Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Model Matching Problems for Positive Systems

Giuseppe Conte * **Anna Maria Perdon** * **Elena Zattoni** **

* *Department of Information Engineering, Polytechnic University of Marche, Ancona, Italy (e-mail: perdon@univpm.it, gconte@univpm.it)*
** *Department of Electrical, Electronic, and Information Engineering "G. Marconi", Alma Mater Studiorum · University of Bologna, Bologna, Italy (e-mail: elena.zattoni@unibo.it).*

Abstract: The problem of compensating a given plant by means of a static compensator in such a way that, for any input, the output of the compensated system matches that of a given positive model, when both are initialized at 0, and its state evolves in the positive cone for positive initial conditions and inputs is considered. Under a mild structural assumption for the output-difference system between the plant and the model, a complete characterization of solvability of the problem in terms of necessary and sufficient conditions is obtained by means of structural geometric methods. Solvability conditions are practically checkable by algorithmic procedures and by solving a set of linear inequalities. The problem of asymptotic matching for any initial condition is then considered and solvability is characterized by necessary and sufficient conditions. A necessary condition that is practically checkable is given. Solvability by a dynamic compensator is also studied and a sufficient condition to characterize it is given.

Keywords: Linear positive systems; model matching problem; structural methods; geometric methods.

1. INTRODUCTION

Positive systems are dynamical systems whose state and output variables are non negative at all times if the initial conditions and the inputs are non negative. Phenomena that can be naturally described by means of a positive system appear in various fields, since many physical quantities such as population levels, buffer sizes, charge levels, light intensity levels, prices, etc. are naturally non negative. A mathematical model that correctly describes their time evolution needs to satisfy the positivity constraint. In view of these applications, control and design problems for positive systems have attracted the interest of many researchers.

In the 1980s, Luenberger presented in Luenberger (1979) the first systematic treatment of positive systems. Then, many theoretical results were obtained, making use of classical mathematical results on positive matrices (see, for instance, Berman and Plemmons (1979), Seneta (1981)) and, in particular, Perron (1907), Frobenius (1912), Ito (1997), and Karpelevich (1988) on eigenvalue localization. Stability issues, asymptotic behavior and equilibrium points where investigated (see, for instance, De Leenheer and Aeyels (2001) and De Leenheer and Aeyels (2002)). The problem concerning the existence and the design of controllers that make the closed-loop system asymptotically stable and positive where considered in Heemels et al. (1998), Kaczorek (2002), van den Hof (1998) and several sufficient conditions, based on Gershgorin's theorem, were proposed. In Gao et al. (2005), sufficient constructive conditions for the existence of a stabilizing compensator ensuring the positivity of the compensated system were given in terms of LMI's.

However, several classical results, for instance, on reachability, observability and realization that hold for general linear systems cannot be directly extended to positive systems due to the fact that such systems evolve on cones, rather than on linear spaces. Instead, new mathematical tools, based on graph theory and cone theory, need to be developed and employed. For more details and a complete list of references, see the fundamental books Farina and Rinaldi (2000), Kaczorek (2002) and the survey paper Benvenuti and Farina (2004).

Since compensating a given plant in such a way to match a given model is a powerful control strategy that may help to solve complex problems (Wolovich (1972), Moore and Silverman (1972), Morse (1973), Bao et al. (2012), Seyboth and Allgöwer (2014)), the model matching problem is, among classical control problems, one that deserves to be studied in the framework of positive systems. The problem has been investigated by a number of authors for several classes of dynamical systems (e.g. linear systems Malabre (1982), systems over rings and time-delay systems Conte and Perdon (1995), nonlinear systems Moog et al. (1991), descriptor systems Kučera (1992), 2D systems Picard et al. (1998), switching systems Conte et al. (2014), periodic systems Colaneri and Kucera (1997), hybrid systems Conte et al. (2018a), LPV systems Conte et al. (2018b)) using various methods. In particular, the structural geometric approach has been shown to be capable of providing solvability conditions and viable procedure for constructing solutions.

With the above motivation, in this paper we investigate the problem of compensating a linear plant so that the state of the compensated system evolves in the positive cone for all positive initial conditions and inputs (positivity condition) and its output matches that of a given positive model (matching condition). More precisely, this means that, for all inputs, the output of the compensated plant equals that of the model if both are initialized at 0 (exact model matching). In a more demanding formulation of the problem, in addition to this condition, the output of the compensated plant is required to converge asymptotically to that of the model for any initialization of both (asymptotic model matching). This objective is achieved if the compensator also stabilizes the output-difference system between the plant and the model.

