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Distributed Feedback Control on the SIS Network Model: An Impossibility Result ⋆

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Abstract: This paper considers the deterministic Susceptible-Infected-Susceptible (SIS) epidemic network model, over strongly connected networks. It is well known that there exists an endemic equilibrium (the disease persists in all nodes of the network) if and only if the effective reproduction number of the network is greater than 1. In fact, the endemic equilibrium is unique and is asymptotically stable for all feasible nonzero initial conditions. We consider the recovery rate of each node as a control input. Using results from differential topology and monotone systems, we establish that it is impossible for a large class of distributed feedback controllers to drive the network to the healthy equilibrium (where every node is disease free) if the uncontrolled network has a reproduction number greater than 1. In fact, a unique endemic equilibrium exists in the controlled network, and it is exponentially stable for all feasible nonzero initial conditions. We illustrate our impossibility result using simulations, and discuss the implications on the problem of control over epidemic networks.

Keywords: deterministic epidemic models, Susceptible-Infected-Susceptible (SIS) model, complex networks, control of networked systems, differential topology, monotone systems

1. INTRODUCTION

The control community has recently increased its attention on the mathematical modelling of disease outbreaks in a large population, as it is a fundamental issue in epidemiology and public health studies (Anderson and May, 1991; Nowzari et al., 2016). The salient behaviour of many epidemic models can be characterised by an effective reproduction number, $\mathcal{R}_0$ (the quantitative definition of which may depend on the model). Roughly speaking, the disease is eventually eradicated from the system if $\mathcal{R}_0 \leq 1$, but will persist if $\mathcal{R}_0 > 1$. Since experiments in epidemics are usually expensive and impossible for large human networks, mathematical modelling and analysis can be an economical and effective approach to understand an epidemic process. Then, the motivation to study control techniques for epidemic models becomes clear.

The Susceptible-Infected-Susceptible (SIS) model supposes that each individual in the population is either Infected with a disease of interest, or Susceptible but not Infected, and able to transition between the two states (Pastor-Satorras and Vespignani, 2001; Mieghem et al., 2009). It is a fundamental model, having both probabilistic (Fagnani and Zino, 2017) and deterministic variants (Lajmanovich and Yorke, 1976; Fall et al., 2007; Mieghem et al., 2009). This paper considers the deterministic variant, which has at least two perspectives: (i) a disease spreading on a network of interconnected individuals, or (ii) a disease spreading across a network of interconnected populations, viz. *a metapopulation* (each node represents one well-mixed population). To simplify our exposition, we focus on the metapopulation narrative, but our theoretical conclusions also hold for the first perspective.

On strongly connected SIS networks, $\mathcal{R}_0$ uniquely determines the limiting behaviour of the disease (Fall et al., 2007). If $\mathcal{R}_0 \leq 1$, the healthy equilibrium (also the unique equilibrium) is asymptotically stable for all feasible [1] initial conditions.

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[1] By feasible, we mean initial conditions which are meaningful and of interest in the epidemic context.

tial conditions; the disease is eradicated from every node. If $\mathcal{R}_0 > 1$, then in addition to the healthy equilibrium, there is a unique endemic equilibrium in which the disease is present in each node, and it is asymptotically stable for all feasible nonzero initial conditions.

A number of works have investigated different control approaches. A common, centralised, approach is to formulate an optimisation problem to minimise $\mathcal{R}_0$ given a set of constraints and knowledge of all network parameters, either by (i) setting network parameters such as the recovery rate of each node or the infection rates between nodes (Preciado et al., 2014; Ramírez-Llanos and Martínez, 2014; Torres et al., 2016), or (ii) by removing certain nodes or links in the network (Bishop and Shames, 2011). A recent method avoids this issue, but requires a synchronised stopping time across the network and/or additional consensus process to compute a piece of centralised information (Mai et al., 2018). Optimal control approaches remain a considerable open challenge (Nowzari et al., 2016).

Alternatively, heuristic feedback controllers may be used to dynamically adjust node parameters, often in a distributed manner relying only on local node information: it is this approach that this paper will investigate. A key issue is that the feedback controller changes the closed-loop dynamics, and consequently one must investigate whether there still exists a unique endemic equilibrium, and whether the convergence behaviour has changed. This can make analysis difficult since most existing techniques rely on specialised algebraic calculations of the model equations, e.g. (Fall et al., 2007; Lajmanovich and Yorke, 1976; Mieghem et al., 2009), meaning conclusions might be specific to the controller. Existing results are limited. Indeed, (Liu et al., 2019) establishes an impossibility result on the SIS network model for a specific feedback controller.

We resolve this issue by presenting a novel unified analysis framework based on the Poincaré–Hopf Theorem from differential topology (Milnor, 1997), and monotone dynamical systems (Smith, 1988). We analyse a broad class of distributed feedback algorithms which control the recovery rate of each node. We show that if $\mathcal{R}_0 > 1$ for the underlying uncontrolled network, then it is impossible for the the controlled network to reach the healthy equilibrium. It is proved that from all nonzero feasible initial conditions, the controlled network converges to the unique endemic equilibrium exponentially fast. However, we show feedback control will always control the unique endemic equilibrium to be closer to the healthy equilibrium. Our results highlight the challenges of feedback control for SIS networks, while leaving the door open for other approaches such as controlling the infection rates between nodes, or time-varying or nonsmooth or adaptive controllers.

We conclude this section by introducing notation, and relevant aspects of graph theory. Section 2 introduces the SIS model, and defines the control problem, while Section 3 establishes the main result. Section 4 provides discussions and a simulation, and conclusions are drawn in Section 5.

1.1 Notation

The n-column vector of all ones and zeros is given by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively. The $n \times n$ identity and $n \times m$ zero matrices are given by I_n and $\mathbf{0}_{n\times m}$, respectively. The i^{th} entry of a vector a and $(i,j)^{th}$ entry of a matrix A are a_i and a_{ij}, respectively. For vectors $a, b \in \mathbb{R}^n$, we write $a \geq b$ and $a > b$ if $a_i \geq b_i$ and $a_i > b_i$, respectively, for all i. A matrix $A \in \mathbb{R}^{n\times m}$ is nonnegative or positive if $A \geq \mathbf{0}_{n\times m}$ or $A > \mathbf{0}_{n\times m}$, respectively. For a real square matrix M with spectrum $\sigma(M)$, define $\rho(M) = \max\{|\lambda| : \lambda \in \sigma(M)\}$ and $s(M) = \max\{\text{Re}(\lambda) : \lambda \in \sigma(M)\}$ as the spectral radius of M and the largest real part among the eigenvalues of M, respectively. A matrix M is said to be *Hurwitz* if $s(M) < 0$. A matrix A is called an M-matrix if it can be written as $A = cI_n - B$, with $c > 0$, $B \geq \mathbf{0}_{n\times n}$ and $c \geq \rho(B)$ (Berman and Plemmons, 1979)

For a set $\mathcal{M}$ with boundary, we denote the boundary as $\partial\mathcal{M}$, and the interior $\text{Int}(\mathcal{M}) \triangleq \mathcal{M} \setminus \partial\mathcal{M}$. We denote by

$$\mathbb{R}^n_{\geq 0} = \{x : x_i \geq 0\,, \forall\, i = 1, \ldots, n\}$$

the positive orthant. We define the set

$$\Xi_n = \{x \in \mathbb{R}^n_{\geq 0} : 0 \leq x_i \leq 1, i \in \{1, \ldots, n\}\}.$$

1.2 Graph Theory

For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, $\mathcal{V} = \{1, \ldots, n\}$ is the set of vertices (or nodes). The set of ordered edges is given by $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and the edge $e_{ij} = (v_i, v_j)$ is said to be incoming with respect to v_j and outgoing with respect to v_i. The matrix $A \geq \mathbf{0}_{n\times n}$ is the weighted adjacency matrix, defined such that $e_{ij} \in \mathcal{E}$ if and only if $a_{ji} > 0$. We will sometimes write "the matrix A associated with $\mathcal{G}$", or write $\mathcal{G}[A]$ to denote $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. We define the neighbour set of v_i as $\mathcal{N}_i \triangleq \{v_j : e_{ji} \in \mathcal{E}\}$. A directed path is a sequence of edges of the form $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), \ldots$, where $v_{p_i} \in \mathcal{V}$ are distinct and $e_{p_i p_{i+1}} \in \mathcal{E}$. A graph $\mathcal{G}[A]$ is *strongly connected* if and only if there is a path from every node to every other node, which is equivalent to A being *irreducible* (Berman and Plemmons, 1979).

