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Finite-time Boundedness of T-S Fuzzy Systems Subject to Injection Attacks: A Sliding Mode Control Method ⋆

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Abstract: This paper studies the sliding mode control for T-S fuzzy systems in the framework of finite-time boundedness. It is assumed that the control signals are transmitted via vulnerable channels, where injection attacks might happen. A fuzzy sliding mode controller is firstly synthesized to guarantee the finite-time reachability of the prescribed sliding surface and attenuate the effect of the injection attacks. By introducing a partitioning strategy, the finite-time boundedness over both the reaching phase and the sliding motion phase are analyzed. Furthermore, sufficient criteria are derived such that the closed-loop system is finite-time bounded over the whole specified finite-time interval despite of the injection attacks. An optimal algorithm is further provided for searching ideal control gains with fewer energy demands. Finally, a simulation example verifies the proposed sliding mode control approach.

Keywords: T-S fuzzy system, sliding mode control, finite-time boundedness, partitioning strategy, injection attacks.

1. INTRODUCTION

Takagi-Sugeno (T-S) fuzzy model has been widely utilized to control complicated nonlinear systems due to its powerful capacity for representing nonlinear plants by a collection of linear subsystems (Sala and Arino (2007); Niu et al. (2010)). In the past decades, considerable research interests have been devoted to T-S fuzzy systems using different control approaches, e.g., state feedback control (Zhou and Han (2019)), H_∞ control (Song et al. (2018a)), sampled control (Wu et al. (2015)), sliding mode control (SMC) (Li et al. (2018)) and so on.

As an effective robust control approach, SMC has received substantial attention owing to its simplicity and insensitivity to matched disturbances and parameter variations (Song et al. (2018b); Chen et al. (2013); Utkin and Shi (1996); Zhang et al. (2019)). Taking the advantages of T-S fuzzy modeling method and the SMC, fruitful research results have been reported. For example, the SMC for T-S fuzzy systems were studied in (Li et al. (2018)) and Wang et al. (2018) based on uniformly ultimately boundedness and asymptotically stability, respectively.

Recently, some initial efforts have been devoted to the control/filtering problem subject to cyber attacks, e.g. Denial-of-Service attacks (Zhao et al. (2019)), replay attacks (Zhu and Martinez (2014)), injection attacks (Jin et al. (2017)) and so on. Especially, the adaptive SMC for T-S fuzzy systems under injection attacks was investigated in Zhang et al. (2020), where the input-to-state stability were achieved for the closed-loop fuzzy systems.

It is worthy to notice that the above results were derived within an infinite time interval. In practice, many industrial applications (e.g., robot control system, chemical reaction process, and guided missile launch system) only have a finite or short working time, during which the states or outputs are expected to avoid exceeding the prescribed physical threshold. This has triggered the research on finite-time stability (FTS)/boundedness (FTB). FTS means that the state norm remains in a given region during a finite-time interval under bounded initial states (Michel and Wu (1969)). Amato et al. (2001) extended the definition of FTS to FTB. A system is FTB if for bounded initial states and bounded disturbances, the states stay in the predefined bound in a fixed time interval. Recently, the authors in Song et al. (2017) investigated the SMC for nonlinear systems with external disturbance based on FTB. And then, this SMC approach on FTB has been extended to various systems, e.g., Markovian jump systems (Cao et al. (2019)), switched systems (Zhao and Niu (2019)), T-S fuzzy systems (Jiang et al. (2019)) and so on. Nevertheless, to the authors' best knowledge, the FTB via SMC for T-S fuzzy system subject to injection attacks has not been fully investigated and remains open. Moreover, the existing results cannot be directly applied to the above case due to the existence of injection attacks, which motivates the present work.

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This paper focuses on the finite-time boundedness of SMC for T-S fuzzy systems subject to injection attacks. An SMC law is proposed such that the effect of the cyber attacks can be attenuated and the state trajectories can be forced onto the specified sliding surface in a finite time interval $[0, \mathcal{T}^\star]$. By introducing a partitioning approach, sufficient conditions are derived to ensure the FTB over both the reaching phase $[0, \mathcal{T}^\star]$ and the sliding motion phase $[\mathcal{T}^\star, \mathcal{T}]$. Control gains with fewer energy requirements are obtained by an optimal algorithm.

