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A Generalized Minimum Phase Property for Finite-Dimensional Continuous-Time MIMO LTI Systems with Additive Disturbances

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Abstract: In this paper, we further generalize the definition of the extended zero dynamics to finite-dimensional continuous-time MIMO LTI systems with additive disturbances; and then introduce the concept of minimum phase for this class of systems. We show that the extended zero dynamics is invariant under the application of dynamic extension to its input, and therefore, a minimum phase system remains minimum phase after a finite number of steps of dynamic extension. We further introduce the extended zero dynamics canonical form for square MIMO LTI systems with uniform vector relative degree. We prove that a system is minimum phase according to our definition if its zero dynamics is asymptotically stable. The converse of the statement holds if the system is stabilizable from the control input.

The objective of this research is to solve the model reference robust adaptive control problem for finite-dimensional continuous-time square MIMO LTI systems that is minimum phase according to the generalized definition using an appropriately vectorized version of Pan and Başar (2000), and results here constitute important building blocks. In a subsequent paper, Başar and Pan (2019), we further connect the dots: starting with a square MIMO LTI system that is minimum phase with respect to admissible initial conditions and admissible disturbance waveforms, we must obtain a true system representation that admits both the extended zero dynamics canonical form representation and the strict observer canonical form representation. Toward this end, we need to be able to extend the given system to one with uniform vector relative degree and with uniform observability indices without changing its minimum phase property, and this has been done in Başar and Pan (2019), fully resolving this issue.

Keywords: Minimum phase, zero dynamics canonical form, extended zero dynamics, extended zero dynamics canonical form, dynamic extension, vector relative degree.

1. INTRODUCTION

The minimum phase concept for linear systems is of paramount importance in their model reference control. The classical definition of the minimum phase property for SISO linear systems is that the transfer function of such a system has all zeros in $\mathbb{C}_-$ (Ioannou and Sun, 1996). The concept has been generalized to affine nonlinear systems as asymptotic stability property of their zero dynamics. When the system is free of disturbances, the zero dynamics has been defined in Isidori (1995) as the dynamics of the system when the input is designed to keep the output of the system to be identically zero. When the system is further subject to an additive disturbance input, the minimum phase property is an assumption on the extended zero dynamics of the system. The extended zero dynamics of the system is simply the zero dynamics of the system as defined in Isidori (1995) together with driving terms involving the output of the system and the additive disturbance inputs (see Pan and Başar (2018) for the single-input and single-output (SISO) case). Generalization of the minimum phase concept to nonlinear systems had been studied earlier in Liberzon et al. (2002), which is in line with the development of Sontag and Wang (1997).

In this paper, we investigate a generalization of the minimum phase concept of Pan and Başar (2018), defined for SISO LTI systems with well-defined relative degree and additive disturbance inputs, to multiple-input and multiple-output (MIMO) LTI systems with additive disturbance inputs. The objective is to define the minimum phase concept in such a way that it is both necessary and sufficient for model reference control. Motivated by the results of Pan and Başar (2018) and Isidori (1995), we define the extended zero dynamics of any MIMO LTI system as the maximal dimensional linear subdynamics, driven by the output of the system and the disturbance inputs, but independent of the rest of the system states or the control input. For square MIMO LTI systems with vector relative degree, the zero dynamics canonical form of the system as defined in Isidori (1995) reveals the extended zero dynamics of such systems. We prove in this paper that the extended zero dynamics of the system is invariant (modulo linear state transformation) if we apply a step of the dynamic extension (Isidori, 1995) to it. Therefore, for a square MIMO LTI system which, after a finite number of dynamic extensions, admits vector relative degree, the extended zero dynamics of the system can be computed and checked for our definition of the concept of minimum phase. This means that our concept of minimum phase can be checked for square MIMO LTI systems that are right invertible (See Remark 1). We further introduce the extended zero dynamics canonical form for square MIMO LTI systems with uniform vector relative degree. This extended zero dynamics canonical form is essential in the proof of robust adaptive control design (Pan and Başar, 2000). We prove that if the extended zero dynamics is asymptotically stable (i. e., minimum phase according to Isidori (1995)), then it is minimum phase according to our definition. The converse holds if the original system is stabilizable from the control input.

In a subsequent paper, Başar and Pan (2019), we build on the results of this paper, and prove that the generalized minimum phase property is necessary for model reference control. We further connect the dots: starting with a square MIMO LTI system that is minimum phase with respect to the admissible initial condition and admissible disturbance waveform, we must obtain a true system representation that admits both the extended zero dynamics canonical form representation and the strict observer canonical form representation in order to apply the (appropriately vectorized version of) robust adaptive control design of Pan and Başar (2000) to the system. For a system to admit the extended zero dynamics canonical form, it must admit uniform vector relative degree. This can be achieved by dynamic extensions that are independent of the unknown parameters in the system. This condition on a system may be too restrictive. Actually, we just need the system to admit vector relative degree (not necessarily uniform) after a finite number of dynamic extensions that are independent of the unknown parameters in the system. After that, we can further achieve the requirement of uniform vector relative degree by appropriately integrating the output channels of the system, and thus leading to an extended system that admits the extended zero dynamics canonical form (see Başar and Pan (2019)). For the system to admit the strict observer canonical form, it must admit uniform observability indices. We therefore have to further extend the system without changing its relative degree and minimum phase property. This can be done by adding dummy state variables into the system. These results are summarized in two lemmas in Başar and Pan (2019) that fully resolve the model reference robust adaptive control problem for minimum phase finite-dimensional continuous-time square MIMO LTI systems.

The balance of the paper is as follows. In the next section, we introduce the notations used in the paper. In Section 3, we introduce the definition of the extended zero dynamics of an MIMO LTI system and the concept of minimum phase for the system based on the property of its extended zero dynamics. Then, we present the extended zero dyanmics canonical form representation for a square MIMO LTI system with uniform vector relative degree in four exhaustive cases based on the relationship of the relative degree r, the number of output channels m, and the dimension of the system n. The paper ends with some concluding remarks in Section 4, and an appendix.

