find solutions to the LMIs (23) in a numerical sense, the following minimization problem can be employed by the MATLAB solver, feasp (Gahinet et al., 1995):

$$\begin{aligned}&\min_{\eta,\widetilde{\boldsymbol{G}},\widetilde{\boldsymbol{P}}^{(k,k')},\forall k,k'} \quad \eta\\ &\text{s.t.} \quad \forall\, k \in \{1,\ldots,K\}, k' \in \{1,\ldots,K\},\\ &\quad \eta\boldsymbol{I}_{(n(n+1))} + \begin{bmatrix}\rho^4\widetilde{\boldsymbol{P}}^{(k,k')} & \mathcal{C}(\widetilde{\boldsymbol{A}}^{(k,k')})^\top\widetilde{\boldsymbol{G}}^\top\\ \widetilde{\boldsymbol{G}}\mathcal{C}(\widetilde{\boldsymbol{A}}^{(k,k')}) & \widetilde{\boldsymbol{G}}+\widetilde{\boldsymbol{G}}^\top-\widetilde{\boldsymbol{P}}^{(k,k')}\end{bmatrix} \succeq 0,\\ &\quad \eta\boldsymbol{I}_{(n(n+1)/2)} + \widetilde{\boldsymbol{P}}^{(k,k')} \succeq 0. \quad \text{(F.1)}\end{aligned}$$

If the minimal value of $\eta$ in (F.1) is not positive, the feasible solutions $\widetilde{\boldsymbol{G}}$ and $\widetilde{\boldsymbol{P}}^{(k,k')} \succ 0$ satisfying the LMIs (23) are found.