Two-degree-of-freedom Design Between Intersample and Sample Responses: Closed-loop State-space Approach Based on Steady-state Characteristics

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Abstract: A dual-rate sampled-data control system is designed in which the sampling interval of measured signals is twice as long as the holding interval of a control input. In the proposed method, a control law is extended based on the null space and intersample ripples are eliminated without changing the sampled output trajectory in the steady state. Polynomial and state-space approaches are conventional open-loop design methods based on the null space. However, conventional methods are ineffective for unstable systems and are unavailable when the null space of the plant model is only the zero vector because the controlled plant is not arbitrarily selectable. On the other hand, the proposed method is effective for unstable systems, as well as stable systems, even when the null space of the plant model is only the zero vector, because the proposed controller is designed based on a closed-loop system.

Keywords: null space, steady state, intersample, sampled data, multirate, continuous time, discrete time, digital computer

1. INTRODUCTION

In the sampled-data control system (Araki and Yamamoto, 1986; Araki and Hagiwara, 1986; Albertos, 1990; Apostolakis, 1992; Chen and Francis, 1995), the continuous-time system is controlled using the discrete-time controller, and hence the continuous-time signal and the discrete-time signal are sampled and held, respectively. Therefore, there are two intervals, which are the sampling and holding intervals, and these intervals are influenced by the hardware and/or the environment of the control system. When these two intervals are equivalent, the system is referred to as a single-rate system. On the other hand, when the intervals are different, the system is referred to as a multi-rate or dual-rate system. The present study proposes a new design method for controlling a dual-rate system, in which the sampling interval of the measured signals is restricted to twice as long as the holding interval of a control input, as illustrated in Fig. 1.

A dual-rate system can be converted to a single-rate system by setting the holding interval to be the same as the restricted sampling interval. Generally, the control performance of a dual-rate system is superior to that of a converted single-rate system. However, a dual-rate system may have ripples between the sampling instants, even if the discrete-time output at the sampling instant converges to the reference input without steady-state error. This is because the control input can be changed between the sampling instants (Tangirala et al., 1999). The condition for eliminating such intersample ripples is that the steady-state gains from the reference input to the control inputs be equivalent (Tangirala et al., 2001; Jia et al., 2002; Jia, 2005).

When the control system is re-designed such that the ripple-free condition is satisfied, the steady-state intersample ripples are eliminated, but the sampled output trajectory is simultaneously changed and might be deteriorated. Therefore, the intersample ripple problem should be resolved independent of the sampled response. For the achievement of such an independent design, an extension for a dual-rate system has been proposed by Sato (2008). In this design method, a control law is re-designed using the controlled plant dynamics, and the intersample ripples are eliminated independent of both the transient and

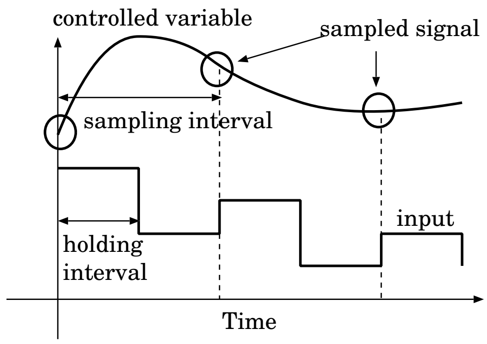

Fig. 1. Dual-rate sampled-data control system

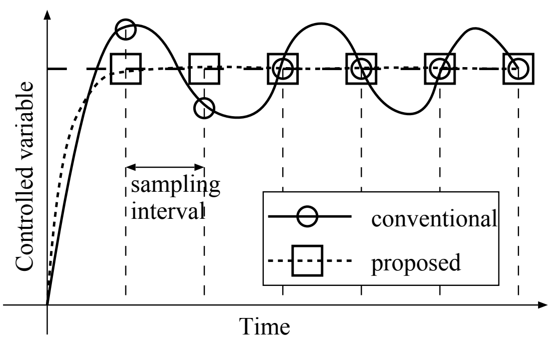

Fig. 2. Improvement in the steady-state response

steady-state responses in discrete time. On the other hand, since the design purpose is improved steady-state performance, the important discrete-time property is not the transient response, but rather the steady-state response (Fig. 2). Therefore, design methods are based on the null space of an over-actuated system (Galeani et al., 2011; Cocetti et al., 2018) in order to eliminate the steady-state intersample ripples without changing the sampled response (Sato et al., 2017; Yasui et al., 2019a,b). However, conventional methods are useful only when the controlled plant is stable because the design methods are based on an open-loop system. On the other hand, in the proposed method, a control law is extended using the null space of the closed-loop system. Since a closed-loop system can be stabilized even when a controlled plant is unstable, the proposed method is effective for unstable plants.

