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Initial state design for suppressing undesirable effects of controller switches

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Abstract: We propose a new initial state design procedure for a newly-activated controller at a controller switch. By minimizing the value of the state-dependent switching $\mathcal{L}_2$ gain presented in this paper, we can obtain the optimal initial state for suppressing the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur.

Keywords: Control systems; Controllers; Initial states; Linear systems; Switches.

1. INTRODUCTION

The switches from an operating controller to a more desirable one have been widely performed for letting a control system adjust to the changes in control objective, operating conditions, surrounding circumstances, and so on. Here, it is needless to say that we should reduce the undesirable effects of a controller switch. A practical technique for reducing the undesirable effect is to appropriately initialize a newly-activated controller.

Many studies have addressed the issue of suppressing the fluctuations in transient responses after a controller switch by designing the initial state of a newly-activated controller, e.g., Hanus *et al.* (1987), Kothare *et al.* (1994), Edwards and Postlethwaite (1998), Turner and Walker (2000), and Paxman and Vinnicombe (2000). Most of them aimed at making the output of a control system after a controller switch close to the virtual output in the case where the switch does not occur.

Asai (2003) designed a control system by reducing the value of the Hankel-type switching $\mathcal{L}_2$ gain presented in Asai (2005) to suppress the fluctuations in transient responses. Suyama and Sebe (2019a) obtained the optimal switching matrix, which determines the initial state of a newly-activated controller, for suppressing the fluctuations in transient responses after a controller switch by minimizing the value of another switching $\mathcal{L}_2$ gain. Moreover, Suyama and Sebe (2019b) presented a procedure for directly obtaining the optimal initial state by using the state-dependent switching $\mathcal{L}_2$ gain (Suyama and Sebe, 2018b).

For taking a desirable reference signal into consideration, Zaccarian and Teel (2005) and Hespanha *et al.* (2007) directly minimized the $\mathcal{L}_2$ norm of the error between the plant output and reference signal to obtain the optimal initial state. Under the assumption that only the state of an integrator is assigned in a state-feedback control system, Saito *et al.* (1998) approximated its step response by assigning an initial value by using the observability Gramian. Also Nakano *et al.* (2018) directly evaluated the $\mathcal{L}_2$ norm of the output signal by using the observability Gramian in an observer-based servo system to show that transient responses caused by state-feedback and observer gains switch can be suppressed by appropriately initializing a newly-activated observer.

In this paper, we first present a new state-dependent switching $\mathcal{L}_2$ gain that focuses on the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur. Then, for a given switching situation, by minimizing the value of the state-dependent switching $\mathcal{L}_2$ gain, we obtain the optimal initial state for suppressing the difference between the actuality and virtual situation. We can make the output of a control system after a controller switch close to the virtual output in the case where the switch does not occur more effectively. The proposed initial state design procedure has the following advantages.

- The switching time does not affect its design result, and need not be known in advance.
- We need only solving a linear matrix inequality (LMI) problem. Complicated calculation is not necessary.
- We can always obtain the solution. It is not necessary to discuss the solvability.

They are important especially when the initial state should be determined immediately after a non-preplanned controller switch. Moreover, by the proposed initial state design, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017).

Notations. $\mathcal{F}_\ell(G, K)$ denotes the lower linear fractional transformation (LFT) of G and K, and $\mathcal{L}_2(a, b)$: the Lebesgue space of all square-integrable and vector-valued functions defined on an interval (a, b), i.e., $\mathcal{L}_2(a, b) =$

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$\{\, x(t) \mid \|x(t)\|_{2\,(a,\,b)} < \infty \,\}$, where $\|x(t)\|_{2\,(a,\,b)}$ denotes the $\mathcal{L}_2$ norm defined by $\|x(t)\|_{2\,(a,\,b)} = \left[\int_a^b x^{\mathrm{T}}(t)x(t)dt \right]^{\frac{1}{2}}$.

2. STATE-DEPENDENT SWITCHING $\mathcal{L}_2$ GAIN

2.1 System switch

Suppose that a linear time-invariant (LTI) system H_p switches to another LTI system H_f with a state transition at the switching time $t = t_0$.

Suppose that the pre-switch system is represented by

$$H_p\colon \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p w(t) \\ z(t) = C_p x_p(t) + D_p w(t), \end{cases} \quad t \le t_0, \tag{1}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ $(t \le t_0)$ is the state-variable vector, $w(t) \in \mathbb{R}^{n_i}$ is the input, and $z(t) \in \mathbb{R}^{n_o}$ is the output. We assume the following.

Assumption 2.1.

(a) A_p is stable.
(b) (A_p, B_p) is controllable and (C_p, A_p) is observable.
(c) $x_p(-\infty) = 0$.

Suppose that the post-switch system is represented by

$$H_f\colon \begin{cases} \dot{x}_f(t) = A_f x_f(t) + B_f w(t) \\ z(t) = C_f x_f(t) + D_f w(t), \end{cases} \quad t > t_0, \tag{2}$$

where $x_f(t) \in \mathbb{R}^{n_f}$ $(t > t_0)$ is the state-variable vector and $w(t)$, $z(t)$ are the same input and output as in the pre-switch system H_p. We assume the following.

Assumption 2.2.

(a) A_f is stable.
(b) (A_f, B_f) is controllable and (C_f, A_f) is observable.

Note that n_f is not always equal to n_p, because the system order can change. For example, in a controller switch, there can be the difference in orders between an operating controller and a newly-activated one.

Suppose that the following state transition occurs around the system switch:

$$x_f(t_{0+}) = S x_p(t_0), \tag{3}$$

where $S \in \mathbb{R}^{n_f \times n_p}$ is a constant matrix.