2. NOTATIONS

We let $\mathbb{R}$ denote the real line; let $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; and $\mathbb{C}$ be the set of complex numbers. Unless otherwise specified, all signals, constants, and matrices are real. For a function f, we say that it belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously differentiable (Fréchet differentiability), which is equivalent to all partial derivatives up to kth order being continuous when the domain of f is open, $k \in \mathbb{N} \cup \{\infty\}$. We say that a function is L_∞ if it is bounded. For any matrix A, A' denotes its transpose. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{Z}_+$, $|z|$ denotes the Euclidean norm $\sqrt{z'z}$. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any matrix M, $\|M\|_{p,p}$ denotes its p-induced norm, $1 \le p \le \infty$. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m\times n}$.

3. THE MINIMUM PHASE PROPERTY

In this section, we will introduce a generalized definition of the minimum phase property for finite-dimensional continuous time MIMO LTI systems. We first present a canonical form that reveals the extended zero dynamics for square MIMO LTI systems with vector relative degree.

Lemma 1. Consider a square MIMO LTI system

$$\dot{x} = Ax + Bu + Dw; \quad x(0) = x_0 \in \mathcal{D}_0 \tag{1a}$$
$$y = Cx + Fu + Ew \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}^m$ is the control input, $m \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (Pan and Başar, 2018), A, B, D, C, F, and E are constant matrices of appropriate dimensions.

Let the system admit vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ from u to y, that is, $i = 1, \ldots, m$,

$$F_{i,:} = C_{i,:}B = \cdots = C_{i,:}A^{r_i-2}B = \mathbf{0}_{1\times m}$$

where $F_{i,:}$ and $C_{i,:}$ are the ith row vectors of the matrices F and C, respectively, and

$$\begin{bmatrix} C_{1,:}A^{r_1-1}B \\ \vdots \\ C_{m,:}A^{r_m-1}B \end{bmatrix} =: B_0$$

is an invertible matrix (for those $i = 1, \ldots, m$ with $r_i = 0$, the corresponding row in B_0 is replaced by $F_{i,:}$). The matrix B_0 is said to be the *high frequency gain matrix*. Then, there exists an invertible matrix T_o such that, in

$$\bar{x} := T_o^{-1}x = [\, x_z' \; x_{1,1} \ldots x_{1,r_1} \ldots x_{m,1} \ldots x_{m,r_m} \,]'$$

coordinates, the system (1) admits the representation

$$\dot{x}_z = A_z x_z + \sum_{i=1}^{m} A_{z1,i} y_i + D_z w \tag{2a}$$
$$\dot{x}_{i,j} = x_{i,j+1} + D_{i,j} w; \tag{2b}$$
$$1 \le i \le m \text{ with } r_i > 0, 1 \le j < r_i$$
$$\dot{x}_{i,r_i} = A_i \bar{x} + C_{i,:}A^{r_i-1}Bu + D_{i,r_i} w; \tag{2c}$$
$$1 \le i \le m \text{ with } r_i > 0$$
$$y_i = x_{i,1} + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i > 0 \tag{2d}$$
$$y_i = \bar{C}_{i,:}\bar{x} + F_{i,:}u + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i = 0 \tag{2e}$$

where $x_z \in \mathbb{R}^{n-\sum_{i=1}^m r_i}$; $x_{i,j} \in \mathbb{R}$, $1 \le i \le m$ with $r_i > 0$, $1 \le j \le r_i$. (2) is called the *zero dynamics canonical form* of system (1). (Note that here (2) is not the extended zero dynamics canonical form.) The dynamics (2a) is said to be the *extended zero dynamics* of system (1).

Proof It is proved in (Isidori, 1995, Chapter 5) that the following row vectors are linearly independent:

$$Q_{i,j} := C_{i,:}A^{j-1}, 1 \le i \le m \text{ with } r_i > 0, 1 \le j \le r_i$$

Let K_1 be any $(n - \sum_{i=1}^{m} r_i) \times n$-dimensional real matrix such that

$$T_1 =: \left[K_1' \, Q_{1,1}' \cdots Q_{1,r_1}' \cdots Q_{m,1}' \cdots Q_{m,r_m}' \right]'$$

is an invertible matrix. Let

$$\hat{x} := T_1 x =: (\hat{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots, x_{m,r_m})$$

where $\hat{x}_z \in \mathbb{R}^{n-\sum_{i=1}^{m} r_i}$, and the rest are scalars. In $\hat{x}$ coordinates, (1) admits the representation

$$\dot{\hat{x}}_z = \hat{A}_z \hat{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i} \hat{A}_{zj,i} x_{i,j} + \hat{B}_z u + \hat{D}_z w \tag{3a}$$

$$\dot{x}_{i,j} = x_{i,j+1} + D_{i,j} w; \tag{3b}$$
$$1 \le i \le m \text{ with } r_i > 0, 1 \le j < r_i$$

$$\dot{x}_{i,r_i} = \hat{A}_i \hat{x} + C_{i,:}A^{r_i-1}Bu + D_{i,r_i} w; \tag{3c}$$
$$1 \le i \le m \text{ with } r_i > 0$$

$$y_i = x_{i,1} + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i > 0 \tag{3d}$$

$$y_i = \hat{C}_{i,:}\hat{x} + F_{i,:}u + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i = 0 \tag{3e}$$

By (3c) and (3e), we may solve for

$$u = \hat{K}_1 y + \tilde{K}_1 \hat{x} + \hat{H} w + \sum_{i=1, r_i > 0}^{m} \tilde{H}_i \dot{x}_{i,r_i} = B_0^{-1} S$$

where S is an m-dimensional vector with the ith row being $\dot{x}_{i,r_i} - \hat{A}_i \hat{x} - D_{i,r_i} w$, when $r_i > 0$, or $y_i - \hat{C}_{i,:}\hat{x} - E_{i,:}w$, when $r_i = 0$, and $\hat{K}_1$ has nonzero column i only for y_i with $r_i = 0$. Substituting this into (3a), we have

$$\dot{\hat{x}}_z = \hat{A}_z \hat{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i} \hat{A}_{zj,i} x_{i,j} + \hat{B}_z (\sum_{i=1, r_i > 0}^{m} \tilde{H}_i \dot{x}_{i,r_i} + \hat{K}_1 y + \tilde{K}_1 \hat{x} + \hat{H} w) + \hat{D}_z w$$

Introduce the coordinate transformation

$$\tilde{x}_z = \hat{x}_z - \sum_{i=1, r_i > 0}^{m} \hat{B}_z \tilde{H}_i x_{i,r_i}$$