2. THE SIS MODEL AND CONTROL PROBLEM

This section will introduce the SIS network model, and then formulate the feedback control problem.

2.1 The Deterministic SIS Network Model

The network Susceptible-Infected-Susceptible model is fundamental within the deterministic epidemic modelling literature. To keep the paper concise and focused on the feedback control problem, we refer the interested reader to (Lajmanovich and Yorke, 1976; Nowzari et al., 2016) for details on modelling derivations.

As explained in the introduction, this SIS model has at least two popular contexts under which it is studied, and we focus on the metapopulation context. Each individual resides in a well-mixed population, with the size of each population being large and constant. It is assumed that each individual is either Infected (I) with some disease of interest, or is Susceptible (S) but not infected. Each individual can transition between the two states. There is a network of $n \geq 2$ such populations (forming a metapopulation), captured by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$, with each node representing a population. Associated with node $i \in \mathcal{V}$ is the variable $x_i(t) \in [0, 1]$, which represents the proportion of population i that is Infected (and thus $1 - x_i(t)$ represents the proportion of population i that is Susceptible). The SIS dynamics are given by

$$\dot{x}_i(t) = -d_i x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} b_{ij} x_j(t), \qquad (1)$$

where $d_i > 0$ is called the recovery rate of node i, and for a node $j \in \mathcal{N}_i$, $b_{ij} > 0$ is called the infection rate from node j to node i. If $j \notin \mathcal{N}_i$, then $b_{ij} = 0$. Defining $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ yields

$$\dot{x}(t) = (-D + B - X(t)B)x(t), \qquad (2)$$

with $X(t) = \text{diag}(x_1(t), \ldots, x_n(t))$, $D = \text{diag}(d_1, \ldots, d_n)$ being diagonal matrices. The nonnegative matrix B is associated with the graph $\mathcal{G}$. One can prove that if $x(0) \in \Xi_n$, then $x(t) \in \Xi_n \, \forall t \geq 0$. Thus, Eq. (2) is well defined and $x(t)$ has a physically consistent meaning for all $x(0) \in \Xi_n$. From the modelling context, we say an initial condition $x(0)$ is feasible if $x(0) \in \Xi_n$, and from here on, we only consider feasible initial conditions.

Obviously, $x = \mathbf{0}_n$ is an equilibrium of Eq. (2), and we call this the healthy equilibrium. Any other equilibrium $x^* \in \Xi_n \setminus \mathbf{0}_n$ is said to be an endemic equilibrium, as the disease persists in at least one population. Consistent with the literature, we define

$$\mathcal{R}_0 \triangleq \rho(D^{-1}B) \qquad (3)$$

as the effective reproduction number of the disease on the network. The following result fully characterises the number of equilibria and the limiting behaviour of Eq. (2) using $\mathcal{R}_0$, with different proofs in (Lajmanovich and Yorke, 1976; Fall et al., 2007; Mieghem et al., 2009).

Proposition 1. Consider the system Eq. (2), and suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ is strongly connected.

(1) If $\mathcal{R}_0 \leq 1$, then $\mathbf{0}_n$ is the unique equilibrium of Eq. (2), and for all $x(0) \in \Xi_n$, $\lim_{t\to\infty} x(t) = \mathbf{0}_n$.
(2) If $\mathcal{R}_0 > 1$, then in addition to the equilibrium $\mathbf{0}_n$, there is a unique endemic equilibrium $x^* \in \text{Int}(\Xi_n)$. Moreover, $\lim_{t\to\infty} x(t) = x^*$ for all $x(0) \in \Xi_n \setminus \mathbf{0}_n$.

Proposition 1 states that $\mathcal{R}_0 \leq 1$ is equivalent to the disease eventually being eradicated, since $\lim_{t\to\infty} x(t) = \mathbf{0}_n$ for all $x(0) \in \Xi_n$. There is an endemic equilibrium if and only if $\mathcal{R}_0 > 1$, and then in fact $x^* \in \text{Int}(\Xi_n)$ is the unique endemic equilibrium that is *asymptotically stable for all feasible nonzero initial conditions* (the Jacobian of Eq. (2) at x^* is Hurwitz).

2.2 Problem Formulation: Distributed Feedback Control

From the conclusions of Proposition 1, it is obviously of interest in the epidemic spreading context to develop control methods to drive the SIS networked system Eq. (2) to the healthy equilibrium $\mathbf{0}_n$ when $\mathcal{R}_0 > 1$. In the introduction, we detailed several existing approaches for controlling the SIS networked system Eq. (2). Some involve adjusting of parameters in D and B, perhaps by optimisation, to ensure that $\mathcal{R}_0$ is minimised. If in fact one reduces $\mathcal{R}_0$ to be less than 1, then convergence to $\mathbf{0}_n$ follows as per Proposition 1. In this paper, we consider a form of feedback control.

In the metapopulation modelling context, the value $d_i > 0$ in Eq. (1) represents the *recovery rate* of the population i against the disease in question. Suppose that we can dynamically control (and in particular increase) the recovery rate at node i, e.g. by increasing medical resources at node i, using a feedback controller. Specifically, let us replace d_i in Eq. (1) with $\bar{d}_i(t) = d_i + u_i(t)$, where $d_i > 0$ is the *base recovery rate* if no additional recovery resources are provided, and $u_i(t)$ the control input at node i.

Consider the class of local feedback controllers of the form

$$u_i(t) = h_i(x_i(t)), \qquad (4)$$

with the following property:

P1 For all i, $h_i : [0, 1] \to \mathbb{R}_{\geq 0}$ with $h_i(0) = 0$ is bounded, of class $\mathcal{C}^\infty$, and monotonically nondecreasing.

Obviously, Eq. (4) satisfying P1 contains a broad class of controllers, and one can assume $h_i(0) = 0$ without loss of generality. We are motivated to consider Eq. (4) for practical reasons. The controller in Eq. (4) is distributed, since only the local state x_i is required for population i. This contrasts with many existing approaches described in the Introduction which require global (and in some instances complete) information regarding D and B. Also, such controllers are intuitive: we increase the recovery rate $\bar{d}_i(t)$ as the infection $x_i(t)$ in node i increases. The work (Liu et al., 2019) considers a controller of the special form $h_i(x_i) = k_i x_i$ with $k_i > 0$ and $d_i = 0$, and this paper significantly expands on that result [2].