2.2 Definition of state-dependent switching $\mathcal{L}_2$ gain

Under the condition that the switch to the post-switch system with the state transition does not occur at $t = t_0$ (strictly speaking, at any $t \in [\,t_0, \infty)$), let us extend the pre-switch system as follows:

$$H_{p,\,\mathrm{ext}}\colon \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p w(t) \\ \begin{cases} z(t) = C_p x_p(t) + D_p w(t), & t \le t_0 \\ z_{\mathrm{vir}}(t) = C_p x_p(t) + D_p w(t), & t > t_0, \end{cases} \end{cases} \tag{4}$$

where $z_{\mathrm{vir}}(t) \in \mathbb{R}^{n_o}$ $(t > t_0)$ is the virtual output under the assumption that no system switch occurs.

The state-dependent switching $\mathcal{L}_2$ gain used in this paper is defined as follows.

Definition 2.3. For an $x_0 \in \mathbb{R}^{n_p}$,

$$\hat{\gamma}_{\text{sd-dif}}(x_0) = \sup_{\substack{w(t) \in \mathcal{L}_2(-\infty,\,\infty)\setminus\{0\} \\ \text{s.t. } x_p(t_0) = x_0}} \frac{\|z(t) - z_{\mathrm{vir}}(t)\|_{2\,(t_0,\,\infty)}}{\|w(t)\|_{2\,(-\infty,\,\infty)}}. \tag{5}$$

Note that the switching time t_0 does not affect the gain value as shown in Theorem 2.4 later.

Suyama and Sebe (2016) presented the following switching $\mathcal{L}_2$ gain to analyze the magnitude of a system switch:

$$\hat{\gamma}_{\mathrm{dif}} = \sup_{w(t) \in \mathcal{L}_2(-\infty,\,\infty)\setminus\{0\}} \frac{\|z(t) - z_{\mathrm{vir}}(t)\|_{2\,(t_0,\,\infty)}}{\|w(t)\|_{2\,(-\infty,\,\infty)}}. \tag{6}$$

The relationship between $\hat{\gamma}_{\text{sd-dif}}(x_0)$ and $\hat{\gamma}_{\mathrm{dif}}$ is as follows:

$$\hat{\gamma}_{\mathrm{dif}} = \max_{x_0 \in \mathbb{R}^{n_p}} \hat{\gamma}_{\text{sd-dif}}(x_0). \tag{7}$$

2.3 Difference system

Consider the following difference system between the post-switch system H_f and the extended pre-switch system $H_{p,\,\mathrm{ext}}$ on the post-switch side:

$$H_d\colon \begin{cases} \dot{x}_d(t) = A_d x_d(t) + B_d w(t) \\ z_d(t) = z(t) - z_{\mathrm{vir}}(t) = C_d x_d(t) + D_d w(t), \end{cases} \quad t > t_0, \tag{8}$$

where

$$x_d(t) = \begin{bmatrix} x_f(t) \\ x_p(t) \end{bmatrix}, \quad t > t_0 \tag{9}$$

and

$$A_d = \begin{bmatrix} A_f & O \\ O & A_p \end{bmatrix}, \quad B_d = \begin{bmatrix} B_f \\ B_p \end{bmatrix}, \quad C_d = [\, C_f \;\; -C_p \,], \quad D_d = D_f - D_p. \tag{10}$$

From Assumptions 2.1 (a) and 2.2 (a), H_d is stable. However, H_d is not always controllable and observable. If H_p and H_f have a common pole, there is the possibility that the pole is uncontrollable and/or unobservable. Thus, H_d is stabilizable and detectable in general.

We then consider the system switch from H_p to H_d with the state transition

$$x_d(t_{0+}) = S_d x_p(t_0), \quad S_d = \begin{bmatrix} S \\ I \end{bmatrix}. \tag{11}$$

It occurs at the switching situation $x_p(t_0) = x_0$ to discuss the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$.

2.4 Equation-based $\mathcal{L}_2$ gain condition

The following theorem presents an equation-based $\mathcal{L}_2$ gain condition. It shows that the switching time does not affect the value of the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$.

Theorem 2.4. Let $\gamma > 0$ and $x_0 (\neq 0) \in \mathbb{R}^{n_p}$. The state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$ satisfies $\hat{\gamma}_{\text{sd-dif}}(x_0) < \gamma$ if and only if the following conditions are satisfied.
(a) $\bar{\sigma}(D_d) < \gamma$.
(b) There exists the stabilizing solution $X_d \succeq O$ to the Riccati equation

$$X_d A_d + A_d^{\mathrm{T}} X_d + C_d^{\mathrm{T}} C_d + (X_d B_d + C_d^{\mathrm{T}} D_d) \times (\gamma^2 I - D_d^{\mathrm{T}} D_d)^{-1} (X_d B_d + C_d^{\mathrm{T}} D_d)^{\mathrm{T}} = O. \tag{12}$$