Then, in $\tilde{x} := (\tilde{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots x_{m,r_m})$ coordinates, we have

$$\dot{\tilde{x}}_z = \tilde{A}_z \tilde{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{r_i} \tilde{A}_{zj,i} x_{i,j} + \sum_{i=1, r_i = 0}^{m} \tilde{A}_{z1,i} y_i + \tilde{D}_z w \tag{4a}$$

$$\dot{x}_{i,j} = x_{i,j+1} + D_{i,j} w; \tag{4b}$$
$$1 \le i \le m \text{ with } r_i > 0, 1 \le j < r_i$$

$$\dot{x}_{i,r_i} = \tilde{A}_i \tilde{x} + C_{i,:}A^{r_i-1}Bu + D_{i,r_i} w; \tag{4c}$$
$$1 \le i \le m \text{ with } r_i > 0$$

$$y_i = x_{i,1} + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i > 0 \tag{4d}$$

$$y_i = \tilde{C}_{i,:}\tilde{x} + F_{i,:}u + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i = 0 \tag{4e}$$

Note that $x_{i,r_i} = \dot{x}_{i,r_i-1} - D_{i,r_i-1} w$, $1 \le i \le m$ with $r_i \ge 2$. Then, we can substitute these into (4a) and introduce a state transformation:

$$\tilde{x}_z - \sum_{i=1, r_i \ge 2}^{m} \tilde{A}_{zr_i,i} x_{i,r_i-1} =: \check{x}_z$$

to arrive at the following representation in

$$\check{x} =: (\check{x}_z, x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{m,1}, \ldots x_{m,r_m})$$

coordinates for (1)

$$\dot{\check{x}}_z = \check{A}_z \check{x}_z + \sum_{i=1}^{m}\sum_{j=1}^{0 \vee (r_i-1)} \check{A}_{zj,i} x_{i,j} + \sum_{i=1, r_i \in \{0,1\}}^{m} \check{A}_{z1,i} y_i + \check{D}_z w \tag{5a}$$

$$\dot{x}_{i,j} = x_{i,j+1} + D_{i,j} w; \tag{5b}$$
$$1 \le i \le m \text{ with } r_i > 0, 1 \le j < r_i$$

$$\dot{x}_{i,r_i} = \check{A}_i \check{x} + C_{i,:}A^{r_i-1}Bu + D_{i,r_i} w; \tag{5c}$$
$$1 \le i \le m \text{ with } r_i > 0$$

$$y_i = x_{i,1} + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i > 0 \tag{5d}$$

$$y_i = \check{C}_{i,:}\check{x} + F_{i,:}u + E_{i,:}w; \; 1 \le i \le m \text{ with } r_i = 0 \tag{5e}$$

Recursively, we can eliminate all $x_{i,j}$'s in $\check{x}_z$ dynamics for all $j > 1$. Note that $x_{i,1} = y_i - E_{i,:}w$, $1 \le i \le m$ with $r_i > 0$, and we arrive at the canonical form (2). This completes the proof of the lemma. □

For a square MIMO LTI system, which is without vector relative degree, but can be dynamically extended to one with vector relative degree, we will show (which will be proved later in the paper) that the extended zero dynamics of such a system is identical to the extended zero dynamics for the corresponding dynamically extended system with vector relative degree. Instead of obtaining the canonical form for this more general class of systems, we seek to directly obtain the extended zero dynamics of a given system by noting the structure in (2a).

Consider a MIMO LTI system (not necessary square)

$$\dot{x} = Ax + Bu + Dw; \quad x(0) = x_0 \in \mathcal{D}_0 \tag{6a}$$

$$y = Cx + Fu + Ew \tag{6b}$$

where $x \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$, A, B, D, C, F, and E are constant matrices of appropriate dimensions. For MIMO systems, we tend to require $\mathcal{D}_0$ be a subspace of $\mathbb{R}^n$ rather than just a nonempty subset of $\mathbb{R}^n$. We will make our assumption explicit in the various results if we do require $\mathcal{D}_0$ to be a subspace. Suppose that we can find a full row rank real matrix K, which is $s \times n$-dimensional, and real matrices $A_z \in \mathbb{R}^{s \times s}$ and $A_{z1} \in \mathbb{R}^{s \times m}$ such that

$$KA = A_z K + A_{z1} C \tag{7a}$$

$$A_{z1} F = KB \tag{7b}$$

Then, defining $x_z := Kx$, it evolves according to

$$\dot{x}_z = K(Ax + Bu + Dw) = A_z K x + A_{z1} C x + KBu + KDw = A_z x_z + A_{z1} y + (KD - A_{z1} E) w$$

This looks like the extended zero dynamics we seek. Let K, A_z, A_{z1} be a solution to (7) and $\bar{K}$, $\bar{A}_z$, $\bar{A}_{z1}$ be another solution to (7); then it is easy to check that

$$\begin{bmatrix} \bar{K} \\ K \end{bmatrix}, \begin{bmatrix} \bar{A}_z & \mathbf{0} \\ \mathbf{0} & A_z \end{bmatrix}, \begin{bmatrix} \bar{A}_{z1} \\ A_{z1} \end{bmatrix}$$

is also a solution to (7), except that this solution might not be full row rank for $\begin{bmatrix} \bar{K} \\ K \end{bmatrix}$, even though $\bar{K}$ and K are. Now, let $\tilde{K}$ be a matrix that consists of a set of maximal linearly independent row vectors in $\begin{bmatrix} \bar{K} \\ K \end{bmatrix}$. Then, there exists a real matrix T such that

$$\begin{bmatrix} \bar{K} \\ K \end{bmatrix} = T\tilde{K}.$$

Clearly, T is of full column rank. Now, it is easy to check that

$$\tilde{K},\ \tilde{A}_z := (T'T)^{-1}T' \begin{bmatrix} \bar{A}_z & \mathbf{0} \\ \mathbf{0} & A_z \end{bmatrix} T,$$

and

$$\tilde{A}_{z1} := (T'T)^{-1}T' \begin{bmatrix} \bar{A}_{z1} \\ A_{z1} \end{bmatrix}$$

is a full row rank solution to (7). Thus, (7) admits a maximal solution K, A_z, A_{z1}, in the sense that K is of full row rank and for any other solution to (7), $\tilde{K}$, $\tilde{A}_z$ and $\tilde{A}_{z1}$, we have all row vectors of $\tilde{K}$ linearly dependent on row vectors of K. Thus, $\tilde{K} = T_z K$, and, if $\tilde{K}$ is also a maximal solution, then T_z is an invertible real matrix. Thus, these two solutions will yield extended zero dynamics that are similar to each other.