The paper is organized as follows. In order to tackle the problems, we formulate it in Section 2 as a disturbance decoupling problem with the additional constraint of positivity, as explained above, and we consider solvability both by a static state feedback and by a dynamic compensator. The constraint of positivity makes the problem more difficult than the classical disturbance decoupling problem for linear systems and its solution requires the introduction of specific tools and concepts. Therefore, in Section 3, we define a geometric notion of positive and strongly positive controlled invariance and we use it for stating necessary and suffcicient solvability conditions for the exact matching problem in the framework of positive systems with static state feedback in Theorem 12. This result is obtained under a structural hypothesis that corresponds to left invertibility of the output-difference system between the plant and the model. An algorithmic procedure that makes it possible to check practically the solvability conditions and to construct solutions, if any exists, is given. Solvability of the asymptotic matching problem with static state feedback is characterized in a similar way in Theorem 14 and a necessary condition that can be practically checked by an algorithmic procedure is given in Theorem 15. Finally, a sufficient condition for solvability of the matching problem in the framework of positive systems by means of a dynamic compensator is given in Theorem 16. An illustrative example is then provided. Section 4 contains conclusions and indications of future work.

2. PRELIMINARIES AND PROBLEM STATEMENT

A linear system Σ defined by equations of the form

$$\Sigma \equiv \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^M$ is the input and $y \in \mathbb{R}^p$ is the input, A, B and C are real matrices of suitable dimensions, is called *positive* if $x(t)$ and $y(t)$ are non negative for $t \geq 0$ for non negative initial conditions and inputs. It is well known that this is equivalent to the fact that the dynamic matrix A is a Metzler matrix, i.e. all its off-diagonal elements are non negative (see Berman and Plemmons (1979)), and B, C are non negative matrices, i.e. all their entries are non negative. In the rest of the paper, we will assume that B and C are, respectively, full column rank and full row rank matrices.

Given a linear system Σ_P and a positive linear system Σ_M, called respectively the *plant* and the *model* and described by the equations

$$\Sigma_P \equiv \begin{cases} \dot{x}_P(t) = A_P\,x_P(t) + B_P\,u_P(t) \\ y_P(t) = C_P\,x_P(t) \end{cases} \tag{2}$$

$$\Sigma_M \equiv \begin{cases} \dot{x}_M(t) = A_M\,x_M(t) + B_M\,u_M(t) \\ y_M(t) = C_M, x_M(t) \end{cases} \tag{3}$$

with $dim(y_P) = dim(y_M) = p$, the basic problem we consider consists in compensating the plant in such a way to achieve positivity and to match the model. This second requirement means that, starting with initial conditions $x_P(0) = 0$ and $x_M(0) = 0$ for the plant and for the model, the difference between the output of the compensated plant and that of the model is identically null. A more demanding problem is that of asymptotic matching, in which, in addition to the previous requirement, the difference between the output of the plant and that of the model is required to go asymptotically to 0 for any initial condition $x_P(0)$ and $x_M(0)$ of the plant and of the model. What makes the problems more difficult than the standard model matching problem for linear systems is, of course, the requirement of positivity. The compensator can be assumed to be static or dynamic, giving rise to different formulations of the problem, as stated below.

Problem 1. Given a linear system Σ_P and a positive system Σ_M, called respectively the *plant* and the *model* and described by the equations (2) and (3), assume that the state of the plant and the state of the model are measurable and consider the output-difference system Σ_{diff} described by the equations

$$\Sigma_{diff} \equiv \begin{cases} \dot{x}_P(t) = A_P\,x_P(t) + B_P\,u_P(t) \\ \dot{x}_M(t) = A_M\,x_M(t) + B_M\,u_M(t) \\ y_E(t) = C_P x_P(t) - C_M\,x_M(t) \end{cases} \tag{4}$$

The Exact Model Matching Problem in the framework of positive systems (EMMP+) consists in finding a state feedback law

$$u_P(t) = F_P\,x_P(t) + F_M x_M(t) + D u_M(t), \tag{5}$$

with F_p, F_M and D matrices of suitable dimension, such that, denoting by Σ^F_{diff} the compensated output-difference system given by

$$\Sigma^F_{diff} \equiv \begin{cases} \dot{x}_P(t) = (A_P + B_P F_P)\,x_P(t) + \\ \qquad\qquad B_P F_M\,x_M(t) + B_P D\,u_M(t) \\ \dot{x}_M(t) = A_M\,x_M(t) + B_M\,u_M(t) \\ y_E(t) = C_P x_P(t) - C_M x_M(t) \end{cases}, \tag{6}$$

we have

- **Positivity condition**: the dynamic matrix of Σ^F_{diff}, namely $\begin{pmatrix} A_P + B_P F_P & B_P F_M \\ 0 & A_M \end{pmatrix}$ is a Metzler matrix and its input matrix, namely $\begin{pmatrix} B_P D \\ B_M \end{pmatrix}$ is nonnegative;
- **Matching condition**: the output $y_E(t)$ of Σ^F_{diff}, with initial conditions $(x_P(0)^\top \quad x_M(0)^\top)^\top = 0$, is null for $t \in \mathbb{R}^+$ for all input signals $u_M(t)$.