(c) It holds that

$$x_0^{\mathrm{T}} (\gamma^2 X_p^{-1} - S_d^{\mathrm{T}} X_d S_d) x_0 > 0, \tag{13}$$

where $X_p \succ O$ is a unique solution to the Lyapunov equation

$$A_p X_p + X_p A_p^{\mathrm{T}} + B_p B_p^{\mathrm{T}} = O. \tag{14}$$

Proof: (i) Sufficiency: On the pre-switch side (i.e., $t \leq t_0$), it follows from (14) that $\begin{bmatrix} A_p X_p + X_p A_p^{\mathrm{T}} & B_p \\ B_p^{\mathrm{T}} & -I \end{bmatrix} \preceq O$. Thus,

$$\begin{aligned} &\frac{d}{dt}\big(x_p^{\mathrm{T}}(t) X_p^{-1} x_p(t)\big) - w^{\mathrm{T}}(t) w(t) \\ &= \begin{bmatrix} X_p^{-1} x_p(t) \\ w(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_p X_p + X_p A_p^{\mathrm{T}} & B_p \\ B_p^{\mathrm{T}} & -I \end{bmatrix} \begin{bmatrix} X_p^{-1} x_p(t) \\ w(t) \end{bmatrix} \\ &\leq 0 \end{aligned} \tag{15}$$

for any $w(t) \in \mathcal{L}_2(-\infty, t_0]$ and the corresponding $x_p(t)$. Integrating (15) with respect to $t \in (-\infty, t_0]$ and using Assumption 2.1 (c), we have

$$\|w(t)\|^2_{2\,(-\infty,\,t_0]} \geq x_0^{\mathrm{T}} X_p^{-1} x_0. \tag{16}$$

On the post-switch side (i.e., $t > t_0$), using the stabilizing solution $X_d \succeq O$ to Riccati equation (12), we consider the following equation that holds for any $w(t) \in \mathcal{L}_2(t_0, \infty)$ and the corresponding $x_d(t)$:

$$\begin{aligned} &\int_{t_0}^{\infty} \Big[x_d^{\mathrm{T}}(t) W_{11} x_d(t) + 2 x_d^{\mathrm{T}}(t) W_{12} w(t) \\ &\qquad + w^{\mathrm{T}}(t) W_{22} w(t) + \frac{d}{dt}\big(x_d^{\mathrm{T}}(t) X_d x_d(t)\big)\Big] dt \\ &= \int_{t_0}^{\infty} \begin{bmatrix} x_d(t) \\ w(t) \end{bmatrix}^{\mathrm{T}} \\ &\quad \times \begin{bmatrix} X_d A_d + A_d^{\mathrm{T}} X_d + W_{11} & X_d B_d + W_{12} \\ B_d^{\mathrm{T}} X_d + W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix} \begin{bmatrix} x_d(t) \\ w(t) \end{bmatrix} dt, \end{aligned} \tag{17}$$

where $W_{11} = C_d^{\mathrm{T}} C_d$, $W_{12} = C_d^{\mathrm{T}} D_d$, and $W_{22} = -(\gamma^2 I - D_d^{\mathrm{T}} D_d)$. Here, it follows from the stability of H_d that $x_d(\infty) = 0$ for any $w(t) \in \mathcal{L}_2(t_0, \infty)$. We thus have

$$\begin{aligned} &\text{Left-hand side of (17)} \\ &= \|z_d(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2 \|w(t)\|^2_{2\,(t_0,\,\infty)} - (S_d x_0)^{\mathrm{T}} X_d (S_d x_0). \end{aligned} \tag{18}$$

Next, under Conditions (a), (b) and the detectability of H_d, we have $\begin{bmatrix} X_d A_d + A_d^{\mathrm{T}} X_d + W_{11} & X_d B_d + W_{12} \\ B_d^{\mathrm{T}} X_d + W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix} \preceq O$. Thus, for any $w(t) \in \mathcal{L}_2(t_0, \infty)$ and the corresponding $x_d(t)$,

$$\text{Right-hand side of (17)} \leq 0. \tag{19}$$

Therefore, from (18) and (19) we have

$$\|z_d(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2 \|w(t)\|^2_{2\,(t_0,\,\infty)} \leq (S_d x_0)^{\mathrm{T}} X_d (S_d x_0). \tag{20}$$

Multiplying (16) by $-\gamma^2$, adding it with (20), and using (11) and (13), we have

$$\begin{aligned} &\|z_d(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2 \|w(t)\|^2_{2\,(-\infty,\,\infty)} \\ &\leq -x_0^{\mathrm{T}}\big(\gamma^2 X_p^{-1} - S_d^{\mathrm{T}} X_d S_d\big) x_0 < 0 \end{aligned} \tag{21}$$

for any $w(t) \in \mathcal{L}_2(-\infty, \infty)$. This implies that $\hat{\gamma}_{\mathrm{sd-dif}}(x_0) < \gamma$.

(ii) Necessity: (ii-1) Conditions (a) and (b): Define the $\mathcal{L}_2$ gain of H_d by

$$\gamma_d = \sup_{w \in \mathcal{L}_2\,(t_0,\,\infty) \setminus \{0\}} \frac{\|z_d(t)\|_{2\,(t_0,\,\infty)}}{\|w(t)\|_{2\,(t_0,\,\infty)}}, \tag{22}$$

where $z_d(t) \in \mathcal{L}_2(t_0, \infty)$ is the output of H_d against $w(t) \in \mathcal{L}_2(t_0, \infty)$ under the condition that $x_d(t_{0+}) = 0$. Suppose that Condition (a) and/or (b) is not satisfied. Then, it hold that $\gamma_d \geq \gamma$. Define $\epsilon_1 = \gamma_d - \gamma\,(\geq 0)$. Let $w_{f\infty}(t)$ $(t > t_0)$ denote an input providing the value of γ_d under $x_d(t_{0+}) = 0$. Let $z_{d\infty}(t)$ denote the output against $w_{d\infty}(t)$ under $x_d(t_{0+}) = 0$. We consider the input $\tilde{w}(t) \in \mathcal{L}_2(-\infty, \infty)$ given by

$$\tilde{w}(t) = \begin{cases} w_{p0}(t), & t \leq t_0 \\ \eta \cdot w_{d\infty}(t), & t > t_0, \end{cases} \tag{23}$$

where $\eta > 0$, and $w_{p0}(t)$ $(t \leq t_0)$ realizes $x_p(t_0) = x_0$. Then, the output against $\tilde{w}(t)$ is given by

$$\tilde{z}(t) = C_d e^{A_d(t - t_0)} S_d x_0 + \eta \cdot z_{d\infty}(t), \quad t > t_0. \tag{24}$$