We now introduce the following definition of minimum phase for continuous-time MIMO LTI systems.

Definition 1. Consider an MIMO LTI system (6). Let $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, $A_{z1} \in \mathbb{R}^{s\times m}$ be a maximal solution to (7). Then, $x_z := Kx$ satisfies the dynamics

$$\dot{x}_z = A_z x_z + A_{z1} y + (KD - A_{z1}E)w; \qquad (8)$$
$$x_z(0) = Kx_0 \in K(\mathcal{D}_0)$$

This is said to be the *extended zero dynamics* of (6). (Note that $s = 0$ is also a possible solution, which corresponding to the case when the extended zero dynamics is absent) We will say that (6) is *minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ if the extended zero dynamics is absent; or if (8) satisfies $\forall c_w \geq 0$, $\exists c_c \geq 0$, $\forall x_z(0) = Kx_0 \in K(\mathcal{D}_0)$ with $|x_z(0)| \leq c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, we have $\|x_{z[0,\infty)}\|_\infty \leq c_c$.

For square MIMO LTI systems with vector relative degree, Lemma 1 then guarantees that they admit the zero dynamics canonical form (2). Therefore, the maximal solution to (7) for the system is K, which is equal to the matrix consisting of the first $n - \sum_{i=1}^m r_i$ rows of the matrix T_o^{-1}, A_z, and $A_{z1} := [A_{z1,1} \cdots A_{z1,m}]$ as in Lemma 1.

It is straightforward to see that Definition 1 subsumes Definition 3 of Pan and Başar (2018) for SISO systems, since SISO systems considered in that paper were assumed to admit (vector) relative degree.

In the following, we will show that for the MIMO LTI system (6), a step of dynamic extension (Isidori, 1995, Chapter 5) does not alter its extended zero dynamics. Therefore, the original system (before dynamic extension) is minimum phase if, and only if, the dynamically extended system is minimum phase.

Consider an MIMO LTI system (6). Let $r_i \in \mathbb{Z}_+$, $1 \leq i \leq m$, be such that, $i = 1, \ldots, m$,

$$F_{i,:} = C_{i,:}B = \cdots = C_{i,:}A^{r_i-2}B = \mathbf{0}_{1\times p}$$

and $C_{i,:}A^{r_i-1}B \neq \mathbf{0}_{1\times p}$, (or $F_{i,:} \neq \mathbf{0}_{1\times p}$ if $r_i = 0$), where $F_{i,:}$ and $C_{i,:}$ are the ith row vectors of the matrices F and C, respectively. Let

$$H := \begin{bmatrix} C_{1,:}A^{r_1-1}B \\ \vdots \\ C_{m,:}A^{r_m-1}B \end{bmatrix}$$

(if $r_i = 0$, then the ith row of H is $F_{i,:}$), which may or may not be invertible. Assume that (6) admits a maximal solution $K \in \mathbb{R}^{s\times n}$, A_z, A_{z1} to (7).

Let the elements of H be $(H_{i,j})_{m\times p}$ and $H_{i_0,j_0} \neq 0$. Let us do a step of dynamic extension for u_{j_0} with pivot H_{i_0,j_0} as

$$u_j = v_j; \quad \forall 1 \leq j \leq p \text{ with } j \neq j_0; \qquad (9a)$$
$$u_{j_0} = H_{i_0,j_0}^{-1}(\xi - \sum_{\substack{j=1 \\ j\neq j_0}}^{p} H_{i_0,j}v_j) \qquad (9b)$$
$$\dot{\xi} = v_{j_0}; \quad \xi(0) \in \mathbb{R} \qquad (9c)$$

The composition of (6) and (9) defines the extended system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = A_e \begin{bmatrix} x \\ \xi \end{bmatrix} + B_e v + D_e w; \ \begin{bmatrix} x(0) \\ \xi(0) \end{bmatrix} \in \mathcal{D}_0 \times \mathbb{R} =: \bar{\mathcal{D}}_0 \qquad (10a)$$
$$y = C_e \begin{bmatrix} x \\ \xi \end{bmatrix} + F_e v + Ew \qquad (10b)$$

where $v = [v_1 \cdots v_p]$, $B_{:,j}$ is the jth column of B, $F_{:,j}$ is the jth column of F, $C_{i,:}$ is the ith row of C,

$$A_e = \begin{bmatrix} A & H_{i_0,j_0}^{-1}B_{:,j_0} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}; \quad D_e = \begin{bmatrix} D \\ \mathbf{0}_{1\times q} \end{bmatrix};$$
$$B_e = [B_{e,1} \cdots B_{e,p}]; \ B_{e,j} = \begin{bmatrix} B_{:,j} - \frac{H_{i_0,j}}{H_{i_0,j_0}}B_{:,j_0} \\ 0 \end{bmatrix},$$
$$\forall 1 \leq j \leq p \text{ with } j \neq j_0;$$
$$B_{e,j_0} = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ 1 \end{bmatrix}; \quad C_e = \begin{bmatrix} C & \frac{1}{H_{i_0,j_0}}F_{:,j_0} \end{bmatrix}$$
$$F_e = [F_{e,1} \cdots F_{e,p}]; \ F_{e,j} = F_{:,j} - \frac{H_{i_0,j}}{H_{i_0,j_0}}F_{:,j_0},$$
$$\forall 1 \leq j \leq p$$

It takes simple algebra to verify that $\bar{K} := [K\ \mathbf{0}_{s\times 1}]$, A_z, A_{z1} is a full row rank solution to (7) for the extended system (10). Let $\hat{K} \in \mathbb{R}^{\hat{s}\times(n+1)}$, $\hat{A}_z$, and $\hat{A}_{z1}$ be any maximal solution to (7) for the extended system (10). Then, $\hat{s} \geq s$. Partition $\hat{K} = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix}$, where $\hat{K}_2$ is a column vector. Substitute this structure into (7) for the extended system (10), to obtain

$$\hat{K}A_e = \hat{A}_z\hat{K} + \hat{A}_{z1}C_e; \quad \hat{A}_{z1}F_e = \hat{K}B_e$$

which yields

$$\begin{bmatrix} \hat{K}_1 A & \hat{K}_1 B_{:,j_0}/H_{i_0,j_0} \end{bmatrix}$$
$$= \begin{bmatrix} \hat{A}_z\hat{K}_1 + \hat{A}_{z1}C & \hat{A}_z\hat{K}_2 + \hat{A}_{z1}F_{:,j_0}/H_{i_0,j_0} \end{bmatrix}; \ \mathbf{0}_{\hat{s}\times 1} = \hat{K}_2;$$
$$\hat{A}_{z1}(F_{:,j} - F_{:,j_0}\frac{H_{i_0,j}}{H_{i_0,j_0}})$$
$$= \hat{K}_1(B_{:,j} - B_{:,j_0}\frac{H_{i_0,j}}{H_{i_0,j_0}}); \ 1 \leq j \leq p \text{ with } j \neq j_0$$