The dynamics for population i then become

$$\dot{x}_i(t) = -\Big(d_i + h_i(x_i(t))\Big)x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} b_{ij} x_j(t), \qquad (5)$$

and the network dynamics become

$$\dot{x}(t) = (-D - H(x(t)) + B - X(t)B)x(t), \qquad (6)$$

where $H(x(t)) = \text{diag}(h_1(x_1(t)), \ldots, h_n(x_n(t)))$ is a nonnegative diagonal matrix. The following result establishes that Ξ_n is a positive invariant set of Eq. (6). Since the right hand side of Eq. (6) is smooth in x, the solution for any $x(0) \in \Xi_n$ exists for all $t \geq 0$ and is unique.

Proposition 2. Consider the system Eq. (6) with strongly connected $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$. If $x(0) \in \Xi_n$, then $x(t) \in \Xi_n \, \forall t \geq 0$. Any endemic equilibrium x^* satisfies $x^* \in \text{Int}(\Xi_n)$.

Proof. The simple calculations closely mirror those in existing works, e.g. (Lajmanovich and Yorke, 1976), and we omit the steps. □

Define $\mathcal{R}_0$ as in Eq. (3) as the effective reproduction number of the *uncontrolled network*, and observe the following:

Theorem 1. Consider the system Eq. (6), with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ strongly connected. Suppose that $\mathcal{R}_0 \leq 1$ and for all $i \in \mathcal{V}$, h_i satisfies P1. Then, $\mathbf{0}_n$ is the unique equilibrium of Eq. (6) in Ξ_n, and for all $x(0) \in \Xi_n$, $\lim_{t\to\infty} x(t) = \mathbf{0}_n$.

Proof. Suppose that x^* is a nonzero equilibrium, and thus $x^* > \mathbf{0}_n$ according to Proposition 2. If $\mathcal{R}_0 < 1$, then $s(-D + B) < 0$ (the case where $\mathcal{R}_0 = 1$ and thus $s(-D + B) = 0$ can be similarly treated). One can use the result of (Berman and Plemmons, 1979, Theorem 2.3) to prove that (i) $D - B$ is an irreducible nonsingular M-matrix, and then subsequently (ii) $D + H(x^*) - (I_n - X^*)B$

[2] We assume $d_i > 0$ for simplicity and consistency with Eq. (1).

is also an irreducible nonsingular M-matrix. However, the nonsingularity property contradicts the assumption that $x^* > \mathbf{0}_n$ satisfies $(D + H(x^*) - (I_n - X^*)B)x^* = \mathbf{0}_n$ according to Eq. (6). Thus, there are no endemic equilibria when $\mathcal{R}_0 \leq 1$.

From Eq. (6), we obtain that $\dot{x} \leq \dot{y} = (-D + B)y$ because $I_n - X(t)$ is a diagonal matrix with entries in $[0, 1]$, and $H(x(t))$ is nonnegative. Since $s(-D+B) < 0$ means $-D+B$ is Hurwitz, initialising $\dot{y} = (-D + B)y$ with $y(0) = x(0)$ yields $\lim_{t\to\infty} x(t) = \mathbf{0}_n$. □

The problem of this paper is summarised as follows.

Problem 1. Establish and characterise the behaviour of Eq. (6), including the limiting behaviour $\lim_{t\to\infty} x(t)$ if it exists, for the system Eq. (6) under the assumptions

(1) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ is strongly connected and $\mathcal{R}_0 > 1$,
(2) for all $i \in \mathcal{V}$, $h_i : [0, 1] \to \mathbb{R}_{\geq 0}$ satisfies P1.

3. MAIN RESULT

In Section 3.1, we present a sufficient condition for a nonlinear autonomous system to have a unique equilibrium, which we use in Section 3.2 to help solve Problem 1.

3.1 Uniqueness of Equilibrium For Nonlinear Systems

The condition presented in this subsection is derived from the Poincaré–Hopf Theorem, which is a classical result from differential topology. Since the focus of this paper is on feedback control of epidemic networks, details on differential topology are differed Appendix B.

Consider the autonomous system

$$\dot{x}(t) = f(x(t)) \tag{7}$$

where $f = [f_1(x), \ldots, f_n(x)]^\top \in \mathbb{R}^n$ is a nonlinear vector-valued function, and $x = [x_1, \ldots, x_n]^\top$. We assume that $f_i, i = 1, \ldots, n$, belongs to the class of $\mathcal{C}^\infty$ functions. The Jacobian of f evaluated at a point x is denoted by df_x.

For a topological space X, we introduce the Euler characteristic $\chi(X)$ (Guillemin and Pollack, 2010; Milnor, 1997), an integer number associated [3] with X. A key property is that distortion or bending of X (specifically a homotopy) leaves $\chi(X)$ invariant. Euler characteristics are known for a great many topological spaces. A manifold $\mathcal{M}$ is contractible if $\mathcal{M}$ is homotopy equivalent to a single point, and has Euler characteristic $\chi(\mathcal{M}) = 1$. Any compact and convex subset of $\mathbb{R}^n$ is contractible.

Theorem 2. Consider the system Eq. (7), and suppose that $\mathcal{M} \subset \mathbb{R}^n$ is an m-dimensional compact, contractible and smooth manifold with boundary $\partial\mathcal{M}$, with $m \leq n$. Suppose further that f points inward to $\mathcal{M}$ at every $x \in \partial\mathcal{M}$. If $df_{\bar{x}}$ is Hurwitz for every $\bar{x} \in \mathcal{M}$ satisfying $f(\bar{x}) = 0$, then Eq. (7) has a unique equilibrium $x^* \in \text{Int}(\mathcal{M})$. Moreover, x^* is locally exponentially stable.

Proof. The focus of this paper is on feedback control for the SIS network system; we defer the proof to Appendix B.

[3] While the Euler characteristic can be extended to noncompact X, this paper will only consider the Euler characteristic for compact X.

3.2 Distributed Feedback Control: An Impossibility Result

We are interested in applying Theorem 2 for $\mathcal{R}_0 > 1$. In order to do so, we first need to find a contractible manifold $\mathcal{M}$ for the system Eq. (6) with the property that at all points on $\partial\mathcal{M}$, $f(x)$ is pointing inward. To identify one such $\mathcal{M}$, we first define a *Metzler* matrix as a matrix with all off-diagonal entries nonnegative (Berman and Plemmons, 1979), and state the following result.

Lemma 1. ((Varga, 2009, Section 2.1)). Let A be an irreducible Metzler matrix. Then, $s(A)$ is a simple eigenvalue of A and there exists a unique (up to scalar multiple) vector $x > \mathbf{0}_n$ such that $Ax = s(A)\boldsymbol{x}$. If $Az = \lambda z$ for some scalar λ and nonzero vector $z \geq \mathbf{0}_n$, then $s(A) = \lambda$.

Let $\phi \triangleq s(-D + B)$, and suppose $y > \mathbf{0}_n$ satisfies $(-D + B)y = \phi y$ as in Lemma 1. Without loss of generality, assume $\max_i y_i = 1$. For $\epsilon \in (0, 1)$, define the set

$$\mathcal{M}_\epsilon \triangleq \{x : \epsilon y_i \leq x_i \leq 1, \forall i = 1, \ldots, n\} \subset \Xi_n \tag{8}$$

The boundary $\partial\mathcal{M}_\epsilon$ is the union of the faces

$$P_i = \{x : x_i = \epsilon y_i, x_j \in [y_j, 1] \,\forall j \neq i\}, \tag{9a}$$
$$Q_i = \{x : x_i = 1, x_j \in [\epsilon y_j, 1] \,\forall j \neq i\}. \tag{9b}$$

It can be shown that $s(-D + B) > 0 \Leftrightarrow \rho(D^{-1}B) = \mathcal{R}_0 > 1$ and $s(-D + B) = 0 \Leftrightarrow \mathcal{R}_0 = 1$ (Liu et al., 2019, Proposition 1). Then, the following invariance result holds (see Appendix C for the proof).