Notations: $\mathbb{R}$ refers to the set of reals. $\|\cdot\|$ is the Euclidean vector (or induced matrix) norm. For a real symmetric matrix, $P > 0 (\geq 0)$ represents that P is a positive definite (or positive semidefinite) matrix; I is the identity matrix of proper dimensions. $\lambda_{min}(P)$ ($\lambda_{max}(P)$) stands for the smallest (largest) eigenvalue of P.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a class of nonlinear systems described by:

$$\dot{x}(t) = f(x(t), u_c(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u_c(t) \in \mathbb{R}^m$ is the control input; According to the standard T-S fuzzy modeling method in Tanaka and Wang (2001), the original nonlinear systems with external disturbances can be represented by the following T-S fuzzy systems:

$$\dot{x}(t) = \sum_{i=1}^{p} \beta_i(\xi(t)) \big[A_i x(t) + B_i u_c(t) + C\omega(t)\big] \tag{2}$$

where $\xi(t)$ is the premise variable; A_i, B_i, C are known system matrices, and B_i is with full column rank; The external disturbance $\omega(t)$ is assumed to satisfy $\Omega_{[t_1,t_2],\omega} \triangleq \{\omega^T(t)\omega(t) \leq \varpi^2 \text{ for } t \in [t_1, t_2]\}$, where ϖ is a known positive scalar. The nonlinear terms are captured by the membership functions $\beta_i(\xi(t)) \geq 0$ with properties $\sum_{i=1}^{p} \beta_i(\xi(t)) = 1$. In this work, it is required that $0 < \underline{\beta}_i \leq \beta_i(\xi(t))$, which can be achieved by appropriately building the fuzzy plant (more details see Lam and Leung (2005)).

Now, we model the injection attacks as follows:

$$u_c(t) = u(t) + \sigma_c(x(t)), \tag{3}$$

where the control signal $u(t)$ is to be designed later; $\sigma_c(x(t)) \triangleq \zeta(t)\Psi(x(t), t)$ denotes the injection attacks with $\zeta(t)$ being an unknown weighting matrix and $\Psi(x(t), t)$ being the system information utilized by the attackers. Furthermore, it is assumed that $\|\zeta(t)\| \leq \zeta$, $\|\Psi(x(t), t)\| \leq \psi(x(t), t)$, where $\zeta > 0$ is a known scalar and $\psi(x(t), t) > 0$ is a known function. In the following, $\beta_i(\xi(t))$, $\sigma_c(x(t))$, $\Psi(x(t), t)$, $\psi(x(t), t)$ will be rewritten as $\beta_i(\xi)$, $\sigma_c(t)$, $\Psi(t)$, $\psi(t)$, respectively, for notational simplification.

Remark 1. The attack model (3) is motivated by (Jin et al. (2017)), where $\sigma_c(x(t))$ is considered as a faulty (or malicious) signal. Hence, it is natural to consider this cyber attack as a special case of actuator faults. Besides, the control signal is assumed to be continuous-time to reduce the complexity of the analysis and synthesis.

By means of (2) and (3), the fuzzy system subject to injection attacks can be formulated as follows:

$$\dot{x}(t) = \sum_{i=1}^{p} \beta_i(\xi) \big[A_i x(t) + B_i(u(t) + \sigma_c(t)) + C\omega(t)\big] \tag{4}$$

Definition 1. Amato et al. (2001) Given scalars $c_1, c_2 > 0$ with $c_1 < c_2$, a weighted matrix $\mathcal{R} > 0$ and a time interval $[t_1, t_2]$. The fuzzy system (4) is FTB w.r.t. $(c_1, c_2, [t_1, t_2], \mathcal{R}, \Omega_{[t_1,t_2],\omega})$ if

$$x^T(t_1)\mathcal{R}x(t_1) \leq c_1 \Rightarrow x^T(t_2)\mathcal{R}x(t_2) \leq c_2 \tag{5}$$

for any $t \in [t_1, t_2]$.

The objective of this work is to realize the FTB of the controlled nonlinear systems and eliminate the effect of the injection attacks via a sliding mode controller.

3. MAIN RESULTS

Design the following integral sliding function $s(t) \in \mathbb{R}^m$:

$$s(t) = Gx(t) - G\int_0^t \sum_{i=1}^{p}\sum_{j=1}^{p} \beta_i(\xi)\beta_j(\xi) A_{ij} x(\tau) d\tau, \tag{6}$$

where $A_{ij} \triangleq A_i + B_i K_j$ and K_j will be designed later. G is chosen such that $GB_i > 0$ (or $GB_i < 0$) for all $i = 1, 2, ..., p$. Then, it follows from (4) and (6) that

$$\begin{aligned}\dot{s}(t) =& \sum_{i=1}^{p} \beta_i(\xi) GB_i(u(t) + \sigma_c(t)) + GC\omega(t) \\ & - G\sum_{i=1}^{p}\sum_{j=1}^{p} \beta_i(\xi)\beta_j(\xi) B_i K_j x(t)\end{aligned} \tag{7}$$