By straightforward algebra, we have that $\hat{K}_1$, $\hat{A}_z$, and $\hat{A}_{z1}$ satisfy (7) for the original system (6) and $\hat{K} = \begin{bmatrix} \hat{K}_1 & \mathbf{0}_{\hat{s}\times 1} \end{bmatrix}$, which implies that $\hat{K}_1$ has full row rank. This shows that $\hat{s} \le s$ by the maximality of K, A_z, and A_{z1} solution to (7) for the original system. Hence, we have $s = \hat{s}$ and

$$\hat{K} = [T_z K \, \mathbf{0}_{s\times 1}] \, , \hat{A}_z = T_z A_z T_z^{-1}, \hat{A}_{z1} = T_z A_{z1}$$

where T_z is a real $s \times s$-dimensional invertible matrix. The extended zero dynamics for the extended system (10) is

$$\begin{aligned} \dot{x}_z &= T_z A_z T_z^{-1} x_z + T_z A_{z1} y + T_z (KD - A_{z1}E) w; \\ x_z(0) &= T_z K x_0 \in T_z K(\mathcal{D}_0) \end{aligned}$$

which is identical (modulo linear transformations) to the extended zero dynamics (8) for the original system (6). Hence, the process of dynamic extension does not alter the extended zero dynamics or the minimum phase property of the system.

Remark 1. Based on Theorem 2.1 of Sannuti and Saberi (1987) and Proposition 5.4.1 of Isidori (1995), we can prove that a square MIMO LTI system is right invertible if, and only if, it can be dynamically extended to admit vector relative degree. For such a system, we may determine the extended zero dynamics by Lemma 1 for the dynamically extended system. This extended zero dynamics is itself the one for the original system. The preceding discussion also gives a way to compute the solution K, A_z and A_{z1} to (7) for the original system given the solution to (7) for the dynamically extended system. For nonsquare MIMO LTI systems or square MIMO LTI systems that are not right invertible, the Definition 1 applies to these systems, but we haven't been able to offer a way to calculate its extended zero dynamics, which may in fact not be useful in the first place.

Next, we present the extended zero dynamics canonical form representation for a finite-dimensional continuous-time square MIMO LTI system with uniform vector relative degree in four different cases.

Lemma 2. Consider the square MIMO LTI system (1). Let the system have uniform vector relative degree $r \in \mathbb{N}$, $r < n/m$, from u to y, that is,

$$F = CB = \cdots = CA^{r-2}B = \mathbf{0}_{m\times m}$$

and $CA^{r-1}B =: B_0$ is an invertible matrix. Then, there exists a real invertible matrix T_o such that, in

$$[x_z' \, x_1 \cdots x_r]' = T_o^{-1} x$$

coordinates, the system (1) admits the state space representation

$$\dot{x}_z = A_z x_z + A_{z1} x_1 + D_z w \tag{11a}$$
$$\dot{x}_i = A_{i1} x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{11b}$$
$$\dot{x}_r = A_{rz} x_z + A_{r1} x_1 + B_0 u + D_r w \tag{11c}$$
$$y = x_1 + Ew \tag{11d}$$

where $x_z \in \mathbb{R}^{n-rm}$; $x_i \in \mathbb{R}^m$, $i = 1, \ldots, r$; B_0 is the high-frequency gain matrix of the system. The representation (11) is called the *extended zero dynamics canonical form* of system (1). Note that

$$x_1 = y - Ew$$

from (11d), and therefore (11a) can be written as

$$\dot{x}_z = A_z x_z + A_{z1} y + (D_z - A_{z1}E) w$$

which is the extended zero dynamics of (1) as defined in Definition 1.

Proof We will follow the argument for SISO system in Pan and Başar (2018), and adapt it to our present context. Let

$$V = [B \cdots A^{r-1}B]_{n\times(rm)} \, , \; U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{(rm)\times n}$$

Then, we have

$$UV = \begin{bmatrix} \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star_{m\times m} \\ \mathbf{0}_{m\times m} & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star_{m\times m} & \cdots & \star_{m\times m} \end{bmatrix}_{(rm)\times(rm)}$$

which is clearly invertible. Hence, U and V are of rank rm. Note that $\operatorname{rank}(V) = rm$ implies that

$$\dim(\mathcal{N}(V')) = n - rm$$

Hence, there exists a real nonsingular $(n - rm) \times n$-dimensional matrix K such that

$$V'K' = \mathbf{0}_{(rm)\times(n-rm)}, \; \operatorname{rank}(K) = n - rm.$$

Define

$$\bar{U} = \begin{bmatrix} K \\ U \end{bmatrix}_{n\times n} \quad \text{and} \quad \bar{V} = [K' \, V]_{n\times n} \, .$$

Then, we have

$$\bar{U}\bar{V} = \begin{bmatrix} KK' & \mathbf{0}_{(n-rm)\times(rm)} \\ UK' & UV \end{bmatrix}$$

Since K is nonsingular, KK' is invertible. Hence, $\bar{U}\bar{V}$ is block lower triangular and is invertible. Then, we have $\bar{U}$ and $\bar{V}$ as invertible matrices.