We thus have

$$\begin{aligned} &\hat{\gamma}_{\text{sd-dif}}(x_0) \\ &\geq \frac{\|\tilde{z}(t)\|_{2\,(t_0,\,\infty)}}{\|\tilde{w}(t)\|_{2\,(-\infty,\,\infty)}} \\ &\geq \frac{\eta\,\|z_{d\infty}(t)\|_{2\,(t_0,\,\infty)} - \|C_d e^{A_d(t-t_0)} S_d x_0\|_{2\,(t_0,\,\infty)}}{\|w_{p0}(t)\|_{2\,(-\infty,\,t_0]} + \eta\,\|w_{d\infty}(t)\|_{2\,(t_0,\,\infty)}} \\ &= \gamma_d - \epsilon_2, \end{aligned} \tag{25}$$

where $\epsilon_2 \geq 0$. By choosing η sufficiently large, we can achieve $\epsilon_2 \leq \epsilon_1$. This implies $\hat{\gamma}_{\text{sd-dif}}(x_0) \geq \gamma$. Thus, $\hat{\gamma}_{\text{sd-dif}}(x_0) < \gamma$ does not hold.

(ii-2) Condition (c): Suppose that Condition (c) is not satisfied. Consider the following input:

$$\hat{w}(t) = \begin{cases} B_p^{\mathrm{T}} e^{-A_p^{\mathrm{T}}(t - t_0)} X_p^{-1} x_0, & t \leq t_0 \\ K_d e^{(A_d + B_d K_d)(t - t_0)} S_d x_0, & t > t_0, \end{cases} \tag{26}$$

where

$$K_d = (\gamma^2 I - D_d^{\mathrm{T}} D_d)^{-1} (X_d B_d + C_d^{\mathrm{T}} D_d)^{\mathrm{T}}. \tag{27}$$

Here, $-A_p^{\mathrm{T}}$ is anti-stable by Assumption 2.1 (a); $A_d + B_d K_d$ is stable because X_d is the stabilizing solution to the Riccati equation (12). Thus, $\hat{w}(t)$ belongs to $\mathcal{L}_2(-\infty, \infty)$.

Since on the pre-switch side, the controllability Gramian $X_p = \int_0^{\infty} e^{A_p t} B_p B_p^{\mathrm{T}} e^{A_p^{\mathrm{T}} t} dt \succ O$ is the solution to the Lyapunov equation (14), we have

$$\|\hat{w}(t)\|^2_{2\,(-\infty,\,t_0]} = x_0^{\mathrm{T}} X_p^{-1} x_0. \tag{28}$$

Furthermore, using Assumption 2.1 (c), we have $x_p(t_0) = x_0$.

On the post-switch side, using K_d given in (27) and Condition (b), we have

$$\begin{aligned} &\begin{bmatrix} X_d A_d + A_d^{\mathrm{T}} X_d + W_{11} & X_d B_d + W_{12} \\ B_d^{\mathrm{T}} X_d + W_{12} & W_{22} \end{bmatrix} \\ &= \begin{bmatrix} I & O \\ -K_d & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} O & O \\ O & I \end{bmatrix} \begin{bmatrix} I & O \\ -K_d & I \end{bmatrix}. \end{aligned} \tag{29}$$

Here, the input $\hat{w}(t)$ and corresponding $\hat{x}_d(t)$ with the initial condition $x_d(t_{0+}) = S_d x_0$ satisfy $\hat{w}(t) = K_d \hat{x}_d(t)$, $t > t_0$. We then have

$$\begin{bmatrix} I & O \\ -K_d & I \end{bmatrix} \begin{bmatrix} \hat{x}_d(t) \\ \hat{w}(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_d(t) \\ 0 \end{bmatrix}, \quad t > t_0. \tag{30}$$

By applying (29) and (30) to (19), we have that the inequality (19) has equality; thus, the inequality (20) also has equality as

$$\|\hat{z}(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2\|\hat{w}(t)\|^2_{2\,(t_0,\,\infty)} = (S_d x_0)^{\mathrm{T}} X_d S_d x_0, \quad (31)$$

where $\hat{z}(t)$ is the output corresponding to $\hat{w}(t)$.

Multiplying (28) by $-\gamma^2$ and adding it with (31), we have

$$\begin{aligned} &\|\hat{z}(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2\|\hat{w}(t)\|^2_{2\,(-\infty,\,\infty)} \\ &= -\, x_0^{\mathrm{T}}(\gamma^2 X_p^{-1} - S_d^{\mathrm{T}} X_d S_d)x_0. \end{aligned} \quad (32)$$

Since Condition (c) is not satisfied, $\|\hat{z}(t)\|^2_{2\,(t_0,\,\infty)} - \gamma^2\|\hat{w}(t)\|^2_{2\,(-\infty,\,\infty)} \geq 0$. Therefore, $\hat{\gamma}_{\text{sd-dif}}(x_0) < \gamma$ does not hold. □

Let $x_0 \neq 0$. Suppose that Conditions (a) and (b) in Theorem 2.4 are satisfied, and $x_0^{\mathrm{T}}(\gamma^2 X_p^{-1} - S_d^{\mathrm{T}} X_d S_d)x_0 = 0$. Then, $\hat{\gamma}_{\text{sd-dif}}(x_0) = \gamma$. Furthermore, an input providing the value of $\hat{\gamma}_{\text{sd-dif}}(x_0)$ is given by (26).