Problem 2. In the same hypotheses as in Problem 1, the Asymptotic Model Matching Problem in the framework of positive systems (AMMP+) consists in finding a state feedback law of the form (5), with F_p, F_M and D matrices of suitable dimension, that solves the EMMM+ and, in addition, stabilizes the compensated output-difference system Σ_{diff}^F.

Remark 3. The Positivity condition assures that, for nonegative initial conditions and inputs, the state of the compensated plant evolves in the positive cone of $\mathbb{R}^n$, as that of the model. The above problems make sense, in particular, if the plant is itself positive. The above formulation encompasses the more general situation in which positivity has to be achieved, not only preserved, by the feedback, as e.g. in the problem of stabilizing and making positive a given linear system that is considered in Gao et al. (2005).

Remark 4. Viewing u_P as the control input and u_M as a measurable disturbance for Σ_{diff}, the above formulations of the matching problems makes it clear that they are equivalent, respectively, to a disturbance decoupling problem with measurable disturbance and to the same problem with stability by means of a static state feedback which achieves positivity of the plant. The equivalence between the model matching problem and a suitable disturbance decoupling one was exploited in most of the paper dealing with model matching that have been quoted in the introduction. As already remarked, the positivity requirement complicates the problems with respect to the classical disturbance decoupling ones dealt with in Basile and Marro (1992) and Wonham (1985).

In the framework of linear systems, measurability of the state of the plant and of the model is not a restrictive assumption if dynamic compensators are allowed. The situation is more complicated to handle in the framework of positive systems. In case states are not measurable, but there exists a static state feedback law of the form (5) that satisfies the Positivity condition and the Matching condition, one could consider the dynamic compensator Σ_C given by the equation

$$\Sigma_C \equiv \begin{cases} \dot z_1(t) = (A_P + B_P F_P)\, z_1(t) + \\ \qquad\quad B_P F_M\, z_2(t) + B_P D\, u_M(t) \\ \dot z_2(t) = A_M\, z_2(t) + B_M\, u_M(t) \\ u_P(t) = F_P z_1(t) + F_M z_2(t) + D u_M(t) \end{cases} \tag{7}$$

whose dynamic matrix is a Metzler matrix and whose input matrix is non negative. Applying Σ_C to Σ_{diff}, the resulting system takes the form

$$\Sigma_{diff}^C \equiv \begin{cases} \dot x_P(t) = A_P\, x_P(t) + B_P F_P\, z_1(t) \\ \qquad\quad B_P F_M\, z_2(t) + B_P D\, u_M(t) \\ \dot x_M(t) = A_M\, x_M(t) + B_M\, u_M(t) \\ \dot z_1(t) = (A_P + B_P F_P)\, z_1(t) + \\ \qquad\quad B_P F_M\, z_2(t) + B_P D\, u_M(t) \\ \dot z_2(t) = A_M\, z_2(t) + B_M\, u_M(t) \\ y_E(t) = C_P x_P(t) - C_M x_M(t) \end{cases} \tag{8}$$

and it is possible to show that, initializing the system at $(x_P(0)^\top\, x_M(0)^\top\, z_1(0)^\top\, z_2(0)^\top)^\top = 0$, for any input $u_M(t)$ one has $y_E(t) = 0$ for $t \geq 0$. In other words, Σ_C forces the output of the plant to equal that of the model, so achieving an exact matching. However, the Positivity condition for the compensated system Σ_{diff}^C does not necessarily hold, even if the plant is positive, since $B_P F_P$ is not necessarily nonnegative. This means that a static feedback solution may loose the property of guaranteeing positivity for the dynamics of the compensated system if it is implemented by means of a dynamical compensator. In view of this remark, we can state the following problem.