Let $x_z = Kx$ and $z_i = CA^{i-1}x$, $i = 1, \ldots, r$. Consider the coordinate transformation

$$z := [x_z' \, z_1' \cdots z_r']' = \bar{U}x$$

In z coordinates, system (1) admits the state space representation

$$\begin{aligned} \dot{z} &= \bar{U}A\bar{U}^{-1}z + \bar{U}Bu + \bar{U}Dw =: \tilde{A}z + \tilde{B}u + \tilde{D}w \\ y &= C\bar{U}^{-1}z + Ew =: \tilde{C}z + Ew \end{aligned}$$

Note that

$$\begin{aligned} C &= [\mathbf{0}_{m\times(n-rm)} \, I_m \, \mathbf{0}_{m\times(rm-m)}] \, \bar{U} \\ \Rightarrow \quad \tilde{C} &= [\mathbf{0}_{m\times(n-rm)} \, I_m \, \mathbf{0}_{m\times(rm-m)}] \end{aligned}$$

$$\tilde{A} = \bar{U}A\bar{U}^{-1} =: \begin{bmatrix} \tilde{A}_z & \tilde{A}_{z1} & \cdots & \tilde{A}_{zr} \\ \tilde{A}_{1z} & \tilde{A}_{11} & \cdots & \tilde{A}_{1r} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{rz} & \tilde{A}_{r1} & \cdots & \tilde{A}_{rr} \end{bmatrix}$$

$$\tilde{B} = \bar{U}B = \begin{bmatrix} K \\ U \end{bmatrix} B = \begin{bmatrix} \mathbf{0}_{(n-rm)\times m} \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ B_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-m)\times m} \\ B_0 \end{bmatrix}$$

where $\tilde{A}_z$ is $(n - rm) \times (n - rm)$-dimensional; $\tilde{A}_{ij}$, $i, j = 1, \ldots, r$, are $m \times m$-dimensional. Next, we calculate the matrix $\tilde{A}$, using the equality

$$\bar{U}A = \tilde{A}\bar{U}$$

Thus, we have

$$\bar{U}A = \begin{bmatrix} KA \\ CA \\ \vdots \\ CA^r \end{bmatrix} = \begin{bmatrix} \tilde{A}_z K + \sum_{i=1}^r \tilde{A}_{zi} CA^{i-1} \\ \tilde{A}_{1z} K + \sum_{i=1}^r \tilde{A}_{1i} CA^{i-1} \\ \vdots \\ \tilde{A}_{rz} K + \sum_{i=1}^r \tilde{A}_{ri} CA^{i-1} \end{bmatrix} = \tilde{A}\bar{U}$$

Equating

$$CA^j \quad \text{and} \quad \tilde{A}_{jz}K + \sum_{i=1}^r \tilde{A}_{ji} CA^{i-1},\ j = 1, \ldots, r-1,$$

we have

$$\tilde{A}_{jz} = \mathbf{0}_{m\times(n-rm)},\ \tilde{A}_{j\,j+1} = I_m,\ \text{ and }\ \tilde{A}_{ji} = \mathbf{0}_{m\times m},$$

when $j = 1, \ldots, r-1$ and $i = 1, \ldots, r$, and $i \neq j+1$.

Equating

$$KA \quad \text{and} \quad \tilde{A}_z K + \sum_{i=1}^r \tilde{A}_{zi} CA^{i-1},$$

we have the following line of argument. Note that

$$\begin{aligned} KAV &= K\,[AB \cdots A^r B] \\ &= [\mathbf{0}_{(n-rm)\times m} \cdots \mathbf{0}_{(n-rm)\times m}\ KA^r B] = KA^r BB_0^{-1} CV \end{aligned}$$

Therefore, we have

$$(KA - KA^r BB_0^{-1} C)V = \mathbf{0}_{(n-rm)\times(rm)}$$

Then,

$$V'\,(A'K' - C'(B_0')^{-1} B'A'^r K') = \mathbf{0}_{(rm)\times(n-rm)}$$

Denote the row vectors of K by K_i, $i = 1, \ldots, n-rm$. Then, the column vectors of $A'K' - C'(B_0')^{-1}B'A'^r K'$, that is

$$A'K_i' - C'(B_0')^{-1}B'A'^r K_i',\ i = 1, \ldots, n-rm,$$

are in the null space of V', and therefore in the span of K'. Hence, there exists an $(n-rm)\times(n-rm)$-dimensional real matrix $\tilde{\tilde{A}}_z$ such that

$$A'K' - C'(B_0')^{-1}B'A'^r K' = K'\tilde{\tilde{A}}_z$$

which implies that

$$KA = \tilde{\tilde{A}}_z' K + KA^r BB_0^{-1} C.$$

Then, we have

$$\tilde{A}_z = \tilde{\tilde{A}}_z',\ \tilde{A}_{z1} = KA^r BB_0^{-1},$$

and $\tilde{A}_{zj} = \mathbf{0}_{(n-rm)\times m}$, $j = 2, \ldots, r$.

Hence, in z coordinates, system (1) may be represented by

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} z_1 + \tilde{D}_z w \\ \dot{z}_i &= z_{i+1} + \tilde{D}_i w; \qquad i = 1, \ldots, r-1 \\ \dot{z}_r &= \tilde{A}_{rz} x_z + \sum_{i=1}^r \tilde{A}_{ri} z_i + B_0 u + \tilde{D}_r w \\ y &= z_1 + Ew \end{aligned}$$

Let $z_f = [z_1' \cdots z_r']'$. Then, the dynamics for z_f is

$$\begin{aligned} \dot{z}_f &= \left[\begin{array}{c|c} \mathbf{0}_{(rm-m)\times m} & I_{rm-m} \\ \hline \tilde{A}_{r1} & \tilde{A}_{r2} \cdots \tilde{A}_{rr} \end{array}\right] z_f + \begin{bmatrix} \mathbf{0}_{(rm-m)\times m} \\ B_0 \end{bmatrix} u \\ &\quad + \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + \begin{bmatrix} \tilde{D}_1 \\ \vdots \\ \tilde{D}_r \end{bmatrix} w \\ &=: A_f z_f + B_f u + A_{fz} x_z + D_f w \end{aligned} \tag{12a}$$

$$y = [I_m\ \mathbf{0}_{m\times(rm-m)}]\, z_f + Ew =: C_f z_f + Ew \tag{12b}$$

It is clear that the dynamics (12) with inputs u, x_z, w, and output y is observable with uniform observability indices r and admits uniform vector relative degree r with respect to the input u. By Corollary 1 in the Appendix A, there exists a real invertible coordinate transformation $x_f = T_f^{-1} z_f$ that transforms (12) into the observer canonical form.