Lemma 2. Consider the system Eq. (6), and suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ is strongly connected. Suppose further that $\phi \triangleq s(-D + B) > 0$. Then, there exist sufficiently small $\epsilon_u \in (0, 1)$ and $\epsilon_v \in (0, \epsilon_u]$ such that:

(1) for every $\epsilon \in (0, \epsilon_u]$, $\mathcal{M}_\epsilon$ in Eq. (8) and $\text{Int}(\mathcal{M}_\epsilon)$ are both positive invariant sets of Eq. (6), and
$$-\mathbf{e}_i^\top \dot{x} < 0 \qquad \forall\, x \in P_i\,, i = 1, \ldots n \tag{10a}$$
$$\mathbf{e}_i^\top \dot{x} < 0 \qquad \forall\, x \in Q_i\,, i = 1, \ldots n \tag{10b}$$
where $\mathbf{e}_i$ is the ith canonical unit vector, and
(2) for all $x(0) \in \partial\Xi_n \setminus \mathbf{0}_n$, $x(\bar{\kappa}) \in \mathcal{M}_{\epsilon_v}$ for some finite $\bar{\kappa} > 0$. Any endemic equilibrium $\bar{x}$ must satisfy $\bar{x} \in \text{Int}(\mathcal{M}_{\epsilon_v})$.

We now explain the intuition behind Lemma 2, and refer the reader to the helpful diagram in Fig. 1 for an illustrative example, with $\epsilon \in [\epsilon_v, \epsilon_u]$. The inequalities Eq. (10) imply that the vector field

$$f(x) = (-D + H(x) + B - XB)x \tag{11}$$

points inward at all points on $\partial\mathcal{M}_\epsilon$. Notice that $\mathcal{M}_\epsilon$ is an n-dimensional hypercube, so it is contractible, but it is not *smooth* on the edges and corners formed by the intersections of the faces defined in Eq. (9). In order to apply Theorem 2, we consider the system Eq. (6) on the manifold $\tilde{\mathcal{M}}_\epsilon$, which is simply $\mathcal{M}_\epsilon$ as defined in Eq. (8) for some $\epsilon \in (0, \epsilon_u]$, but with each edge and corner rounded by arbitrarily small amounts so that $\tilde{\mathcal{M}}_\epsilon$ is a smooth, contractible manifold with boundary $\partial\tilde{\mathcal{M}}_\epsilon$. By continuity, $f(x)$ in Eq. (11) will also *point inward* at all points on $\partial\tilde{\mathcal{M}}_\epsilon$. Nagumo's Theorem (Aubin, 1991) implies that $\tilde{\mathcal{M}}_\epsilon$ is a positive invariant set of Eq. (6). We now establish an impossibility result to answer Problem 1 in Section 2.2.

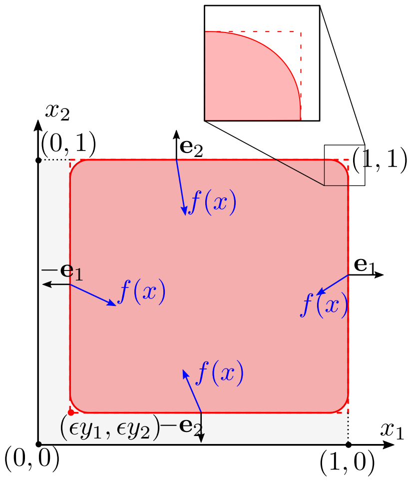

Fig. 1. An illustration of the manifolds $\mathcal{M}_\epsilon$ and $\tilde{\mathcal{M}}_\epsilon$ for Eq. (6), with $n = 2$ and $\epsilon \in [\epsilon_v, \epsilon_u]$. The cube Ξ_n is in light grey, with dotted black boundaries. The dashed red line identifies $\mathcal{M}_\epsilon$, and notice the lower corner $(\epsilon y_1, \epsilon y_2)$ with exaggerated size (in reality, ϵ is small). The solid red line and shaded red area identify $\tilde{\mathcal{M}}_\epsilon$ and $\text{Int}(\tilde{\mathcal{M}}_\epsilon)$, respectively. Then, $\tilde{\mathcal{M}}_\epsilon$ is simply $\mathcal{M}_\epsilon$ with the corners rounded by an arbitrarily small amount (the corner $(1,1)$ is magnified and exaggerated for clarity). Referring to Eq. (10), black arrows denote vectors $\mathbf{e}_1, \mathbf{e}_2$ (with direction), and blue arrows show the vector field f pointing inward at example points on the boundary $\partial\tilde{\mathcal{M}}_\epsilon$.

Theorem 3. Consider the system Eq. (6), with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ strongly connected. Suppose that $\mathcal{R}_0 > 1$ and for all $i \in \mathcal{V}$, h_i satisfies P1. Then,

(1) In Ξ_n, Eq. (6) has two equilibria: $x = \mathbf{0}_n$ and a unique endemic equilibrium $x^* \in \text{Int}(\Xi_n)$, which is unstable and locally exponentially stable, respectively.
(2) For all $x(0) \in \Xi_n \setminus \mathbf{0}_n$, there holds $\lim_{t\to\infty} x(t) = x^*$ at an exponentially fast rate.

Proof. The proof consists of two parts. In *Part 1*, we establish the existence and uniqueness of the equilibrium $x^* \in \text{Int}(\Xi_n)$, and the local stability properties of x^* and $\mathbf{0}_n$. In *Part 2*, we establish the convergence to x^*.

Part 1: Lemma 2 states that there exists a sufficiently small $\epsilon_v > 0$ such that $\mathcal{M}_{\epsilon_v}$ in Eq. (8) and $\text{Int}(\mathcal{M}_{\epsilon_v})$ are both positive invariant sets of Eq. (6), and for every $x \in \partial\mathcal{M}_{\epsilon_v}$, the vector field $f(x)$ in Eq. (11) points inward. As discussed above the statement of Theorem 3, we can obtain from $\mathcal{M}_{\epsilon_v}$ a smooth, contractible and compact manifold $\tilde{\mathcal{M}}_{\epsilon_v}$ by rounding the corners and edges of $\mathcal{M}_{\epsilon_v}$ by arbitrarily small amounts. By continuity, $f(x)$ in Eq. (11) points inward at every $x \in \partial\tilde{\mathcal{M}}_{\epsilon_v}$, and thus $\tilde{\mathcal{M}}_{\epsilon_v}$ and $\text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$ are both positive invariant sets of Eq. (6). Moreover, if $x(t) \in \partial\Xi_n \setminus \mathbf{0}_n$ for some time $t \geq 0$, then $x(t+\tau) \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$ for some $\tau > 0$. Thus, any nonzero equilibrium of Eq. (6) must be in $\text{Int}(\tilde{\mathcal{M}}_{\epsilon_v}) \subset \text{Int}(\Xi_n)$.