Problem 5. Given a positive plant Σ_P and a positive model Σ_M, the Exact Model Matching Problem by Dynamical Conpensator in the framework of positive systems (DEMMP+) consists in finding a linear system Σ_C of the form

$$\Sigma_C \equiv \begin{cases} \dot z(t) = A_C\, z(t) + B_C u_M(t) \\ u_P(t) = F_C z(t) + D u_M(t) \end{cases} \tag{9}$$

where A_C is a Metzler matrix and B_C is a non negative matrix such that, denoting by Σ_{diff}^C the compensated output-difference system given by

$$\Sigma_{diff}^C \equiv \begin{cases} \dot x_P(t) = A_P\, x_P(t) + B_P F_C\, z(t) + B_P D\, u_M(t) \\ \dot x_M(t) = A_M\, x_M(t) + B_M\, u_M(t) \\ \dot z(t) = A_C\, z(t) + B_C u_M(t) \\ y_E(t) = C_P x_P(t) + C_M x_M(t) \end{cases}, \tag{10}$$

we have

- **Positivity condition**: the dynamic matrix of Σ_{diff}^C, namely $\begin{pmatrix} A_P + B_P F_P & 0 & B_P F_M \\ 0 & A_M & 0 \\ 0 & 0 & A_C \end{pmatrix}$ is a Metzler matrix and its input matrix, namely $\begin{pmatrix} B_P D \\ B_M \\ B_C \end{pmatrix}$ is non negative;
- **Matching condition**: for all input signals $u_M(t)$, the output $y_E(t)$ of Σ_{diff}^C, with initial conditions $(x_P(0)^\top \quad x_M(0)^\top \quad x_C(0)^\top)^\top = 0$, is null for $t \in \mathbb{R}^+$.

3. STRUCTURAL ANALYSIS AND PROBLEM SOLUTION

Let us write the equations of the ouput-difference system Σ_{diff} in the more compact form

$$\Sigma_{diff} \equiv \begin{cases} \dot x_E(t) = A_E\, x_E(t) + B_1\, u_P(t) + \\ \qquad\quad B_2 u_M(t) \\ y_E(t) = C_E\, x_E(t) \end{cases} \tag{11}$$

with $x_E = (x_P^\top, x_M^\top)^\top$, $A_E = \begin{pmatrix} A_P & 0 \\ 0 & A_M \end{pmatrix}$, $B_1 = \begin{pmatrix} B_P \\ 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ B_M \end{pmatrix}$, $C_E = (C_P \quad -C_M)$. We recall that a subspace $\mathcal{V} \subseteq \mathcal{X}_E = \mathcal{X}_P \oplus \mathcal{X}_M$ is an (A_E, B_1)- invariant subspace if the relation

$$A_E \mathcal{V} \subseteq \mathcal{V} + Im\, B_1 \tag{12}$$

holds. It is well known that (12) is equivalent to the existence of a feedback matrix $F : \mathcal{X}_E \to \mathcal{U}$ such that the relation

$$(A_E + B_1 F)\mathcal{V} \subseteq \mathcal{V} \tag{13}$$

holds and that there exists a maximal (A_E, B_1)-invariant subspace, denoted by $\mathcal{V}^*$ contained in $Ker\, C_E$ (see Basile and Marro (1992)). Any feedback matrix for which (13) holds is called a *friend* of $\mathcal{V}$.

Note that if $V = \begin{pmatrix} V_P \\ V_M \end{pmatrix}$ is a matrix whose columns are a basis of the (A_E, B_1)-invariant subspace $\mathcal{V}$, then $span\,(V_P)$ is an (A_P, B_P)-invariant subspace of $\mathcal{X}_P$ and $span\,(V_M)$ is an A_M-invariant subspace of $\mathcal{X}_M$.

In dealing with positive systems, we are interested in the following notion.

Defition 6. Given a linear system Σ defined by equation of the form (1), an (A, B)-invariant subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be *positive* if it has a friend F such that $(A + BF)$ is a Metzler matrix.

Defition 7. Given a linear system Σ defined by equation of the form (1), an (A, B)-invariant subspace $\mathcal{V}$ of $\mathcal{X}$ is said to be *strongly positive* if it has a positive friend F such that $(A + BF)$ is a Metzler matrix.

Given a linear system Σ defined by equations of the form (1), in order to study the positivity of an (A, B)-invariant subspace $\mathcal{V}$, let us assume that the condition

$$\mathcal{V} \cap Im\, B = \{0\} \tag{14}$$

holds. In that case, if V is a matrix whose columns are a basis of $\mathcal{V}$, there exists a unique pair of matrices (L, M) of suitable dimensions such that

$$AV = VL + BM \tag{15}$$

and, letting P be a permutation matrix such that $PV = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ with V_1 square of dimensions $dim(\mathcal{V}) \times dim(\mathcal{V})$ and non singular, any friend F of $\mathcal{V}$ has the two-block form

$$F = (-MV_1^{-1} - KV_2V_1^{-1} \quad K)P$$

where K is an arbitrary matrix of suitable dimensions. We can therefore state the following proposition.