Also, it follows from Conditions (a) and (b) in Theorem 2.4 that for any $x_0(\neq 0) \in \mathbb{R}^{n_p}$, the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$ satisfies

$$\hat{\gamma}_{\text{sd-dif}}(x_0) \geq \gamma_d. \quad (33)$$

2.5 LMI-based $\mathcal{L}_2$ gain condition

The LMI conditions in the theorem presented below will play an essential role in the proposed initial state design.

Theorem 2.5. Let $\gamma > 0$ and $x_0\,(\neq 0) \in \mathbb{R}^{n_p}$. The state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$ satisfies $\hat{\gamma}_{\text{sd-dif}}(x_0) < \gamma$ if and only if there exist $\tilde{X}_p \succ O$ and $\tilde{X}_d \succ O$ satisfying the following conditions:

$$\begin{bmatrix} \tilde{X}_p A_p + A_p^{\mathrm{T}}\tilde{X}_p & \tilde{X}_p B_p \\ B_p^{\mathrm{T}}\tilde{X}_p & -\gamma I \end{bmatrix} \prec O \quad (34)$$

$$\begin{bmatrix} \tilde{X}_d A_d^{\mathrm{T}} + A_d\tilde{X}_d & B_d & \tilde{X}_d C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}} & -\gamma I & D_d^{\mathrm{T}} \\ C_d\tilde{X}_d & D_d & -\gamma I \end{bmatrix} \prec O \quad (35)$$

$$\begin{bmatrix} x_0^{\mathrm{T}}\tilde{X}_p x_0 & x_0^{\mathrm{T}} S_d^{\mathrm{T}} \\ S_d x_0 & \tilde{X}_d \end{bmatrix} \succ O. \quad (36)$$

Proof: The proof of the sufficiency is entirely analogous to that in Theorem 2.4. Thus, we only prove the necessity by obtaining $\tilde{X}_p \succ O$ and $\tilde{X}_d \succ O$ satisfying (34)–(36) from Conditions (a)–(c) in Theorem 2.4.

On the pre-switch side, it follows from Assumption 2.1 (a) that there exists $X'_p \succ O$ satisfying

$$A_p X'_p + X'_p A_p^{\mathrm{T}} \prec O. \quad (37)$$

Multiplying (37) by $\epsilon > 0$ and adding it with (14), we have

$$A_p(X_p + \epsilon X'_p) + (X_p + \epsilon X'_p)A_p^{\mathrm{T}} + B_p B_p^{\mathrm{T}} \prec O, \quad (38)$$

where $X_p + \epsilon X'_p \succ O$. Left- and right-multiplying (38) by $(X_p + \epsilon X'_p)^{-1} \succ O$ and using the Schur complement, we have

$$\begin{bmatrix} (X_p + \epsilon_p X'_p)^{-1}A_p + A_p^{\mathrm{T}}(X_p + \epsilon_p X'_p)^{-1} & (X_p + \epsilon_p X'_p)^{-1}B_p \\ B_p^{\mathrm{T}}(X_p + \epsilon_p X'_p)^{-1} & -I \end{bmatrix} \prec O. \quad (39)$$

Multiplying (39) by $\gamma > 0$ and defining $\tilde{X}_p \succ O$ by

$$\tilde{X}_p = \gamma(X_p + \epsilon X'_p)^{-1}, \quad (40)$$

we can have (34).

By using $\tilde{X}_p$ given in (40) with sufficiently small $\epsilon > 0$, we can have

$$x_0^{\mathrm{T}}(\gamma\tilde{X}_p - S_d^{\mathrm{T}} X_d S_d)x_0 > 0 \quad (41)$$

from Condition (c) in Theorem 2.4. Consider the following Riccati matrix inequality related to (12):

$$\begin{aligned} &X'_d A_d + A_d^{\mathrm{T}} X'_d + C_d^{\mathrm{T}} C_d + (X'_d B_d + C_d^{\mathrm{T}} D_d) \\ &\quad \times (\gamma^2 I - D_d^{\mathrm{T}} D_d)^{-1}(X'_d B_d + C_d^{\mathrm{T}} D_d)^{\mathrm{T}} \prec O \end{aligned} \quad (42)$$

Since there exists the stabilizing solution $X_d \succ O$ to (12), there exists a solution $X'_d \succ O$ to (42). Furthermore, $X_d \succ O$ is the minimum solution, i.e., $X'_d \succeq X_d$ (Zhou *et al.*, 1996). Since $\tilde{X}_p$ satisfying (34) and X_f also satisfy (41), there exists X'_d such that

$$x_0^{\mathrm{T}}(\gamma\tilde{X}_p - S_d^{\mathrm{T}} X'_d S_d)x_0 > 0. \quad (43)$$

We take $\tilde{X}_d \succ O$ as

$$\tilde{X}_d = \frac{1}{\gamma}X'_d. \quad (44)$$

Then, from (42) we can have

$$\begin{bmatrix} \tilde{X}_d A_d + A_d^{\mathrm{T}}\tilde{X}_d & \tilde{X}_d B_d & C_d^{\mathrm{T}} \\ B_d^{\mathrm{T}}\tilde{X}_d & -\gamma I & D_d^{\mathrm{T}} \\ C_d & D_d & -\gamma I \end{bmatrix} \prec O. \quad (45)$$