$$\begin{aligned} \dot{x}_f &= \left[\begin{array}{c|c} \hat{A}_{11} & \\ \vdots & I_{rm-m} \\ \hat{A}_{r-1,1} & \\ \hline \hat{A}_{r1} & \mathbf{0}_{m\times(rm-m)} \end{array}\right] x_f + \begin{bmatrix} \mathbf{0}_{(rm-m)\times m} \\ B_0 \end{bmatrix} u \\ &\quad + T_f^{-1} \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} x_z + T_f^{-1} D_f w \\ y &= [I_m\ \mathbf{0}_{m\times(rm-m)}]\, x_f + Ew \end{aligned}$$

Note that

$$T_f^{-1}\,[\mathbf{0}_{m\times(rm-m)}\ B_0']' = [\mathbf{0}_{m\times(rm-m)}\ B_0']'$$

implies that

$$T_f^{-1} \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(rm-m)\times(n-rm)} \\ \tilde{A}_{rz} \end{bmatrix}$$

Partition x_f as $[x_1 \cdots x_r]'$, with x_i being m-dimensional, $i = 1, \ldots, r$. Clearly,

$$y = z_1 + Ew = x_1 + Ew,$$

and, we have $z_1 = x_1$. Then, the system (1) admits the following state space representation, in $x := [x_z'\ x_1 \cdots x_r]'$ coordinates,

$$\begin{aligned} \dot{x}_z &= \tilde{A}_z x_z + \tilde{A}_{z1} x_1 + \tilde{D}_z w \\ \dot{x}_i &= \hat{A}_{i1} x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \\ \dot{x}_r &= \tilde{A}_{rz} x_z + \hat{A}_{r1} x_1 + B_0 u + D_r w \\ y &= x_1 + Ew \end{aligned}$$

where $[D_1' \cdots D_r']' = T_f^{-1} D_f$. Clearly, the above is in the form of (11). Hence, the desired matrix

$$T_o = \bar{U}^{-1} \begin{bmatrix} I_{n-rm} & \mathbf{0} \\ \mathbf{0} & T_f \end{bmatrix}$$

This completes the proof of the lemma. □

Remark 2. We observe in the previous lemma that the zero dynamics of the system (1) according to Isidori (1995) is exactly $\dot{x}_z = A_z x_z$. The extended zero dynamics is simply the zero dynamics together with driving terms which include the output of the system and the disturbance input.

Lemma 3. Consider the square MIMO LTI system (1). Let the system have relative degree $r = n/m \in \mathbb{N}$, from u to y, that is,

$$F = CB = \cdots = CA^{r-2}B = \mathbf{0}_{m\times m}$$

and $CA^{r-1}B =: B_0$ is invertible. Then, there exists an invertible matrix T_o such that, in

$$[x_1 \cdots x_r]' = T_o^{-1} x$$

coordinates, the system (1) admits the state space representation

$$\dot{x}_i = A_{i1} x_1 + x_{i+1} + D_i w; \qquad i = 1, \ldots, r-1 \tag{13a}$$

$$\dot{x}_r = A_{r1} x_1 + B_0 u + D_r w \tag{13b}$$

$$y = x_1 + Ew \tag{13c}$$

where $x_i \in \mathbb{R}^m$, $i = 1, \ldots, r$; B_0 is the high-frequency gain matrix of the system. The representation (13) is called the *extended zero dynamics canonical form* of system (1) (which is also the observer canonical form). The *extended zero dynamics* for the system is clearly absent.

Proof Define

$$V = [B \cdots A^{r-1}B]_{n\times n}\,,\ U = \begin{bmatrix} C \\ \vdots \\ CA^{r-1} \end{bmatrix}_{n\times n}$$

Then, we have

$$UV = \begin{bmatrix} \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & CA^{r-1}B \\ \vdots & \iddots & \iddots & \star_{m\times m} \\ \mathbf{0}_{m\times m} & \iddots & \iddots & \vdots \\ CA^{r-1}B & \star_{m\times m} & \cdots & \star_{m\times m} \end{bmatrix}_{n\times n}$$

which is clearly invertible. Hence, U and V are invertible. Then, the system (1) is observable, and is controllable from u. Clearly, the observability indices of (1) all equal to r. By Corollary 1, there exists a real invertible transformation

$$T_o^{-1}x =: [x_1' \cdots x_r']' ,$$

where x_i is m-dimensional, $i = 1, \ldots, r$, that transforms the system into observer canonical form.

$$\begin{aligned} \dot{x}_i &= A_{i1}x_1 + x_{i+1} + B_i u + D_i w; \qquad i = 1, \ldots, r-1 \\ \dot{x}_r &= A_{r1}x_1 + B_r u + D_r w \\ y &= x_1 + Ew \end{aligned}$$

Because the system (1) admits uniform vector relative degree r from u to y, we have

$$B_1 = \cdots = B_{r-1} = \mathbf{0}_{m\times m}$$

It is straightforward to obtain that

$$B_r = CA^{r-1}B = B_0.$$

This completes the proof of the lemma. □

Lemma 4. Consider a square MIMO LTI system (1) with $n > 0$ and $F = B_0$ invertible, i. e., that the system admits uniform vector relative degree 0 from u to y, and B_0 is the high-frequency gain matrix of the system. Then, the system (1) admits the following representation:

$$\begin{aligned} \dot{x} &= (A - BB_0^{-1}C)x + BB_0^{-1}(y - Ew) + Dw \\ &=: \hat{A}x + \hat{B}(y - Ew) + Dw \qquad (14a) \\ y &= Cx + B_0 u + Ew \qquad (14b) \end{aligned}$$

The representation (14) is called the *extended zero dynamics canonical form* of (1). The dynamics (14a) is called the *extended zero dynamics* of (1).

Proof Note that

$$u = B_0^{-1}(y - Cx - Ew).$$

Substitution of this expression into (1a) readily leads to (14a). Hence, (14) is a representation of (1).

By Lemma 1, we identify that (14a) is the extended zero dynamics for (1) according to Definition 1.

This completes the proof of the lemma. □

Definition 2. Consider a MIMO LTI system

$$y = B_0 u + Ew \tag{15}$$

where $u \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; and B_0 and E are constant matrices of appropriate dimensions. Let B_0 be of full row rank, which implies that the system admits uniform vector relative degree 0 from u to y, and B_0 is the high-frequency gain matrix of the system. Then, (15) is called the *extended zero dynamics canonical form.* Clearly, the *extended zero dynamics* is absent in this case.