In the present paper, $\boldsymbol{R}$ denotes the set of real numbers, and I is an identify matrix.

2. PROBLEM STATEMENT

As a controlled plant, consider the following single-input/single-output time-invariant linear continuous-time system:

$$\dot{\boldsymbol{x}}(t) = A_c\boldsymbol{x}(t) + \boldsymbol{b}_c u(t) \tag{1}$$

$$y(t) = \boldsymbol{c}_c^T\boldsymbol{x}(t) \tag{2}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$, $u(t) \in \boldsymbol{R}$, and $y(t) \in \boldsymbol{R}$ are the state vector, the control input, and the plant output, respectively, in continuous time, and $A_c \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{b}_c \in \boldsymbol{R}^n$, and $\boldsymbol{c}_c \in \boldsymbol{R}^n$. Since the continuous-time system is controlled using a discrete-time digital computer, the continuous-time measured signals are sampled and the discrete-time control input calculated in the digital computer is held to be converted to the continuous-time signal.

In the present study, the discrete-time control input is converged to a continuous-time signal using the zero-order holder, and the discrete-time model is described as follows:

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + \boldsymbol{b}_d u(k) \tag{3}$$

$$y(k) = \boldsymbol{c}_d^T\boldsymbol{x}(k) \tag{4}$$

$$A_d = e^{A_c T_s}$$

$$\boldsymbol{b}_d = \int_0^{T_s} e^{A_c\sigma} d\sigma \boldsymbol{b}_c$$

$$\boldsymbol{c}_d = \boldsymbol{c}_c$$

where $\boldsymbol{x}(k) \in \boldsymbol{R}^n$, $u(k) \in \boldsymbol{R}$, and $y(k) \in \boldsymbol{R}$ are the state vector, the input vector, and the plant output, respectively, in discrete time, and T_s denotes the holding interval. This discrete-time model is referred to as a single-rate system because the sampling interval is equivalent to the holding interval. Generally, the sampling interval is not always equivalent to the holding interval, because of hardware specifications, for example. The present study discusses the design method based on the following assumption.

Assumption 1.

- The sampling interval is twice as long as the holding interval

In this case, the controlled plant model is expressed as follows:

$$\boldsymbol{x}(k+2) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \tag{5}$$

$$y(k) = \boldsymbol{c}^T\boldsymbol{x}(k) \tag{6}$$

where

$$A = A_d^2$$

$$B = [A_d\boldsymbol{b}_d \ \ \boldsymbol{b}_d]$$

$$\boldsymbol{c} = \boldsymbol{c}_d$$

$$\boldsymbol{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$u_i(k) = u(k+i-1)$$

Assumption 2.

- Continuous-time model Eq. (1), Eq. (2) is unknown
- Single-rate model Eq. (3), Eq. (4) is unknown
- Dual-rate model Eq. (5), Eq. (6) is unknown
- State vector $\boldsymbol{x}(k)$ is measured.

Based on the above assumption, the proposed control system is designed using the following state feedback control law:

$$\boldsymbol{u}(k) = Kr(k) - F\boldsymbol{x}(k) \tag{7}$$

where $r(k) \in \boldsymbol{R}$ is the step-type reference input to be followed by the plant output. Here, $K \in \boldsymbol{R}^2$ and $F \in \boldsymbol{R}^{2\times n}$ are the feed-forward gain and the feedback gain, respectively. The block diagram of the designed control system is illustrated in Fig. 3. The closed-loop system using the control law is written as follows:

$$\boldsymbol{x}(k+2) = (A - BF)\boldsymbol{x}(k) + BKr(k) \tag{8}$$

where F and K are designed so as to satisfy the following assumption.