Suppose that $\tilde{x} \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$ is an equilibrium of Eq. (6). Then, $\tilde{x}$ must satisfy $\mathbf{0}_n < \tilde{x} < \mathbf{1}_n$ and

$$\mathbf{0}_n = (-D + H(\tilde{x}) + B - \tilde{X}B)\tilde{x}. \tag{12}$$

This implies that $I_n - \tilde{X}$ is a positive diagonal matrix, and because $B \geq \mathbf{0}_{n\times n}$ is irreducible, $(I_n - \tilde{X})B$ is also an irreducible nonnegative matrix. Define for convenience $F(x) \triangleq D + H(x) - (I_n - X)B$. Obviously, $F(x)$ has all off-diagonal entries nonpositive for all $x \in \tilde{\mathcal{M}}_{\epsilon_v}$, implying that $-F(\tilde{x})$ is a *Metzler* matrix for any equilibrium $\tilde{x} \in \tilde{\mathcal{M}}_{\epsilon_v}$. Lemma 1 and Eq. (12) indicate that $s(-F(\tilde{x})) = 0$, and as a consequence, (Qu, 2009, Theorem 4.27) establishes that $F(\tilde{x})$ is a singular irreducible M-matrix.

Define $\Gamma(x) = \text{diag}(\frac{\partial h_1}{\partial x_1}x_1, \ldots, \frac{\partial h_n}{\partial x_n}x_n)$ and because h_i is monotonically nondecreasing in x_i, $\Gamma(x)$ is a nonnegative diagonal matrix for all $x \in \tilde{\mathcal{M}}_{\epsilon_v}$. The Jacobian of Eq. (6) at a point $x \in \tilde{\mathcal{M}}_{\epsilon_v}$ is

$$df_x = -\left(F(x) + \Delta(x) + \Gamma(x)\right) \tag{13}$$

where $\Delta(x) = \sum_{i=1}^n \big(\sum_{j=1}^n b_{ij}x_j\big)\mathbf{e}_i\mathbf{e}_i^\top$ is a diagonal matrix. Because B is irreducible, there exists for all $i = 1, \ldots, n$, a k_i such that $b_{ik_i} > 0$. This implies that $\sum_{j=1}^n b_{ij}x_j \geq b_{ik_i}x_{k_i} > 0$ for all $x \in \tilde{\mathcal{M}}_{\epsilon_v}$. In other words, $\Delta(x)$ is positive diagonal for all $x \in \tilde{\mathcal{M}}_{\epsilon_v}$. It follows from (Qu, 2009, Theorem 4.31) that $F(\tilde{x}) + \Delta(\tilde{x}) + \Gamma(\tilde{x})$ is a non-singular M-matrix, with eigenvalues having strictly positive real parts. This implies that $df_{\tilde{x}}$ is Hurwitz for all $\tilde{x} \in \tilde{\mathcal{M}}_{\epsilon_v}$ satisfying Eq. (12). Application of Theorem 2 establishes that there is in fact a unique equilibrium $x^* \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v}) \subset \text{Int}(\Xi_n)$, and x^* is locally exponentially stable.

For the healthy equilibrium $\mathbf{0}_n$, Eq. (13) yields $df_{\mathbf{0}_n} = -D + B$. Since $\mathcal{R}_0 > 1 \Leftrightarrow s(-D+B) > 0$ by hypothesis, the Linearization Theorem (Sastry, 1999, Theorem 5.42) yields that $\mathbf{0}_n$ is an unstable equilibrium of Eq. (6).

Part 2: To complete the proof, we need only show that $\lim_{t\to\infty} x(t) = x^*$ for all $x(0) \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$. We shall use key results from the theory of monotone dynamical systems, the details of which are presented in Appendix A.

It is clear that for all $x \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$, df_x in Eq. (13) is an irreducible matrix with all nonnegative offdiagonal entries. Thus, Eq. (6) is an $\mathbb{R}^n_{\geq 0}$ monotone system in $\text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$ (see Lemma 3 in the Appendix A). Note that $\text{Int}(\tilde{\mathcal{M}}_{\epsilon_v}) \subset \mathbb{R}^n_{\geq 0}$. Because there is a unique equilibrium $x^* \in \text{Int}(\tilde{\mathcal{M}}_{\epsilon_v})$, one can show using an extension of a well-known monotone systems result that $\lim_{t\to\infty} x(t) = x^*$ asymptotically for all $x(0) \in \tilde{\mathcal{M}}_{\epsilon_v}$ (see below Lemma 4 in Appendix A for details). It remains to prove that convergence is exponentially fast.

Since df_{x^*} is Hurwitz, let $\mathcal{B}$ denote the locally exponentially stable region of attraction of x^*. For every $x_0 \in \tilde{\mathcal{M}}_{\epsilon_v}$, the fact that $\lim_{t\to\infty} x(t) = x^*$ implies that there exists a finite $T_{x_0} \geq 0$ such that $x(0) = x_0$ for Eq. (6) yields $x(t) \in \mathcal{B}$ for all $t \geq T_{x_0}$. Now, $\tilde{\mathcal{M}}_{\epsilon_v}$ is compact, which implies that there exists a $\bar{T} \geq \max_{x_0 \in \tilde{\mathcal{M}}_{\epsilon_v}} T_{x_0}$ such that for all $x(0) \in \tilde{\mathcal{M}}_{\epsilon_v}$, there holds $x(t) \in \mathcal{B}$ for all $t \geq \bar{T}$. In other words, there exists a time $\bar{T}$ independent of $x(0)$ such that any trajectory of Eq. (6) beginning in $\tilde{\mathcal{M}}_{\epsilon_v}$ enters the region of attraction $\mathcal{B}$. Because $\bar{T}$ is independent of $x(0)$, there exist positive constants α_1 and α_2 such that $\|x(t) - x^*\| \leq \alpha_1 e^{-\alpha_2 t}\|x(0) - x^*\|$ for all $x(0) \in \tilde{\mathcal{M}}_{\epsilon_v}$ and

$t \geq 0$. I.e., $\lim_{t\to\infty} x(t) = x^*$ exponentially fast for all $x(0) \in \tilde{\mathcal{M}}_{\epsilon_v}$. □

We conclude the section by comparing the endemic equilibrium of the controlled and uncontrolled SIS network.

Corollary 1. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$ is strongly connected, and $\mathcal{R}_0 > 1$. Suppose further that for all $i \in \mathcal{V}$, h_i satisfies P1 and $\exists j : x_j > 0 \Rightarrow h_j(x_j) > 0$. Let x^* and $\bar{x}^*$ in $\text{Int}(\Xi_n)$ denote the unique endemic equilibrium of Eq. (2) and Eq. (6), respectively. Then, $\bar{x}^* < x^*$.

Proof. The proof is provided on a pre-print in arXiv, see (Ye et al., 2020, Appendix C)

4. SIMULATIONS AND DISCUSSIONS

The introduction of the control input $h_i(x_i(t))$ into the SIS dynamics changes the vector field from

$$f(x) = (-D + (I - X)B)x \tag{14}$$

as in Eq. (2) to

$$\bar{f}(x) = (-D - H(x) + (I - X)B)x \tag{15}$$

as in Eq. (6). However, the changes to the vector field through $h_i(x_i), i \in \mathcal{V}$ are such that the uniqueness of the endemic equilibrium, and the property that it is convergent for all nonzero feasible initial conditions (in fact exponentially stable), are both preserved.

We now use a simple simulation of an SIS system in Eq. (6) with $n = 2$ nodes to help intuitively explain Theorem 3, and to discuss the implications of the impossibility result. The aim is to illustrate the change from Eq. (14) to Eq. (15), so we choose the parameters and controllers arbitrarily; the salient conclusions are unchanged for many other choices of parameters and controllers. We set

$$D = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0.5 \\ 0.7 & 0.1 \end{bmatrix}, \tag{16}$$

which yields $s(-D + B) = 0.2633$ and $\mathcal{R}_0 = 1.6334$.