Now, we design the following SMC law:

$$\begin{aligned}u(t) =& \sum_{j=1}^{p} \beta_j(\xi) K_j x(t) - \varrho(t)\Big(\sum_{i=1}^{p} \beta_i(\xi) GB_i\Big)^{-1} \\ & \times \operatorname{sgn}(s(t)),\end{aligned} \tag{8}$$

$$\varrho(t) = \Big\| \sum_{i=1}^{p} \beta_i(\xi) GB_i \Big\| \zeta\psi(t) + \|GC\|\varpi + \nu, \tag{9}$$

with $\nu > 0$.

In the sequel, it is firstly demonstrated that the specified sliding surface $s(t) = 0$ can be reached during a finite-time interval $[0, \mathcal{T}^\star]$ with $\mathcal{T}^\star < \mathcal{T}$ by using the SMC law in (8)-(9). Meanwhile, the FTB of the closed-loop system over both the reaching phase $[0, \mathcal{T}^\star]$ and the sliding motion phase $[\mathcal{T}^\star, \mathcal{T}]$ can be ensured despite of injection attacks.

3.1 Reachability Analysis

In this subsection, the finite-time reachability of the prescribed sliding surface $s(t) = 0$ will be analyzed.

Theorem 1. For a given finite (possibly short) time $\mathcal{T}$ and a sufficiently small scalar $\epsilon > 0$, if the SMC law is synthesized by (8)-(9) with the parameter ν satisfying

$$\nu \geq \frac{\|Gx(0)\|}{\mathcal{T} - \epsilon}, \tag{10}$$

the states of the fuzzy system (4) will be driven onto the specified sliding surface $s(t) = 0$ in finite time $\mathcal{T}^\star$ with $\mathcal{T}^\star < \mathcal{T}$.

Proof. Chose the Lyapunov function candidate as $V_1(t) \triangleq \frac{1}{2}s^T(t)s(t)$. By using (7)-(9), we have

$$\dot{V}_1(t) = s^T(t)\dot{s}(t) \leq -\nu\|s(t)\|.$$

Noting that $\|s(t)\| = \sqrt{2V_1(t)}$, one has

$$\dot{V}_1(t) \le -\sqrt{2}\nu\sqrt{V_1(t)}. \tag{11}$$

Integrating both sides of (11) from 0 to $\mathcal{T}^\star$ leads to

$$\mathcal{T}^\star \le \frac{\sqrt{2}}{\nu}\sqrt{V_1(0)} = \frac{1}{\nu}\|s(0)\| = \frac{1}{\nu}\|Gx(0)\|. \tag{12}$$

Then, it follows from (10) that

$$\mathcal{T}^\star \le \mathcal{T} - \epsilon < \mathcal{T}. \tag{13}$$

Thus, the state trajectories will be driven onto the specified sliding surface $s(t) = 0$ during the interval $[0, \mathcal{T}^\star]$ with $\mathcal{T}^\star < \mathcal{T}$. □

3.2 FTB Within $[0, \mathcal{T}^\star]$

In the sequel, the FTB conditions during both the reaching phase and the sliding motion phase will be derived by employing a partitioning strategy.

Lemma 1. (Song et al. (2016)) (Partitioning Strategy) For the fuzzy system (4) with the prescribed parameters $(c_1, c_2, [t_1, t_2], \mathcal{R}, \Omega_{[t_1,t_2],\omega})$, the closed-loop fuzzy system is FTB w.r.t. $(c_1, c_2, [t_1, t_2], \mathcal{R}, \Omega_{[t_1,t_2],\omega})$, if and only if there exists an additional scalar $c^\star$ satisfying $c_1 < c^\star < c_2$ such that the closed-loop system is FTB w.r.t. $(c_1, c^\star, [0, \mathcal{T}^\star], \mathcal{R}, \Omega_{[t_1,t_2],\omega})$ in the reaching phase and is FTB w.r.t. $(c^\star, c_2, [\mathcal{T}^\star, \mathcal{T}], \mathcal{R}, \Omega_{[t_1,t_2],\omega})$ in the sliding motion phase.