Proposition 8. Given a linear system Σ of the form (1) and an (A, B)-invariant subspace $\mathcal{V}$ of $\mathcal{X}$ such that condition (14) holds, then

- $\mathcal{V}$ is a positive (A, B)-invariant subspace if and only if, with the above assumptions and notations, there exists a matrix K of suitable dimensions such that the matrix $(A + B(-MV_1^{-1} - KV_2V_1^{-1} \quad K)P)$ is a Metzler matrix.
- $\mathcal{V}$ is a strongly positive (A, B)-invariant subspace if and only if the above conditions holds and the matrix $(-MV_1^{-1} - KV_2V_1^{-1} \quad K)$ is positive.

Remark 9. A parametrization of the family of all friends of $\mathcal{V}$ can be obtained also if condition (14) does not hold and a characterization of positive controlled invariance, similar to the above one, can be derived also in the general case. We limited our analysis to the situation in which (14) holds only for simplifying the formulation of the results about the matching problem in the next section.

More generally, the following result is useful in the specific situation we are interested in.

Proposition 10. Given the plant Σ_P and the positive model Σ_M, consider the resulting output-difference system Σ_{diff} described by (11), let $\mathcal{V}^*$ be the maximal (A_E, B_1)-invariant subspace contained in $Ker\, C_E$ and assume that the condition

$$\mathcal{V}^* \cap Im\, B_1 = \{0\} \tag{16}$$

holds. Then for any subspace $\mathcal{W} \subseteq \mathcal{V}^* + Im\, B_1$, the set of (A_E, B_1)-invariant subspaces $\mathcal{V}$ contained in $Ker\, C_E$ such that $\mathcal{W} \subseteq \mathcal{V} + Im\, B_1 \subseteq \mathcal{V}^* + Im\, B_1$ is a lattice with respect to intersection and sum of subspaces. As a consequence

a) there exists a minimal (A_E, B_1)-invariant subspace contained in $Ker\, C_E$, denoted by $\mathcal{V}_*(\mathcal{W})$ such that $\mathcal{W} \subseteq \mathcal{V}_*(\mathcal{W}) + Im\, B_1 \subseteq \mathcal{V}^* + Im\, B_1$;
b) if $\mathcal{V}_1$ and $\mathcal{V}_2$ are (A_E, B_1)-invariant subspaces such that $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}^*$, any friend of $\mathcal{V}_2$ is also a friend of $\mathcal{V}_1$.

Proof. See Conte and Perdon (1991).

Remark 11. Condition (16) is equivalent to left invertibility of Σ_E and, due to the structure of that system, it implies that also the plant Σ_P is a left invertible linear system. In fact, the maximal controlled (A_P, B_P)-invariant subspace $\mathcal{V}_P^*$ of $\mathcal{X}_P$ contained in $Ker\, C_P$ is such that $\mathcal{V}_P^* \oplus \{0\}$ is an (A_E, B_1)-invariant subspace of $\mathcal{X}_P \oplus \mathcal{X}_M$ contained in $Ker\, C_E = Ker(C_P \quad -C_M)$. It follows that $\mathcal{V}_P^* \oplus \{0\}$ is contained in $\mathcal{V}^*$ and hence $\mathcal{V}_P^* \cap Im\, B_P = \{0\}$.

3.1 Solvability conditions for the EMMP+

We can now state the structural solvability conditions for the EMMP+.

Theorem 12. Given a linear system Σ_P and a positive model Σ_M, consider the output-difference system Σ_{diff} described by (11) and assume that condition (16) holds. Letting V be a matrix whose columns form a basis of $\mathcal{V}^*$, the related EMMP+ is solvable if and only if

a) $B_2 = VH - B_1D$ for a matrix H and a matrix D of suitable dimensions such that B_1D is non negative and
b) $\mathcal{V}_*(ImB_2)$ is a positive (A_E, B_1)-invariant subspace.

Proof. Sufficiency. Condition a) implies $Im\, B_2 \subseteq \mathcal{V}^* + Im\, B_1$ and together with (16) this implies that $\mathcal{V}_*(Im\, B_2)$ exists. Without loss of generality, we can assume that V has been chosen of the form $V = (V_1 \quad V_2)$, where the columns of V_1 form a basis of $\mathcal{V}_*(Im\, B_2)$. Therefore, we have $B_2 = V_1H' - B_1D$ and the static state feedback $u_P(t) = F_Px_P(t) + F_Mx_m(t) + Du_m(t)$, where $(F_P \quad F_M) : \mathcal{X}_P \oplus \mathcal{X}_M \to \mathcal{U}$ is a friend of $\mathcal{V}_*(ImB_2)$, decouples the output $y_E(t)$ of Σ_E from the input $u_M(t)$ (see Basile and Marro (1992)). By b), we can assume that the matrix $\begin{pmatrix} A_P + B_PF_P & B_PF_M \\ 0 & A_M \end{pmatrix}$ is a Metzler matrix and, since $\begin{pmatrix} B_PD \\ B_M \end{pmatrix}$ is non negative by a), this shows that the the mentioned static state feedback law solves the EMMP+.