We first consider Eq. (6) when there is no control, i.e. $h_1(x_1) \equiv h_2(x_2) \equiv 0$. Fig. 2 shows the vector field in Eq. (14). The figure shows that the unique endemic equilibrium $x^* = [0.4413, 0.2973]^\top$ (the red dot) is attractive for all feasible nonzero $x(0)$, which is consistent with existing results (Proposition 1, Item 2) since $\mathcal{R}_0 > 1$.

Next, we introduce the controllers $h_1(x_1) = 0.5{x_1}^{0.5}$ and $h_2(x_2) = 0.9x_2$ into Eq. (6). The resulting vector field in Eq. (15) is presented in Fig. 3; there is a unique endemic equilibrium $\bar{x}^* = [0.15, 0.1142]^\top$ which is attractive for all feasible nonzero $x(0)$. This is consistent with Theorem 3. Comparing Fig. 2 and 3, we see the introduction of the controllers $h_1(x_1)$ and $h_2(x_2)$ has changed the vector field, and shifted the unique endemic equilibrium from x^* to $\bar{x}^*$, with $\mathbf{0}_n < \bar{x}^* < x^*$ as per Corollary 1.

The state feedback controllers taking the form in Eq. (4), with property P1, comprises a large class of controllers. In the context of network SIS epidemic models, Theorem 3 establishes the following broad conclusion on such a class of feedback controllers: If the underlying uncontrolled system (the underlying vector field f in Eq. (14)) has a unique endemic equilibrium x^* that is convergent for all feasible nonzero $x(0)$, i.e. $\mathcal{R}_0 > 1$, then regardless of which controllers $h_i(x_i)$ satisfying P1 we introduce, viz. regardless of how we modify Eq. (14) to become Eq. (15), there will always exist a unique endemic equilibrium $\bar{x}^*$ that is convergent for all feasible nonzero $x(0)$. Moreover, $\bar{x}^*$ is *always* closer to the healthy equilibrium than x^* (as illustrated in a simulation example above). It is worth noting that the presence of a *single* node j with positive control input, i.e. $x_j > 0 \Rightarrow h_j(x_j) > 0$, leads to an improvement for *every* node i, viz. $\bar{x}_i^* < x_i^*$. As per Theorem 1, the healthy equilibrium is globally asymptotically stable for the controlled network if and only if $\mathcal{R}_0 \leq 1$, i.e. the *underlying uncontrolled network* itself has the property that the unique equilibrium is $x = \mathbf{0}_n$.

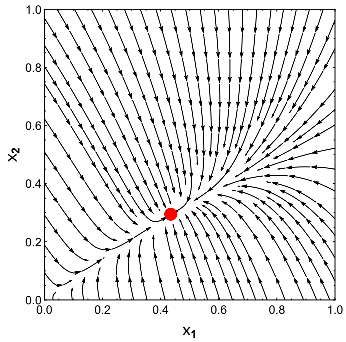

Fig. 2. Vector field of an uncontrolled SIS network with $n = 2$ nodes. The red dot identifies the unique endemic equilibrium $x^* = [0.4413, 0.2973]^\top \in \text{Int}(\Xi_n)$.

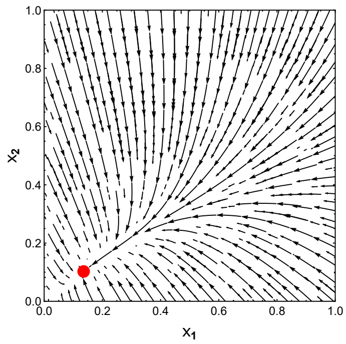

Fig. 3. Vector field of a controlled SIS network with $n = 2$ nodes. The red dot identifies the unique endemic equilibrium $\bar{x}^* = [0.15, 0.1142]^\top \in \text{Int}(\Xi_n)$.

5. CONCLUSION

We have investigated the problem of distributed feedback control for the deterministic SIS epidemic network model when the uncontrolled system converges to a unique endemic equilibrium for all feasible nonzero initial conditions. We considered the recovery rate of each node as a control input. Using tools from differential topology and monotone systems, we proved that for a large class of distributed controllers, the controlled network had a unique endemic equilibrium that was almost globally exponentially stable. Although no solution to the problem of controlling the network to the healthy equilibrium was provided, we have established an impossibility result on a broad class of controllers. Still, we proved that feedback control can help improve the endemic equilibrium, pushing it closer to the origin. Our findings may provide insight on possible future work that consider other approaches, such as control of the infection rates (including infection between nodes), time-varying and/or nonsmooth controllers.

Appendix A. MONOTONE SYSTEMS

We provide a simple introduction to monotone systems, and a convergence result sufficient for the purposes of this paper. For details, the reader is referred to (Smith, 1988).

Consider the system Eq. (7) on a convex, open set $U \subset \mathbb{R}^n$, with f sufficiently smooth such that df_x exists for all $x \in U$ and the solution $x(t)$ is unique for every initial condition in U. We use $\phi_t(x_0)$ to denote the solution $x(t)$ of Eq. (7) with $x(0) = x_0$. If whenever $x_0, y_0 \in U$, satisfying $x_0 \le y_0$, implies $\phi_t(x_0) \le \phi_t(y_0)$ for all $t \ge 0$ for which both $\phi_t(x_0)$ and $\phi_t(y_0)$ are defined, then the system Eq. (7) is said to be an $\mathbb{R}^n_{\ge 0}$ monotone system. The following is a necessary and sufficient condition for a system to be $\mathbb{R}^n_{\ge 0}$ monotone.

Lemma 3. ((Smith, 1988, Lemma 2.1)). Suppose that f in Eq. (7) is of class $\mathcal{C}^1$ in an open and convex $U \subset \mathbb{R}^n$. Then, Eq. (7) is $\mathbb{R}^n_{\ge 0}$ monotone if and only if df_x has nonnegative off-diagonal entries for every $x \in U$.

We say Eq. (7) is an *irreducible* $\mathbb{R}^n_{\ge 0}$ monotone system if df_x is irreducible for all $x \in U$. The following result for $\mathbb{R}^n_{\ge 0}$ monotone systems will be used for analysis in Section 3.2. Let E and $B(e)$ denote the set of equilibria of Eq. (7), and the basin of attraction of $e \in E$, respectively.

Lemma 4. ((Smith, 1988, Theorem 2.6)). Let $\mathcal{M}$ be an open, bounded, and positive invariant set for an irreducible $\mathbb{R}^n_{\ge 0}$ monotone system Eq. (7). Suppose the closure of $\mathcal{M}$, denoted by $\overline{\mathcal{M}}$, contains a finite number of equilibria. Then, $\cup_{e \in E \cap \overline{\mathcal{M}}} \mathrm{Int}(B(e)) \cap \overline{\mathcal{M}}$ is open and dense[4] in $\mathcal{M}$.

Lemma 4 can be strengthened if the irreducible $\mathbb{R}^n_{\ge 0}$ monotone system Eq. (7) has a *unique* equilibrium $e^* \in \overline{\mathcal{M}}$ and no equilibrium in $\overline{\mathcal{M}} \setminus \mathcal{M}$: nonattractive limit cycles can be ruled out, and one can establish that $x_0 \in \mathcal{M}$, $\lim_{t\to\infty} \phi_t(x_0) = e^*$. The details are omitted due to spatial limitations, and provided in an arXiv pre-print, see (Ye et al., 2020, Appendix A).