1) FTB over reaching phase within $[0, \mathcal{T}^\star]$

By substituting the SMC law (8)-(9) into (4), the closed-loop fuzzy system during the reaching phase within $[0, \mathcal{T}^\star]$ is obtained as follows:

$$\begin{aligned}\dot{x}(t) =& \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_i(\xi)\beta_j(\xi)\big[A_{ij}x(t) + B_i\sigma_c(t) + C\omega(t)\\ &- B_i(\sum_{i=1}^{p}\beta_i(\xi)GB_i)^{-1}\varrho(t)\text{sgn}(s(t))\big].\end{aligned} \tag{14}$$

Theorem 2. For all $i, j = 1, 2, \cdots, p$, if there exist scalars ε_i, $c^\star$ and matrices $P > 0$, K_j, $W > 0$ satisfying

$$\Xi_{ii} < 0, \tag{15}$$

$$\Xi_{ij} + \Xi_{ji} < 0, j > i \tag{16}$$

$$3\mu\bar{\lambda}_B(\sum_{i=1}^{p}\|GB_i\|)^2 + \mu - e^{-\mu\mathcal{T}}W < 0, \tag{17}$$

$$\frac{\aleph}{\underline{\delta}_P} < e^{-\mu\mathcal{T}}c^\star \tag{18}$$

with $\Xi_{ij} \triangleq \begin{bmatrix}\Xi_{ij}^{11} & * \\ \Xi_{ij}^{21} & \Xi_{ij}^{22}\end{bmatrix}$, $\Xi_{ij}^{11} \triangleq PA_{ij} + A_{ij}^TP - \mu P + \varepsilon_iPB_iB_i^TP$, $\Xi_{ij}^{21} \triangleq [PC, -PB_i, 0]^T$, $\Xi_{ij}^{22} \triangleq \text{diag}\{-\mu I, -\mu I, \varepsilon_i^{-1}I - \mu I + W\}$, $\aleph \triangleq \bar{\delta}_Pc_1 + \mu\varpi^2\mathcal{T} + 3\mu\bar{\lambda}_B(\|GC\|^2\varpi^2\mathcal{T} + \nu^2\mathcal{T})$, $\bar{\lambda}_B \triangleq \lambda_{max}\Big\{(\sum_{i=1}^{p}\underline{\beta}_iGB_i)^{-T}(\sum_{i=1}^{p}\underline{\beta}_iGB_i)^{-1}\Big\}$, $\bar{\delta}_P \triangleq \lambda_{max}\{\mathcal{R}^{-1/2}P\mathcal{R}^{-1/2}\}$, $\underline{\delta}_P \triangleq \lambda_{min}\{\mathcal{R}^{-1/2}P\mathcal{R}^{-1/2}\}$, the closed-loop fuzzy system (14) is FTB within the interval $[0, \mathcal{T}^\star]$.

Proof. Choose a Lyapunov function as follows:

$$V_2(t) \triangleq x^T(t)Px(t) + \int_0^t \zeta^2\varphi^2(\tau)Wd\tau. \tag{19}$$

By differentiating $V_2(t)$ along the solution of (14), we have

$$\begin{aligned}\dot{V}_2(t) =& 2x^T(t)P\dot{x}(t) + \zeta^2\varphi^2(t)W\\ =& \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_i(\xi)\beta_j(\xi)\big[x^T(t)(PA_{ij} + A_{ij}^TP)x(t)\\ &- 2x^T(t)PB_i\varrho_s(t) + 2x^T(t)PB_i\sigma_c(t)\\ &+ 2x^T(t)PC\omega(t)\big] + \zeta^2\varphi^2(t)W\end{aligned} \tag{20}$$

with $\varrho_s(t) = (\sum_{i=1}^{p}\beta_i(\xi)GB_i)^{-1}\varrho(t)\text{sgn}(s(t))$. Recalling that $GB_i > 0$ (or $GB_i < 0$) for all $i = 1, 2, ..., p$ and $0 < \underline{\beta}_i \le \beta_i(\xi)$, we obtain that $(\sum_{i=1}^{p}\beta_i(\xi)GB_i)^{-T}(\sum_{i=1}^{p}\beta_i(\xi) \times GB_i)^{-1} \le (\sum_{i=1}^{p}\underline{\beta}_iGB_i)^{-T}$ $(\sum_{i=1}^{p}\underline{\beta}_iGB_i)^{-1}$. Thus, by using (9), one has

$$\begin{aligned}\varrho_s^T(t)\varrho_s(t) \le& 3\bar{\lambda}_B\Big((\sum_{i=1}^{p}\|GB_i\|)^2\zeta^2\varphi^2(t)\\ &+ \|GC\|^2\varpi^2 + \nu^2\Big),\end{aligned} \tag{21}$$

Moreover, for $\varepsilon_i > 0$, it has

$$\begin{aligned}&2x^T(t)PB_i\sigma_c(t)\\ &\le \varepsilon_ix^T(t)PB_iB_i^TPx(t) + \varepsilon_i^{-1}\zeta^2\varphi^2(t)\end{aligned} \tag{22}$$