Necessity. Assume that the static state feedback $u_P(t) = F_Px_P(t) + F_Mx_m(t) + Du_m(t)$, solves the EMMM+. By applying it to Σ_{diff}, we get the compensated system Σ_{diff}^F described by the equations

$$\Sigma_{diff}^F \equiv \begin{cases} \dot{x}_P(t) = (A_P + B_PF_P)x_P(t) + B_PF_Mx_M(t) + \\ \qquad\qquad B_PDu_M(t) \\ \dot{x}_M(t) = A_M\, x_M(t) + B_M\, u_M(t) \\ y_E(t) = C_Px_P(t) - C_M\, x_M(t) \end{cases} \tag{17}$$

whose output is identically null for all input $u_M(t)$ if the initial state is null. This means that there exists an $\left(\begin{pmatrix} A_P + B_PF_P & B_PF_M \\ 0 & A_M \end{pmatrix}, \begin{pmatrix} B_P \\ 0 \end{pmatrix} \right)$-invariant subspace $\mathcal{V} \subseteq \mathcal{X}_P \oplus \mathcal{X}_M$ such that $\mathcal{V} \subseteq Ker(C_P \quad -C_M)$ and $Im \begin{pmatrix} B_PD \\ B_M \end{pmatrix} \subseteq \mathcal{V}$. Therefore, with the notation of (11), $\mathcal{V}$ is a positive (A_E, B_1)-invariant subspace and, letting $\bar{V}$ be a matrix whose columns are a basis of $\mathcal{V}$, we have $B_2 = \bar{V}\bar{H} - B_1D$ for some matrix H of suitable dimensions. Then, if V is a matrix whose columns are a basis of V^* of the form $V = (\bar{V} \quad V')$, condition a) follows by taking $H = (\bar{H}^\top 0)^\top$. Condition b) follows by minimality of $\mathcal{V}_*(Im\, B_2)$ and Proposition 10.

Remark 13. The necessary and sufficient conditions of Theorem 12 are stringent and, since they completely characterize solvability, this means that the requirements of the problem are very strong. It is important to note, however, that such conditions can be practically checked. To do this, one has to construct $\mathcal{V}^*$ and $\mathcal{V}_*(ImB_1)$ by means of the algorithms described respectively in Basile and Marro (1992) and in Conte and Perdon (1991) and to solve the inequalities that characterize positivity of $\mathcal{V}_*(ImB_1)$ as described in Proposition 8.

3.2 Solvability conditions for the AMMP+

About the AMMP+, we can give the following result.

Theorem 14. Given a linear system Σ_P and a positive model Σ_M, consider the output-difference system Σ_E described by (11) and assume that condition (16) holds. Letting V be a matrix whose columns form a basis of $\mathcal{V}^*$, the related AMMP+ is solvable if and only if

a) $B_2 = VH - B_1D$ for a matrix H and a matrix D of suitable dimensions such that B_PD is non negative and
b) $\mathcal{V}_*(ImB_2)$ has a friend F such that $(A_E + B_1F)$ is a Metzler matrix (in particular, $\mathcal{V}_*(ImB_2)$ is a positive (A_E, B_1)-invariant subspace) and a Hurwitz matrix (i.e. Σ^F_{diff} is asymptotically stable).

Proof. It follows from Theorem 12 and Proposition 10.

Characterization of (A_E, B_1)-invariant subspaces that satisfies condition b) of the above therorem is not easy, nevertheless the following result, which represents a necessary condition for the solution of the AMMP+, can be given.

Theorem 15. Given a linear systems Σ_P and a positive model Σ_M, consider the output-difference system Σ_E described by (11) and assume that condition (16) holds. Letting V be a matrix whose columns form a basis of $\mathcal{V}^*$, the related AMMP+ is solvable only if

a) $B_2 = VH - B_1D$ for a matrix H and a matrix D of suitable dimensions such that B_PD is non negative and
b) $\mathcal{V}_*(ImB_2)$ is a positive (A_E, B_1)-invariant subspace and the (unique) matrix L such that $A_EV_1 = V_1L + B_1M$, where V_1 is a matrix whose columns are a basis of $\mathcal{V}_*(ImB_2)$, is Hurwitz.