Appendix B. PROOF OF THEOREM 2

We first introduce some definitions and concepts from differential topology, and then recall the Poincaré-Hopf Theorem. For details, see classical texts such as (Milnor, 1997; Guillemin and Pollack, 2010).

Consider a smooth map $f : X \to Y$, where X and Y are manifolds. A point $x \in X$ is said to be a zero of f if $f(x) = \mathbf{0}$. Associated with f at any $x \in X$ is a linear derivative mapping $df_x : T_xX \to T_{f(x)}Y$, where T_xX and $T_{f(x)}Y$ are the tangent space of X at $x \in X$ and Y at $y = f(x) \in Y$, respectively. If X locally at x looks like $\mathbb{R}^m$, then df_x is simply the Jacobian of f evaluated at x in the local coordinate basis. A zero x of f with nonsingular df_x is said to be nondegenerate, and for such a zero, we define sign $\det(df_x)$ as $+1$ or -1 according as the sign of the determinant of df_x is positive or negative. A nondegenerate zero x of f is isolated: there exists an open ball around x with no other zeros.

[4] A set $S \subset A$ is dense in A if every point $x \in A$ is either in S or in the closure of S.

Variations of the Poincaré–Hopf Theorem exist, with differences. To avoid introducing too much material, we state a version slightly adjusted from that in (Milnor, 1997).

Theorem 4. (The Poincaré-Hopf Theorem (Milnor, 1997)). Consider a smooth vector field f on a compact m-dimensional manifold $\mathcal{M}$. If $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, then f must point outwards at every point on $\partial\mathcal{M}$. Suppose that every zero $x_i \in \mathcal{M}$ of f is nondegenerate. Then,

$$\sum_i \mathrm{sign}\ \det(df_{x_i}) = \chi(\mathcal{M}), \tag{B.1}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$.

Proof of Theorem 2: The proof presented here is compressed due to spatial limitations. For an expanded proof and further details on differential topology, see the arXiv preprint: Ye et al. (2020).

We make two remarks. First, one can consider $\dot{x} = f(x)$ as a system in $\mathcal{M}$, or f as a smooth vector field on $\mathcal{M}$, and we are conceptually considering the same thing. Second, $\bar{x}$ is a zero of f if and only if $\bar{x}$ is a zero of $-f$; the possibly empty sets of zeros of f and $-f$ are the same (we have not yet established the existence of a zero $\bar{x} \in \mathcal{M}$). For convenience, denote $g(x) = -f(x)$.

Next, recall that for any square matrix A, the product of its eigenvalues is equal to $\det(A)$. If $df_{\bar{x}}$ is Hurwitz for some $\bar{x} \in \mathcal{M}$, then all eigenvalues of $dg_{\bar{x}} = -df_{\bar{x}}$ have positive real part. It follows that for any $\bar{x} \in \mathcal{M}$ satisfying $f(\bar{x}) = 0$ and $df_{\bar{x}}$ is Hurwitz, we have sign $\det(dg_{\bar{x}}) = +1$.

We are now ready to apply Theorem 4 to the vector field $g = -f$ on the manifold $\mathcal{M}$. We know that *if $\bar{x}$ is a zero of g* (and if it exists), then it is nondegenerate, since $dg_{\bar{x}} = -df_{\bar{x}}$ is nonsingular by hypothesis. Now, the hypothesis that f points inwards at every $x \in \partial\mathcal{M}$ is equivalent to having the vector field g point outwards at every $x \in \partial\mathcal{M}$. From Eq. (B.1), we have that

$$\sum_i \mathrm{sign}\ \det(dg_{\bar{x}_i}) = \chi(\mathcal{M}), \tag{B.2}$$

with $\chi(\mathcal{M}) = 1$ since $\mathcal{M}$ is contractible. Because every zero $\bar{x}_i$ is nondegenerate, sign $\det(dg_{\bar{x}_i}) = \pm 1$. This implies that there must be at least one zero contributing to the left-hand side of Eq. (B.2), i.e. there exists at least one nondegenerate zero $\bar{x} \in \mathcal{M}$. The hypothesis that $df_{\bar{x}_i}$ is Hurwitz for every zero $\bar{x}_i$ implies as established in the preceding paragraph that sign $\det(dg_{\bar{x}_i}) = +1$ for all $\bar{x}_i$, which proves $\bar{x}_1 = x^*$ is unique. Now, f and g have the same set of zeroes, which establishes the theorem claim, with the local exponential stability of x^* given by $df_{\bar{x}}$ being Hurwitz. The analysis also yields $x^* \in \mathrm{Int}(\mathcal{M})$. □

Appendix C. PROOF OF LEMMA 2

Item 1: Fix $i \in \{1, \ldots, n\}$, and consider a point $x \in P_i$, expressed as $x = \epsilon y + z$ where $z = \sum_{j\neq i}^n (x_j - \epsilon y_j)\mathbf{e}_j \ge \mathbf{0}_n$, with $\mathbf{e}_i$ defined below Eq. (10). Note that $\mathbf{e}_i^\top A z = 0$ for any diagonal matrix A, and $\mathbf{e}_i^\top B z \ge 0$ for any $B \ge \mathbf{0}_{n\times n}$.

We drop the argument t from $x(t)$ when there is no risk of confusion, and define $Y = \mathrm{diag}(y_1, \ldots, y_n)$ and $Z = \mathrm{diag}(z_1, \ldots, z_n)$. At a point $x \in P_i$, Eq. (6) yields

$$\begin{aligned}\dot{x} &= (-D - H(x) + B - (\epsilon Y + Z)B)(\epsilon y + z) \\ &= \phi\epsilon y + (-D + B)z - H(x)(\epsilon y + z) \\ &\quad - (\epsilon^2 Y B y + \epsilon Z B y + \epsilon Y B z + Z B z) \qquad (C.1)\end{aligned}$$

since $(-D + B)y = \phi y$. Computing $-\mathbf{e}_i^\top \dot{x}$ from Eq. (C.1) and simplifying yields

$$\begin{aligned}-\mathbf{e}_i^\top \dot{x} &= -\epsilon y_i\big(\phi - h_i(\epsilon y_i) - \epsilon \sum_{j\in\mathcal{N}_i} b_{ij} y_j\big) \\ &\quad - \mathbf{e}_i^\top (I_n - \epsilon Y) B z \qquad (C.2)\end{aligned}$$

because $\mathbf{e}_i^\top(\phi I_n - \epsilon Y B - H(x))y = y_i(\phi - h_i(\epsilon y_i) - \epsilon y_i \sum_j b_{ij} y_j)$ and $\mathbf{e}_i^\top Z = \mathbf{0}_n^\top$. There exists a sufficiently small $\bar{\epsilon}$ such that $(I_n - \bar{\epsilon}Y)$ is nonnegative, which implies that $(I_n - \bar{\epsilon}Y)B$ is nonnegative. Thus, $\mathbf{e}_i^\top(I_n - \bar{\epsilon}Y)Bz \geq 0$. Since $\max_j y_j = 1$, h_i is monotonically nondecreasing and $h_i(0) = 0$, there exists a sufficiently small $\tilde{\epsilon} > 0$ such that

$$\phi - h_i(\tilde{\epsilon} y_i) - \tilde{\epsilon} \sum_{j\in\mathcal{N}_i} b_{ij} y_j > 0, \ \forall x_j \leq 1$$

By selecting $\epsilon_i = \min\{\bar{\epsilon}, \tilde{\epsilon}\}$, we establish from Eq. (C.2) that $-\mathbf{e}_i^\top \dot{x} < 0$ for all $x \in P_i$. This analysis holds for all $i = 1, \ldots n$, and selecting $\epsilon_u = \min_i \epsilon_i$ ensures that Eq. (10a) holds for all $i = 1, \ldots n$, for all $\epsilon \in (0, \epsilon_u]$.