Furthermore, by using (44), from (43) we can have

$$x_0^{\mathrm{T}}(\tilde{X}_p - S_d^{\mathrm{T}}\tilde{X}_d S_d)x_0 > 0. \quad (46)$$

Rewriting $\tilde{X}_d^{-1}$ by $\tilde{X}_d$ and using the Schur complement, we can have (35) and (36) from (45) and (46). □

3. INITIAL STATE DESIGN

3.1 Problem statement

As shown in Fig. 1, we consider the pre-switch system H_p and post-switch system H_f described in the linear fractional transformation (LFT) framework. Here, w is the exogenous input, z is the evaluation output, u is the control input, and y is the measured output. The generalized plant G is not switched. Let $x_g(t) \in \mathbb{R}^{n_g}$ denote its state-variable vector. On the other hand, the controller is switched from the operating K_p to a more desirable K_f at $t = t_0$ in some sense, such as performance and fault tolerance. Let $x_{k_p}(t) \in \mathbb{R}^{n_{k_p}}$ and $x_{k_f}(t) \in \mathbb{R}^{n_{k_f}}$ denote the state-variable vectors of K_p and K_f, respectively. Then, by taking $x_p(t) = \begin{bmatrix} x_g^{\mathrm{T}}(t) & x_{k_p}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, the pre-switch system H_p with the output can be represented as follows:

$$H_p : \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] = \mathcal{F}_\ell(G, K_p). \quad (47)$$

Furthermore, by taking $x_f(t) = \begin{bmatrix} x_g^{\mathrm{T}}(t) & x_{k_f}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, the post-switch system H_f can be represented as follows:

$$H_f : \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] = \mathcal{F}_\ell(G, K_f). \quad (48)$$

We assume that H_p and H_f obtained above satisfy Assumptions 2.1 and 2.2, respectively.

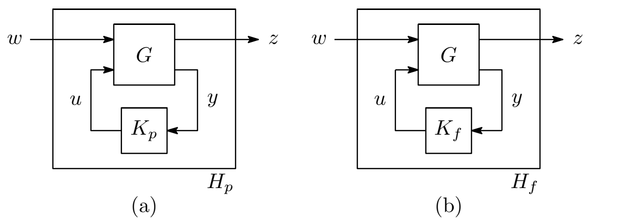

Fig. 1. (a) Pre-switch system and (b) post-switch system in a controller switch.

Since $x_g(t_{0+}) = x_g(t_0)$, the switching matrix is of the following form:

$$S = \begin{bmatrix} I & O \\ S_1 & S_2 \end{bmatrix}, \tag{49}$$

where $S_1 \in \mathbb{R}^{n_{k_f} \times n_g}$ and $S_2 \in \mathbb{R}^{n_{k_f} \times n_{k_p}}$. Then, we have

$$x_f(t_{0+}) = \begin{bmatrix} x_g(t_0) \\ x_{k_f}(t_{0+}) \end{bmatrix} = \begin{bmatrix} x_g(t_0) \\ S_1 x_g(t_0) + S_2 x_{k_p}(t_0) \end{bmatrix}. \tag{50}$$

The initial state design problem considered in this paper is as follows.

Problem 3.1. (Initial state design problem).
For given G, K_p, K_f, and $x_0 = x_p(t_0)$, find the optimal $x_{k_f}(t_{0+})$ that minimizes the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$.

For a given switching situation x_0, this problem provides the optimal initial state of K_f for suppressing the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur. Note that in this problem, the switching matrix S (i.e., S_1 and S_2) is not determined.

On the other hand, we can consider the following problem for assigning the initial state of a newly-activated controller.

Problem 3.2. (Switching matrix design problem).
For given G, K_p, and K_f, find the optimal S_1 and S_2 in (49) that minimizes the switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{dif}}$.

In this problem, the initial state of a newly-activated controller is assigned by $x_{k_f}(t_{0+}) = S_1 x_g(t_0) + S_2 x_{k_p}(t_0)$. This problem focuses on the worst switching situation for the controller switch in the sense of the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$. Thus, it is clear that by Problem 3.1 according to the switching situation, we can more effectively assign the initial state to suppress the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur.

3.2 Proposed initial state design procedure

Consider Problem 3.1. Defining

$$x'_{01} = \begin{bmatrix} 0 \\ x_{k_f}(t_{0+}) \\ 0 \end{bmatrix}, \qquad x'_{02} = \begin{bmatrix} x_g(t_0) \\ 0 \\ x_0 \end{bmatrix}, \tag{51}$$

where $x_{k_f}(t_{0+})$ is the initial state of the newly-activated controller K_f to be obtained, we then have $S_d x_0 = x'_{01} + x'_{02}$. Thus, the LMI (36) is equivalent to

$$\begin{bmatrix} x_0^{\mathrm{T}} \tilde{X}_p x_0 & x'^{\mathrm{T}}_{01} + x'^{\mathrm{T}}_{02} \\ x'_{01} + x'_{02} & \tilde{X}_d \end{bmatrix} \succ O. \tag{52}$$

We can then treat x'_{01} as an LMI variable. Thus, by solving the LMI problem under the constraint conditions (34), (35), and (52) to minimize γ by using the variables $\tilde{X}_p$, $\tilde{X}_d$, and x'_{01}, we can obtain the optimal $x_{k_f}(t_{0+})$ in x'_{01}.