Assumption 3.

- The closed-loop system is stable.
- The sampled plant output follows the reference input without steady-state error.

Furthermore, the closed-loop system in the steady state is decided as follows:

$$\boldsymbol{x}(\infty) = \boldsymbol{G}(1)Kr(\infty) \tag{9}$$

$$\boldsymbol{G}(1) = (I - A + BF)^{-1}B \tag{10}$$

Therefore, the plant output in the steady state is given as follows:

$$y(\infty) = \boldsymbol{G}_c(1)Kr(\infty) \tag{11}$$

$$\boldsymbol{G}_c(1) = \boldsymbol{c}^T\boldsymbol{G}(1) \tag{12}$$

However, the intersample output may oscillate because the control input can be updated between the sampling instants. Here, the steady-state control input is calculated as follows:

$$\boldsymbol{u}(\infty) = (I - F\boldsymbol{G}(1))Kr(\infty) \tag{13}$$

The steady-state input deviation can be suppressed when K and F are appropriately re-designed. However, in this case, the designed closed-loop system is also changed. Hence, the deviation of the control input must be resolved without changing the design parameters.

3. MAIN RESULT

For the steady-state gain matrix of the closed-loop system from the control input to the state vector, $\boldsymbol{G}(1)$ and the steady-state gain vector of the closed-loop system from the control input to the plant output, $\boldsymbol{G}_c(1)$, the following assumption is satisfied.

Assumption 4.

- $\boldsymbol{G}(1)$ is known
- $\boldsymbol{c}$ or $\boldsymbol{G}_c(1)$ is known
- $\mathrm{rank}(\boldsymbol{G}_c(1)) = 1$

Definition 1. $G_{c\perp}$ satisfies the following conditions:

- $\mathrm{Im}(G_{c\perp}) = \mathrm{Ker}(\boldsymbol{G}_c)$
- $\mathrm{Ker}(\boldsymbol{G}_c(1)) := \{G_{c\perp} \in \boldsymbol{R}^2 \mid \boldsymbol{G}_c(1)G_{c\perp} = 0 \}$

Using the defined $G_{c\perp}$, the control law is extended as follows:

$$\boldsymbol{u}(k) = Kr(k) - F\boldsymbol{x}(k) + G_{c\perp}w(k) \tag{14}$$

where $w(k) \in \boldsymbol{R}$ is a newly introduced signal. A block diagram of the extended control system is illustrated in Fig. 4.

The steady-state closed-loop plant output using the extended control law is given as follows:

$$y(\infty) = \boldsymbol{G}_c(1)Kr(\infty) + \boldsymbol{G}_c(1)G_{c\perp}w(\infty) \tag{15}$$

The closed-loop systems using the original and extended control laws are different. However, in the steady state, the sampled output is not affected by the second term on the right-hand side of Eq. (15), because of Definition 1. Therefore, the additional signal $w(k)$ is designed independent of the steady-state sampled output obtained using Eq. (7).

The steady-state control input using the extended control law is calculated as follows:

$$\begin{aligned}\boldsymbol{u}(\infty) =& (I - F(I - A + BF)^{-1}B)Kr(\infty) \\ &- F(I - A + BF)^{-1}BG_{c\perp}w(\infty) \\ =& (I - F\boldsymbol{G}(1))Kr(\infty) - F\boldsymbol{G}(1)G_{c\perp}w(\infty)\end{aligned} \tag{16}$$

Since F is designed so that the closed-loop system is stabilized in designing the original control law, Eq. (16) is calculated. In the proposed method, $w(k)$ is designed so that the control input between the sampling instants is constant in the steady state. Here, Eq. (16) is rewritten as follows:

$$\begin{aligned}\boldsymbol{u}(\infty) &= \boldsymbol{G}_r(1)r(\infty) + \boldsymbol{G}_w(1)w(\infty) \\ \boldsymbol{G}_r(1) &= (I - F\boldsymbol{G}(1))K \\ \boldsymbol{G}_w(1) &= (I - F\boldsymbol{G}(1))G_{c\perp}\end{aligned} \tag{17}$$