Next, fix $i \in \{1, \ldots, n\}$, and consider a point $x \in Q_i$ expressed as $x = \mathbf{e}_i + z$, where $z = \sum_{j\neq i}^n x_j \mathbf{e}_j$. Similar to the above, one can show that at $x \in Q_i$, there holds

$$\dot{x} = (-D - H(x) + B)(\mathbf{e}_i + z) - (E_i B \mathbf{e}_i + E_i B z + Z B \mathbf{e}_i + Z B z)$$

where $E_i = \mathbf{e}_i \mathbf{e}_i^\top$. Using this equality, and observing that $\mathbf{e}_i^\top E_i = \mathbf{e}_i^\top$ and $\mathbf{e}_i^\top Z = \mathbf{0}_n^\top$, one can compute $\mathbf{e}_i^\top \dot{x}$ as

$$\mathbf{e}_i^\top \dot{x} = \mathbf{e}_i^\top(-D - H(x))\mathbf{e}_i = -d_i - h_i(1) < 0 \qquad (C.3)$$

This analysis holds for all $i = 1, \ldots n$, and thus Eq. (10b) holds. It follows from Nagumo's Theorem (Aubin, 1991) that $\mathcal{M}_\epsilon, \forall \epsilon \in (0, \epsilon_u]$ is a positive invariant set of Eq. (6). Moreover, Eq. (10) shows that $\partial\mathcal{M}_\epsilon$ is *not* invariant for Eq. (6). Thus, $\mathrm{Int}(\mathcal{M})$ is also an invariant set of Eq. (6).

Item 2: At some $t \geq 0$, suppose that $x(t) \in \partial\Xi_n \setminus \mathbf{0}_n$. If $x_i(t) = 1$, then Eq. (5) yields $\dot{x}_i = -(d_i + h_i(1)) < 0$. Thus, if $x(t) > \mathbf{0}_n$, then obviously $x(t+\kappa) \in \mathcal{M}_{\epsilon_1}$ for some sufficiently small κ and $\epsilon_1 \in (0, \epsilon_u]$.

Let us suppose then, that $x(t) \in \partial\Xi_n \setminus \mathbf{0}_n$ has at least one zero entry, and define the set $\mathcal{U}_t \triangleq \{i : x_i(t) = 0, i \in \mathcal{V}\}$. The assumption that $\mathcal{G}$ is strongly connected implies that $\exists\, k \in \mathcal{U}_t$ such that $x_j(t) > 0$ for some $j \in \mathcal{N}_k$. Eq. (5) yields $\dot{x}_k = \sum_{l\in\mathcal{N}_k} b_{lk} x_l(t) \geq b_{jk} x_j(t) > 0$. This analysis can be repeated to show that there exists a finite κ such that $\mathcal{U}_{t+\kappa}$ is empty. It follows that $x(t+\kappa) \in \mathcal{M}_{\epsilon_2}$ for some sufficiently small $\epsilon_2 \in (0, \epsilon_u]$.

Since $\partial\Xi_n \setminus \mathbf{0}_n$ is bounded, there exists a finite $\bar{\kappa}$ and sufficiently small $\epsilon_v \in (0, \min\{\epsilon_1, \epsilon_2\}]$ such that $x(\bar{\kappa}) \in \mathcal{M}_{\epsilon_v}$ for all $x(0) \in \partial\Xi_n \setminus \mathbf{0}_n$. Thus, any endemic equilibrium $\bar{x}$ must be in $\mathrm{Int}(\mathcal{M}_{\epsilon_v})$. □

REFERENCES

Anderson, R.M. and May, R.M. (1991). *Infectious Diseases of Humans.* Oxford University Press.

Aubin, J.P. (1991). *Viability Theory.* Systems & Control: Foundations & applications. Birkhauser: Boston.

Berman, A. and Plemmons, R.J. (1979). *Nonnegative Matrices in the Mathematical Sciences.* Computer Science and Applied Mathematics. Academic Press: London.

Bishop, A.N. and Shames, I. (2011). Link operations for slowing the spread of disease in complex networks. *EPL (Europhysics Letters)*, 95(1), 18005.

Fagnani, F. and Zino, L. (2017). Time to Extinction for the SIS Epidemic Model: New Bounds on the Tail Probabilities. *IEEE Transactions on Network Science and Engineering*, 6(1), 74–81.

Fall, A., Iggidr, A., Sallet, G., and Tewa, J.J. (2007). Epidemiological Models and Lyapunov Functions. *Mathematical Modelling of Natural Phenomena*, 2(1), 62–83.

Guillemin, V. and Pollack, A. (2010). *Differential Topology*, volume 370. American Mathematical Soc.

Lajmanovich, A. and Yorke, J.A. (1976). A Deterministic Model for Gonorrhea in a Nonhomogeneous Population. *Mathematical Biosciences*, 28(3-4), 221–236.

Liu, J., Paré, P.E., Nedich, A., Tang, C.Y., Beck, C.L., and Başar, T. (2019). Analysis and Control of a Continuous-Time Bi-Virus Model. *IEEE Transactions on Automatic Control*, 64(12), 4891–4906.

Mai, V.S., Battou, A., and Mills, K. (2018). Distributed Algorithm for Suppressing Epidemic Spread in Networks. *IEEE Control Systems Letters*, 2(3), 555–560.

Mieghem, P.V., Omic, J., and Kooij, R. (2009). Virus spread in networks. *IEEE/ACM Transactions on Networking*, 17(1), 1–14.

Milnor, J.W. (1997). *Topology from the Differentiable Viewpoint.* Princeton University Press.

Nowzari, C., Preciado, V.M., and Pappas, G.J. (2016). Analysis and Control of Epidemics: A Survey of Spreading Processes on Complex Networks. *IEEE Control Systems*, 36(1), 26–46.

Pastor-Satorras, R. and Vespignani, A. (2001). Epidemic spreading in scale-free networks. *Physical Review Letters*, 86(14), 3200–3203.

Preciado, V.M., Zargham, M., Enyioha, C., Jadbabaie, A., and Pappas, G. (2014). Optimal Resource Allocation for Network Protection: A Geometric Programming Approach. *IEEE Transactions on Control of Network Systems*, 1(1), 99–108.

Qu, Z. (2009). *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles.* Springer Science & Business Media.

Ramírez-Llanos, E. and Martínez, S. (2014). A distributed algorithm for virus spread minimization. In *Proceedings of the 2014 American Control Conference*, 184–189.

Sastry, S. (1999). *Nonlinear systems: analysis, stability, and control*, volume 10. Springer New York.

Smith, H.L. (1988). Systems of Ordinary Differential Equations Which Generate an Order Preserving Flow. A Survey of Results. *SIAM Review*, 30(1), 87–113.

Torres, J.A., Roy, S., and Wan, Y. (2016). Sparse Resource Allocation for Linear Network Spread Dynamics. *IEEE Transactions on Automatic Control*, 62(4), 1714–1728.

Varga, R.S. (2009). *Matrix Iterative Analysis*, volume 27. Springer Science & Business Media.

Ye, M., Liu, J., Anderson, B.D.O., and Cao, M. (2020). Applications of the Poincaré–Hopf Theorem: Epidemic Models and Lotka–Volterra Systems. URL `https://arxiv.org/abs/1911.12985`.