Remark 3.3. We can solve Problem 3.2 in an entirely analogous fashion. Defining

$$S_{d1} = \begin{bmatrix} O & O \\ S_1 & S_2 \\ O & O \\ O & O \end{bmatrix}, \qquad S_{d2} = \begin{bmatrix} I & O \\ O & O \\ I & O \\ O & I \end{bmatrix}, \tag{53}$$

we can decompose S_d as $S_d = S_{d1} + S_{d2}$. Then, by solving the LMI problem under the constraint conditions (34), (35), and

$$\begin{bmatrix} \tilde{X}_p & S_{d1}^{\mathrm{T}} + S_{d2}^{\mathrm{T}} \\ S_{d1} + S_{d2} & \tilde{X}_d \end{bmatrix} \succ O \tag{54}$$

to minimize γ by using the variables $\tilde{X}_p$, $\tilde{X}_d$, and S_{d1}, we can obtain the optimal S_1 and S_2 in S_{d1}.

3.3 Features

We can easily show that for any $x_0\,(\neq 0) \in \mathbb{R}^{n_p}$ and $k\,(\neq 0) \in \mathbb{R}$,

$$\hat{\gamma}_{\text{sd-dif}}(x_0) = \hat{\gamma}_{\text{sd-dif}}(k \cdot x_0). \tag{55}$$

Thus, the value of the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$ depends only on the direction in the state space $x_p \in \mathbb{R}^{n_p}$.

Suppose that for a given switching situation $x_0\,(\neq 0) \in \mathbb{R}^{n_p}$, we obtain the optimal initial state $x'_0 \in \mathbb{R}^{n_p+n_f}$ by solving Problem 3.1. Then, for a switching situation $k \cdot x_0$ ($k \neq 0$), the optimal initial state is $k \cdot x'_0$. This implies that the proposed initial state design procedure provides a mapping from a direction in the switching situation $x_0 \in \mathbb{R}^{n_p}$ to a direction in $x_{k_f} \in \mathbb{R}^{n_{k_f}}$.

For a given switching situation $x_0 = x_p(t_0)$, the proposed initial state design provides the optimal initial state $x_{k_f}(t_{0+})$ for suppressing the difference between the actuality that this controller switch occurs and the virtual situation where it does not occur in the sense of the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$. It has the following advantages.

(a) The switching time does not affect its design result, and need not be known in advance.
(b) It needs only solving an LMI problem. That is, complicated calculation is not necessary.
(c) It is not necessary to discuss the solvability, because we can always obtain the solution.

Conversely, the proposed initial state design cannot be applied to tracking control to a specified reference signal. For example, in a servo system for a stepped reference signal, by considering the fluctuations in the error against a controller switch in the steady state, the undesirable effects of the switch can be suppressed by the proposed design. Thus, by using it, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017). However, the virtual output is not always the reference signal to make the actual output track. The proposed design cannot provide an initial state for suppressing the fluctuations in tracking error for a specified reference signal.

Remark 3.4. We should use the initial state design and switching matrix design depending on the situation. For a pre-planned controller switch in the steady state, the proposed initial state design is more efficient for suppressing the undesirable effects of the switch. However, for a non-preplanned controller switch under a violent system fluctuations, the switching matrix design can provide a more desirable initial state.

3.4 Numerical example

The generalized plant G, operating controller K_p, and newly-activated controller K_f are given as

$$G : \left[\begin{array}{cc|c:c} -14 & -2 & 4 & -4 \\ 5 & -9 & -2 & -1 \\ \hline 4 & -12 & 0 & 0 \\ \hdashline -1 & -13 & 0 & 0 \end{array}\right] \tag{56}$$

$$K_p : \left[\begin{array}{cc|c} -5.61 & 2925.70 & 5.39 \\ 926.26 & -4190.34 & 326.88 \\ \hline 4.00 & -719.97 & 0 \end{array}\right] \tag{57}$$

$$K_f : \left[\begin{array}{cc|c} -73.63 & 66562.80 & 73.54 \\ 11257.06 & -726.36 & 2837.09 \\ \hline 86.54 & -16015.40 & 0 \end{array}\right]. \tag{58}$$

Consider the $\mathcal{L}_2$ gain as the performance index. Table 1 shows the performance index values of H_p and H_f; the values imply that this controller switch improves the control performance. Note that $\gamma_d = 0.5962$ is the lower bound of the switching $\mathcal{L}_2$ gains $\hat{\gamma}_{\text{sd-dif}}(x_0)$ and $\hat{\gamma}_{\text{dif}}$.

Table 1. Performance index value.

	$\mathcal{L}_2$ gain
Pre-switch system H_p	1.1077
Post-switch system H_f	0.5635
Difference system H_d	0.5962

Suppose that a controller switch from K_p to K_f occurs at $t = t_0$ and

$$x_0 = x_p(t_0) = \begin{bmatrix} x_g(t_0) \\ x_{k_p}(t_0) \end{bmatrix} = \left[\begin{array}{c} 0.8614 \\ -0.0440 \\ \hdashline 0.4980 \\ 0.0903 \end{array}\right]. \tag{59}$$

In order to obtain the initial state $x_{k_f}(t_{0+})$ of the newly-activated controller K_f, we consider the following three design:

- Proposed initial state design (Problem 3.1)
- Switching matrix design (Problem 3.2)
- Zero initial state.

Table 2 shows the initial state design results. Note that in the switching matrix design, we first obtain the following switching matrix:

$$S_1 = \begin{bmatrix} 0.2920 & 1.1301 \\ -0.0074 & -0.0666 \end{bmatrix}$$
$$S_2 = \begin{bmatrix} 0.4558 & -0.0028 \\ 0.0216 & -0.0018 \end{bmatrix}. \tag{60}$$

The proposed initial state design directly provides the optimal initial state $x_{k_f}(t_{0+})$ of the newly-activated controller K_f in the sense of the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$. That is, the proposed initial state design minimizes $\hat{\gamma}_{\text{sd-dif}}(x_0)$ to obtain the value 0.5991. On the other hand, the switching matrix design minimizes $\hat{\gamma}_{\text{dif}}(x_0)$ to obtain the value 0.7958.