Therefore, $w(k)$ is designed so that the following condition is satisfied:

$$w(\infty) = -\frac{G_{r1}(1) - G_{r2}(1)}{G_{w1}(1) - G_{w2}(1)}r(\infty) \tag{18}$$

where

$$\boldsymbol{G}_r(1) = \begin{bmatrix} G_{r1}(1) \\ G_{r2}(1) \end{bmatrix}$$
$$\boldsymbol{G}_w(1) = \begin{bmatrix} G_{w1}(1) \\ G_{w2}(1) \end{bmatrix}$$

Even if the plant model is unknown, the additional input is decided so that the intersample ripples are eliminated in the steady state because $\boldsymbol{G}(1)$ is known and F and K are design parameters. As a result, the steady-state intersample ripples caused by the control input deviation are eliminated without changing the existing steady-state sampled output trajectory.

4. SIMULATION

Consider the following continuous-time system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t) \tag{19}$$

$$y(t) = [\,1 \;\; 1\,]\,\boldsymbol{x}(t) \tag{20}$$

The sampling interval and the holding interval are set to 2 s and 1 s, respectively. Therefore, the corresponding dual-rate system is derived as follows:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0.34 & 0.81 \\ 0.81 & 2.0 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 0.30 & 0.50 \\ 0.65 & 0.30 \end{bmatrix}\boldsymbol{u}(t) \tag{21}$$

$$y(t) = [\,1 \;\; 1\,]\,\boldsymbol{x}(t) \tag{22}$$

where, in the present study, Eq. (19)–Eq. (22) are unknown.

The stability of a closed-loop system and the convergence of the plant output to the reference input are achieved using the following controller parameters:

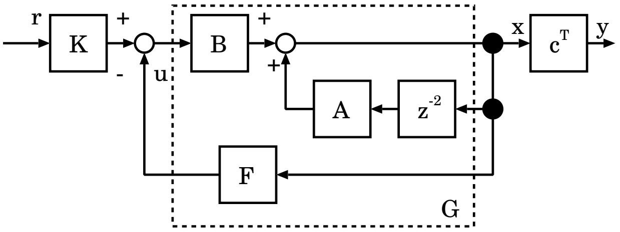

Fig. 3. Block diagram of the original control system

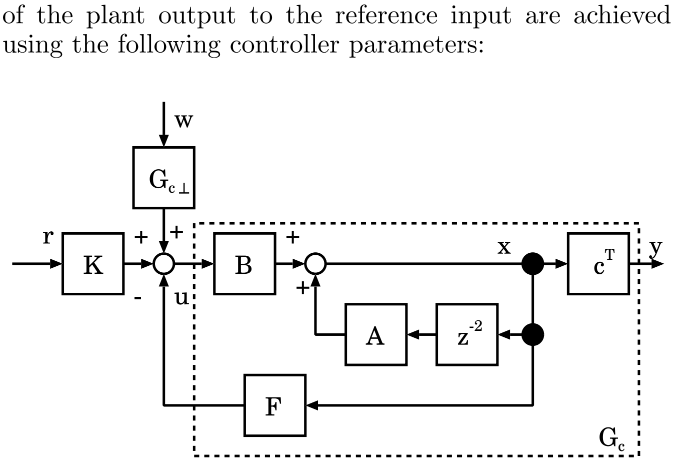

Fig. 4. Block diagram of the extended control system

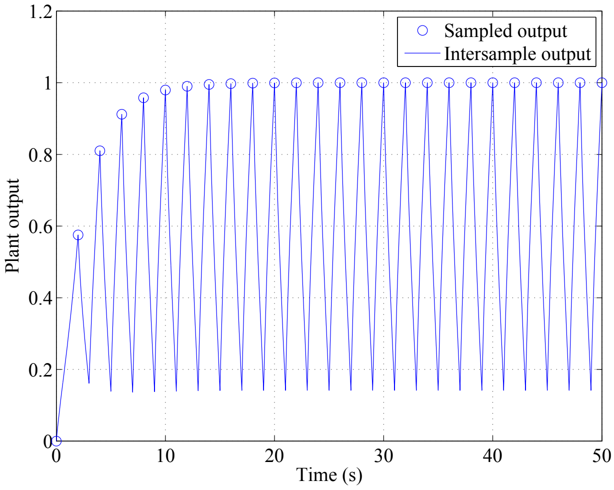

Fig. 5. Output response using the original control law

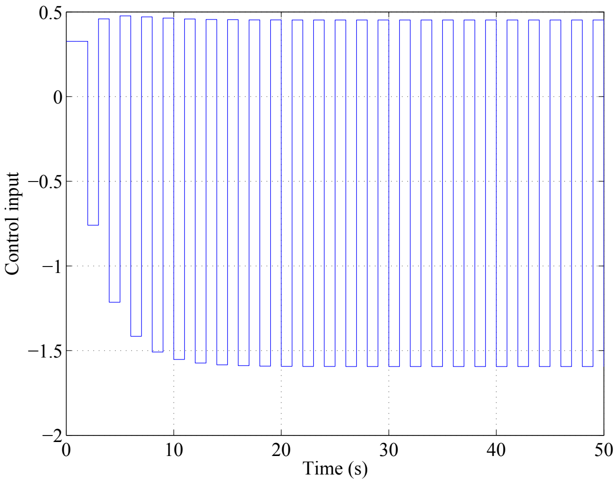

Fig. 6. Input response using the original control law

$$K = \begin{bmatrix} 0.33 \\ 0.33 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.7 & 2.1 \\ -0.92 & 0.36 \end{bmatrix}$$

Using the design parameters, the closed-loop poles are 0.3 and 0.5, and the steady-state closed-loop gain from the reference input to the plant output is 1.0.

The output and input results using Eq. (7) with the designed controller parameters are shown in Fig. 5 and Fig. 6, respectively. In the figures, the output and input trajectories are plotted as blue solid lines, and the sampled output trajectory is plotted as blue circles. Although the sampled output converges to the reference input without steady-state error, the intersample output oscillates between the sampling instants in the steady state. This is because the steady-state gain vector from the reference input to the control input is $[-1.6 \quad 0.45]^T$, and therefore the control input oscillates.

Among conventional methods for resolving the intersample ripples (Sato et al., 2017; Yasui et al., 2019a), the polynomial approach (Sato et al., 2017) is not used, because the controlled plant is unstable, and the state-space approach

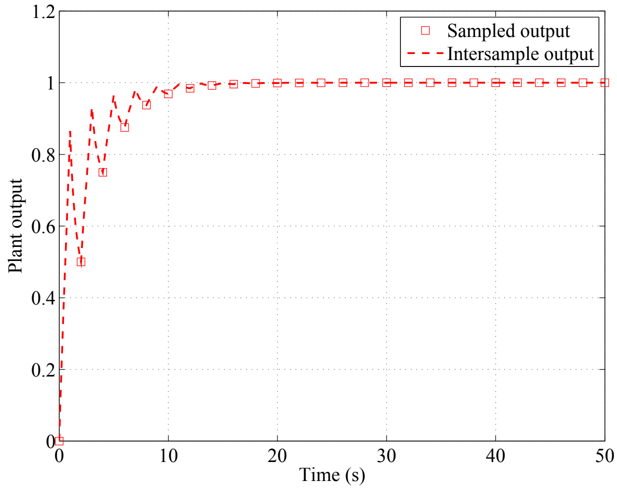

Fig. 7. Output response using the proposed control law

(Yasui et al., 2019a) is also not available, because rank(B = 2) and there is no non-zero null-space element.

On the other hand, the proposed method is available because the rank of $\boldsymbol{G}_c(1)$ is 1. The steady-state gain vector $\boldsymbol{G}_c(1)$ is given as follows:

$$\boldsymbol{G}_c(1) = [\, 1.7 \;\; 1.3 \,]$$

Therefore, $G_{c\perp}$ is designed as follows:

$$G_{c\perp} = \begin{bmatrix} 1.3 \\ -1.7 \end{bmatrix}$$

Furthermore, based on Assumption 4, $\boldsymbol{G}(1)$ is also given as follows:

$$\boldsymbol{G}(1) = \begin{bmatrix} 0.43 & 0.72 \\ 1.3 & 0.61 \end{bmatrix}$$

Using Eq. (18), the steady-state value of the additional input is decided as follows:

$$w(\infty) = 0.56$$

The simulated output and input results using the proposed method are plotted as red dashed lines in Fig. 7 and Fig. 8, and the sampled output trajectory is plotted as red squares. For enhanced visibility, the output and input results using the conventional and proposed control laws are plotted in Fig. 9 and Fig. 10, respectively. Fig. 9 shows that the steady-state sampled output trajectories using the original and proposed control laws are the same even when the additional input is introduced in the proposed method. Furthermore, the intersample ripples are eliminated in the steady state using the proposed method.