Proof. It follows from Theorem 12 by remarking that the dynamics induced on $\mathcal{V}_*(ImB_2)$ in the compensated system is described, in a suitable basis, by the matrix L.

3.3 Solvability conditions for the DEMMP+

About the DEMMP+, we can give the following result.

Theorem 16. Given a positive plant Σ_P and a positive model Σ_M, consider the output-difference system Σ_E described by (11) and assume that condition (16) holds. Then, the related DEMMP+ is solvable if there exists a solution $u_M(t) = F_Px_P(t) + F_Mx_M(t) + Du_M(t)$ of the EMMP+ such that B_PF_P is a non negative matrix. In particular, this holds if $\mathcal{V}(Im\, B_2)$ is a strongly positive (A_E, B_1)-invariant subspace.

Proof. It is sufficient to remark, as done in Section 2, that a solution is given by the compensator (9).

3.4 Example

For sake of illustration, let us consider the plant Σ_P and the model Σ_M defined respectively by the equations

$$\Sigma_P \equiv \begin{cases} \dot{x}_P(t) = \begin{pmatrix} 0 & 2 \\ 3 & -2 \end{pmatrix} x_P(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_P(t) \\ y_P(t) = (1\ 1)x_P(t) \end{cases} \tag{18}$$

$$\Sigma_M \equiv \begin{cases} \dot{x}_M(t) = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} x_M(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_M(t) \\ y_M(t) = (5\ 2)x_M(t) \end{cases} \tag{19}$$

and the resulting system Σ_{diff} of the form (11). Condition (16) holds and we have that $\mathcal{V}_*(Im\, B_2) = span(V)$, where $V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}$, and in particular $B_2 = VH - B_1D$ with $H = (1\ 0)^\top$ and $D = (1)$. Note that $span(V)$ satisfies (15) with $L = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$ and $M = [\,1\ 5\,]$. All friends of $\mathcal{V}$ are of the form

$$F = [\,-5 - k_1 + 2k_2 \quad 4 - 3k_2 + k_1 \quad k_1 \quad k_2\,]$$

for any k_1 and k_2. Therefore, any possible solution of the EMMP+ is of the form $u_P(t) = F\, x_E(t) + u_M(t)$ and the dynamic matrix and the input matrix of the compensated system are respectively of the form

$$A_E + B_1F = \begin{bmatrix} -5 - k_1 + 2k_2 & 6 - 3k_2 + k_1 & k_1 & k_2 \\ -2 - k_1 + 2k_2 & 2 - 3k_2 + k_1 & k_1 & k_2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

and

$$\begin{bmatrix} B_PD \\ B_M \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

We must chose $k_1 > 0$ and $k_2 > 0$ to satisfy the Positivity condition and taking, for instance, $k_1 = 1$ and $k_2 = 2$, the dynamic matrix becomes

$$(A_E + B_1F) = \begin{bmatrix} -2 & 1 & 1 & 2 \\ 1 & -3 & 1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Since it is a Metzler matrix and it is Hurwitz, actually for $F = (-2\ -1\ 1\ 2)$ we have a solution of the AMMP+. Note that $span(V)$ is not a strongly positive (A_E, B_1)-invariant subspace, since any friend F' must satisfy $F'V = FV = M = (-1\ \ -5)$ and this implies that at least one entry of F' is negative.

4. CONCLUSIONS

The structural geometric approach has been applied to the model matching problem in the framework of positive linear system by introducing a suitable notions of positive controlled invariance. This has made it possible to find solvability conditions by static state feedback for the considered problems in a number of situations. In particular, the solvability of the asymptotic model matching problem has been related to the existence of stabilizing friends for positive controlled invariant subspaces and this property needs to be further investigated in the future. Analogously, starting from the sufficient condition that has been found, the complete characterization of solvability by means of dynamical compensators needs to be worked out.

REFERENCES

Bao, W., Qi, Y., Zhao, J., and Chang, J. (2012). Robust dynamic bumpless transfer: An exact model matching approach. *IET Control Theory and Applications*, 6(10), 1341–1350.

Basile, G. and Marro, G. (1992). *Controlled and Conditioned Invariants in Linear System Theory.* Prentice Hall, Englewood Cliffs, New Jersey.

Benvenuti, L. and Farina, L. (2004). A tutorial on the positive realization problem. *IEEE Transactions on Automatic Control*, 497.

Berman, A. and Plemmons, R.J. (1979). *Nonnegative Matrices in the Mathematical Sciences.* Academic, New York.

Colaneri, P. and Kucera, V. (1997). The model matching problem for periodic discrete-time systems. *IEEE Transactions on Automatic Control*, 42(10), 1472–1476.