Table 2. Initial state design results.

	Proposed initial state design	Switching matrix design	Zero initial state
$x_{k_f}(t_{0+})$	$\begin{bmatrix} 0.5025 \\ 0.0592 \end{bmatrix}$	$\begin{bmatrix} 0.4285 \\ 0.0071 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\hat{\gamma}_{\text{sd-dif}}(x_0)$	0.5991	0.6129	1.1004
$\hat{\gamma}_{\text{dif}}$	——	0.7958	4.6482

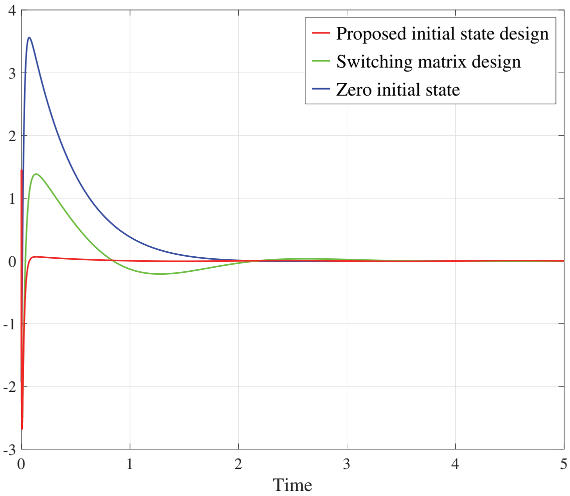

Fig. 2. Responses in $z(t)-z_{\text{vir}}(t)$ against $\hat{w}(t)$ providing the value of $\hat{\gamma}_{\text{sd-dif}}(x_0)$ normalized as $\|\hat{w}(t)\|_{2\,(0,\infty)} = 1$.

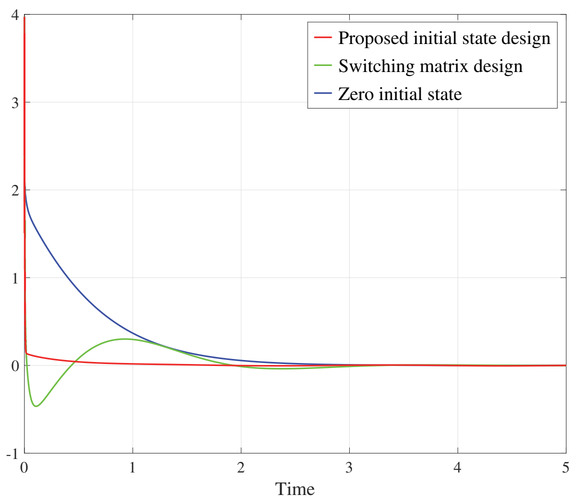

Fig. 3. Actual output $z(t)$ against $\hat{w}(t)$ providing the value of $\hat{\gamma}_{\text{sd-dif}}(x_0)$ normalized as $\|\hat{w}(t)\|_{2\,(0,\infty)} = 1$.

We present the simulation results. We suppose without loss of generality that the controller switch occurs at $t = 0$, i.e., $t_0 = 0$. We can obtain an input $\hat{w}(t)$ given in (26) that provides the value of $\hat{\gamma}_{\text{sd-dif}}(x_0)$ in each case, where it is normalized as $\|w(t)\|_{2\,(0,\infty)} = 1$. Figures 2 and 3 show the responses in $z(t) - z_{\text{vir}}(t)$ and $z(t)$, respectively, after the controller switch against the obtained input $\hat{w}(t)$ in the proposed initial state design case, switching matrix design case, and zero initial state case. As shown in Fig. 2, in the proposed initial state design case, even the worst fluctuation in $z(t) - z_{\text{vir}}(t)$ (i.e., the largest difference between the actuality that this controller switch occurs and the virtual situation where it does not occur) is well suppressed in comparison with the other cases.

This example implies that for given G, K_p, K_f, and x_0, the difference between the actuality and virtual situation can be well suppressed by appropriately designing the initial state $x_{k_f}(t_{0+})$ of K_f according to the situation at a switching time, i.e., x_0. The effectiveness of the initial state design by using the state-dependent switching $\mathcal{L}_2$ gain $\hat{\gamma}_{\text{sd-dif}}(x_0)$ shows its potential applicability as a design index.

4. CONCLUSIONS

We have proposed a new initial state design procedure for a newly-activated controller at a controller switch. By minimizing the value of the state-dependent switching $\mathcal{L}_2$ gain presented in this paper, we can obtain the optimal initial state for suppressing the difference between the actuality that a controller switch occurs and the virtual situation where it does not occur. By the proposed initial state design, we can improve the safety of the operating-state transitions in the procedure of safe preventive maintenance of control systems presented in Suyama and Sebe (2017).

However, the virtual output is not always the reference signal to make the actual output track. The proposed design cannot provide an initial state for suppressing the fluctuations in tracking error for a specified reference signal. This is an important issue to be solved/improved in future.

Khargonekar *et al.* (1991) introduced the worst-case norm of the regulated output over all exogenous inputs and initial states as a performance measure. Balandin and Kogan (2010) obtained optimal time-invariant controllers by minimizing the measure by using LMIs. Another important future work is to clarify the relationship between the proposed initial state design and such $\mathcal{H}_\infty$ controller design with transients.

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