5. CONCLUSION

The present study has proposed a new design method for controlling the dual-rate sampled-data system, where the sampling interval of measured signals is twice as long as the holding interval of a control input. Conventional null-space-based methods are unavailable when a controlled plant is unstable or the null space of the controlled plant has only the zero vector. On the other hand, the proposed method is available because of the closed-loop system-based design. Using the proposed method, the intersample ripples are eliminated such that the sampled output tra-

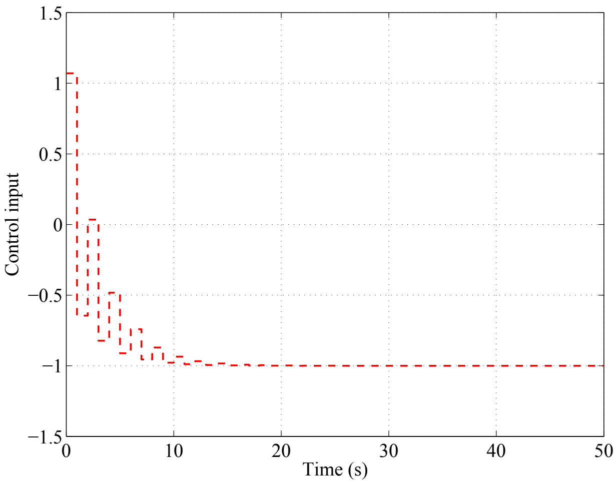

Fig. 8. Input response using the proposed control law

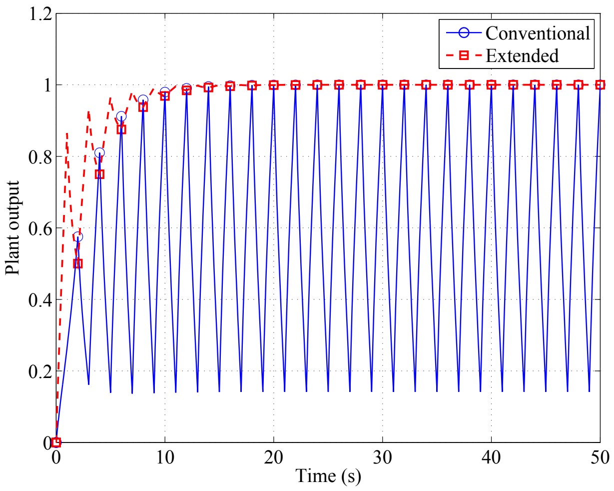

Fig. 9. Output responses using the original control law and the proposed control law

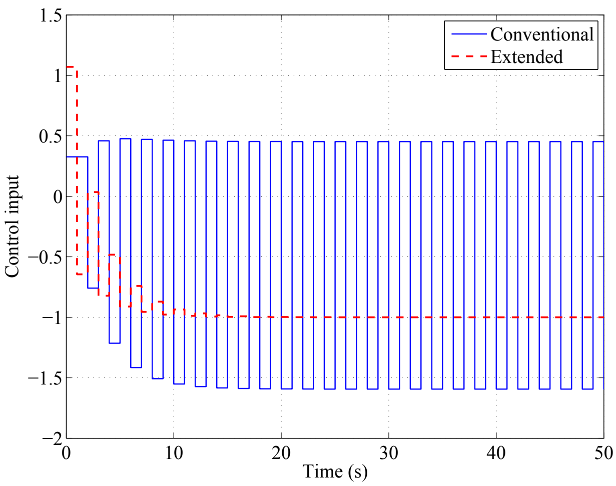

Fig. 10. Input responses using the original control law and the proposed control law

jectory is maintained in the steady state. Finally, the effectiveness of the proposed method is demonstrated through numerical examples.

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