Conte, G. and Perdon, A.M. (1991). Small (a,b)-invariant subspaces in disturbance decoupling problems. In *Proc. 30th IEEE Conference on Decision and Control*. Brighton, England.

Conte, G. and Perdon, A.M. (1995). The model matching problem for systems over a ring and applications to delay-differential systems. In *Proc. IFAC Conference on System Structure and Control*, 319–323. Nantes, France.

Conte, G., Perdon, A.M., and Zattoni, E. (2014). Model matching problems for switching linear systems. *IFAC Proceedings Volumes*, 47(3), 1501–1506.

Conte, G., Perdon, A.M., and Zattoni, E. (2018a). Asymptotic model matching for hybrid linear systems with state jumps. In *Proceedings 57th IEEE Conference on Decision and Control (CDC 2018)*, 2384–2389. Miami Beach, FL.

Conte, G., Perdon, A.M., and Zattoni, E. (2018b). Asymptotic model matching for LPV systems. *IFAC PaperOnLine*, 51(26), 173–178.

De Leenheer, P. and Aeyels, D. (2001). Stabilization of positive linear systems. *Systems & Control Letters*, 44, 259–271.

De Leenheer, P. and Aeyels, D. (2002). Stabilization of positive systems with first integral. *Automatica*, 38, 1583–1589.

Farina, L. and Rinaldi, S. (2000). *Positive linear systems: theory and applications.* Wiley-Interscience, Series on Pure and Applied Mathematics 2000, New York.

Frobenius, G. (1912). Uber matrizen aus nicht negativen elementen. *S.-B. Deutsch Akad. Wiss. Berlin, Math.-Nat. K*, 456–477.

Gao, H., Lam, J., Wang, C., and Xu, S. (2005). Control for stability and positivity: equivalent conditions and computation. *IEEE Trans. on Circuits ans Systems-II, Express Briefs*, 52(9).

Heemels, W.P.M.H., Van Eijndhoven, S.J.L., and Stoorvogel, A.A. (1998). Linear quadratic regulator problem with positive controls. *Int. J. Control*, 70, 551–578.

Ito, H. (1997). A new statement about the theorem determining the region of eigenvalues of stochastic matrices. *Linear Algegra and Applications*, 267(9), 241–246.

Kaczorek, T. (2002). Stabilization of positive linear systems. *Proc. 37th IEEE Conf. Decision Control, Tampa, FL*, 620–6211.

Karpelevich, F.I. (1988). *On the characteristic roots of matrices with nonnegative element*, volume 140 of *Eleven Papers Translated From Russian.* American Math. Soc., Providence, R.

Kučera, V. (1992). Model matching of descriptor systems by proportional state feedback. *Automatica*, 28(2), 423–425.

Luenberger, D.G. (1979). *Positive linear systems.* Introduction to Dynamic Systems: Theory, Models, and Application. Wiley, New York.

Malabre, M. (1982). Structure all'infini des triplets invariants. application à la pursuite parfaite de modèle. In *5ème Conférence Internationale sur l'Analyse et l'Optimisation des Systèmes, INRIA, Versailles*, volume 44 of *Lecture Notes in Control and Information Sciences*, 43–53. Springer Verlag.

Moog, C.H., Perdon, A.M., and Conte, G. (1991). Model matching and factorization for nonlinear systems: A structural approach. *SIAM Journal on Control and Optimization*, 29(4), 769–785.

Moore, B. and Silverman, L.M. (1972). Model matching by state feedback and dynamic compensation. *IEEE Transactions on Automatic Control*, AC–17(4), 491–497.

Morse, A.S. (1973). Structure and design of linear model following systems. *IEEE Transactions on Automatic Control*, AC–18(4), 346–354.

Perron, O. (1907). Zur theorie der matrizen. *Math. Ann.*, 64, 248–263.

Picard, P., Lafay, J.F., and Kucera, V. (1998). Model matching for linear systems with delays and 2D systems. *Automatica*, 34(2), 183–191.

Seneta, E. (1981). *Non-Negative Matrices and Markov Chains 2nd ed.* Springer-Verlag, New York.

Seyboth, G. and Allgöwer, F. (2014). Synchronized model matching: A novel approach to cooperative control of nonlinear multi-agent systems. In *19th World Congress of the International Federation of Automatic Control*, volume 19, 1985–1990.

van den Hof, J.M. (1998). Positive linear observers for linear compartmental systems. *SIAM J. Control Optim.*, 36, 590–608.

Wolovich, W.A. (1972). The use of state feedback for exact model matching. *SIAM Journal on Control*, 10(3), 512–523.

Wonham, W.M. (1985). *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York, 3 edition.