Lemma 5. Consider the system (6). Assume that it admits the extended zero dynamics (8). Then, (6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ implies that the system

$$\dot{z} = A_z z + A_{z1} v; \qquad z(0) = \mathbf{0}_{s\times 1} \tag{16}$$

is bounded input and bounded state stable.

Proof This is a direct consequence of Lemma 9 of Pan and Başar (2018). □

Finally, we present a result that links the generalized minimum phase property to the asymptotic stability property of the extended zero dynamics.

Lemma 6. Consider the system (6). Let the system admit the extended zero dynamics (8). Then, the system is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ if the matrix A_z is Hurwitz. On the other hand, if the system is stabilizable from u and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, then the matrix A_z is Hurwitz.

Proof If the matrix A_z is Hurwitz, the system (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ according to Definition 1.

On the other hand, if the system (6) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and is stabilizable from u, by the stabilizability of the pair (A, B), we have the pair (A_z, A_{z1}) also stabilizable. This is because the uncontrollable part of x_z dynamics from y must be independent of y, u, and the rest of the system states, and hence a part of the uncontrollable part of (6) from u, and therefore Hurwitz. By Lemma 5, the following system

$$\dot{x}_u = A_z x_u + A_{z1} v; \qquad x_u(0) = \mathbf{0}_{s\times 1}$$

is bounded input and bounded state stable. Since the triple (A_z, A_{z1}, I_s) is stabilizable and detectable, by Corollary 1 of Pan and Başar (2018), A_z is Hurwitz.

This completes the proof of the lemma. □

Remark 3. We conclude, based on the previous lemma, that if an MIMO LTI system is minimum phase according to Isidori (1995), then it is also minimum phase according to the generalized definition; on the other hand, if it is minimum phase according to the generalized definition, and it is stabilizable from u, then it is minimum phase according to Isidori (1995).

4. CONCLUSIONS

In this paper, we have generalized the definition of extended zero dynamics (Pan and Başar, 2018) to MIMO LTI systems, which can be computed for systems that are square and right invertible. A system is said to be minimum phase with respect to admissible initial conditions and admissible disturbance waveforms if the extended zero dynamics is absent or admits bounded state trajectory with admissible initial states, arbitrary bounded continuous output waveforms, and admissible bounded disturbance waveforms. We have proved that the extended zero dynamics of an MIMO LTI system is invariant (modulo linear state transformation) under a step of dynamic extension (Isidori, 1995) for one of its inputs. We have also proved that an MIMO LTI system is minimum phase if its zero dynamics is asymptotically stable; the converse holds if the system is further stabilizable from the control input. We presented the extended zero dynamics canonical form for a square MIMO LTI system with uniform vector relative degree, which is different in each of four different cases that mimic the SISO case of Pan and Başar (2018).

In a subsequent paper, Başar and Pan (2019), we prove that this generalized definition of minimum phase is necessary in model reference control. We also connect the dots: starting with a square MIMO LTI system that is minimum phase with respect to the admissible initial condition and admissible disturbance waveform, we must obtain a true system representation that admits both the extended zero dynamics canonical form representation and the strict observer canonical form representation in order to be able to apply the (appropriately vectorized version of) robust adaptive control design of Pan and Başar (2000) to the system. For the system to admit the extended zero dynamics canonical form, it must admit uniform vector relative degree. This can be achieved by dynamic extensions that are independent of the unknown parameters in the system. This assumption on the system may, however, be too restrictive. Actually, we just need the system to admit vector relative degree (not necessarily uniform) after a finite number of dynamic extensions that are independent of the unknown parameters in the system. After that, we can further achieve the requirement of uniform vector relative degree by appropriately integrating the output channels of the system, and thus leading to an extended system that admits the extended zero dynamics canonical form, as shown in Başar and Pan (2019). For the system to admit the strict observer canonical form, it must admit uniform observability indices. We therefore have to further extend the system without changing its relative degree and minimum phase property. This can be done by adding dummy state variables into the system. These results are summarized in two lemmas in Başar and Pan (2019) that fully resolve the model reference robust adaptive control problem for minimum phase finite-dimensional continuous-time square MIMO LTI systems.

Further study of the minimum phase property of composite systems that consist of interconnected LTI systems under suitable assumptions on the component system is a fruitful avenue for future research. For SISO composite systems, comprehensive results have already been obtained in Pan and Başar (2019b,a). The MIMO version of these results are currently under study.

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Appendix A. A USEFUL RESULT

Corollary 1. Consider the MIMO LTI system (6). Let $\nu \in \{0,\ldots,n\}$ be the observability index of the system and $\nu_1,\ldots,\nu_m \in \mathbb{Z}_+$ be the observability indices for the output channels. Assume that $\nu = \nu_1 = \cdots = \nu_m$ and the system admits vector relative degree $r_1,\ldots,r_m \in \{0,\ldots,n\}$ with respect to the control input u such that $r = r_1 = \cdots = r_m$. Then, $r \leq \nu$, there exists an invertible matrix T such that in

$$\bar{x} = [x_{\bar{o}}'\, x_1' \cdots x_\nu']' = T^{-1}x$$

coordinates, we have $n_O := m\nu$, $x_{\bar{o}}$ is $(n-n_O)$-dimensional, x_i is m-dimensional, $i = 1,\ldots,\nu$, and the system (6) admits the *strict observer canonical form* representation, if $\nu > 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}}x_{\bar{o}} + A_{\bar{o},1}x_1 + B_{\bar{o}}u + D_{\bar{o}}w \tag{A.1a}$$
$$\dot{x}_i = A_{i,1}x_1 + x_{i+1} + B_iu + D_iw;\ 1 \leq i < \nu \tag{A.1b}$$
$$\dot{x}_\nu = A_{\nu,1}x_1 + B_\nu u + D_\nu w \tag{A.1c}$$
$$y = x_1 + Fu + Ew \tag{A.1d}$$

or if $\nu = 0$,

$$\dot{x}_{\bar{o}} = \hat{A}_{\bar{o}}x_{\bar{o}} + B_{\bar{o}}u + D_{\bar{o}}w \tag{A.2a}$$
$$y = Fu + Ew \tag{A.2b}$$

where all matrices are constant and of appropriate dimensions, $B_0 := F$, and $B_i = \mathbf{0}_{m\times p}$, $\forall i = 0,\ldots,r-1$, and B_r is of rank m.

Proof This is Corollary 3 of Başar and Pan (2019). □