Consider the following auxiliary function

$$\begin{aligned}\mathcal{J}_1 \triangleq& \dot{V}_2(t) - \mu V_2(t) - \mu\omega^T(t)\omega(t) - \mu\varrho_s^T(t)\varrho_s(t)\\ &- \mu\zeta^2\varphi^2(t).\end{aligned} \tag{23}$$

Utilizing (20) and (22) yields

$$\begin{aligned}\mathcal{J}_1 \le& \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_i(\xi)\beta_j(\xi)\chi^T(t)\Xi_{ij}\chi(t)\\ &- \mu\int_0^t\zeta^2\varphi^2(\tau)Wd\tau\\ =& \sum_{i=1}^{p}\beta_i^2(\xi)\chi^T(t)\Xi_{ii}\chi(t) - \mu\int_0^t\zeta^2\varphi^2(\tau)Wd\tau\\ &+ \sum_{i=1}^{p}\sum_{j>i}\beta_i(\xi)\beta_j(\xi)\chi^T(t)(\Xi_{ij} + \Xi_{ji})\chi(t)\end{aligned} \tag{24}$$

with $\chi(t) = [x^T(t), \omega^T(t), \varrho_s^T(t), \zeta\varphi(t)]^T$. Then, it follows from (15)-(16) that $\mathcal{J}_1 + \mu\int_0^t\zeta^2\varphi^2(\tau)Wd\tau < 0$. Consequently, it has $\mathcal{J}_1 < 0$, i.e.,

$$\begin{aligned}\dot{V}_2(t) <& \mu V_2(t) + \mu\omega^T(t)\omega(t) + \mu\varrho_s^T(t)\varrho_s(t)\\ &+ \mu\zeta^2\varphi^2(t).\end{aligned} \tag{25}$$

Multiplying both sides of (25) by $e^{-\mu t}$ and integrating the obtained inequality from 0 to t with $t \in [0, \mathcal{T}^\star]$, one has

$$\begin{aligned}&e^{-\mu t}V_2(t)\\ &< V_2(0) + \int_0^t\mu\omega^T(\tau)\omega(\tau)d\tau\\ &\quad + \int_0^t\mu\varrho_s^T(\tau)\varrho_s(\tau)d\tau + \int_0^t\mu\zeta^2\varphi^2(t)d\tau\\ &\le \aleph + \int_0^t\mu\Big(3\bar{\lambda}_B(\sum_{i=1}^{p}\|GB_i\|)^2 + 1\Big)\zeta^2\varphi^2(\tau)d\tau\end{aligned} \tag{26}$$

in which the expression (21) is utilized. In addition, it yields from (19):

$$e^{-\mu t}V_2(t) \ge e^{-\mu\mathcal{T}}\underline{\delta}_Px^T(t)Rx(t)$$

$$+\int_0^t e^{-\mu\mathcal{T}}\zeta^2\varphi^2(\tau)Wd\tau. \quad (27)$$

According to condition (17)-(18), one has from (26) and (27):

$$x^T(t)Rx(t)\le \frac{\aleph}{e^{-\mu\mathcal{T}}\underline{\delta}_P} + \int_0^t \frac{(3\mu\bar{\lambda}_B(\sum_{i=1}^p\|GB_i\|)^2+\mu-e^{-\mu\mathcal{T}}W)\zeta^2\varphi^2(\tau)}{e^{-\mu\mathcal{T}}\underline{\delta}_P}d\tau \le \frac{\aleph}{e^{-\mu\mathcal{T}}\underline{\delta}_P}\le c^\star, \quad (28)$$

which implies that the closed-loop fuzzy system (14) is FTB in $[0,\mathcal{T}^\star]$ even in the presence of injection attacks. □

2) FTB over sliding motion phase within $[\mathcal{T}^\star,\mathcal{T}]$

In this part, we will analyze the FTB of the sliding dynamics. Letting $\dot{s}(t)=0$, the equivalent control law is obtained as follows:

$$u_{eq}(t)=\sum_{j=1}^p\beta_j(\xi)K_jx(t)-(\sum_{i=1}^p\beta_i(\xi)GB_i)^{-1}GC\omega(t) - \sigma_c(t).$$

Then, we get the following sliding mode dynamics:

$$\dot{x}(t)=\sum_{i=1}^p\sum_{j=1}^p\beta_i(\xi)\beta_j(\xi)\big[A_{ij}x(t)-B_i(\sum_{i=1}^p\beta_i(\xi)GB_i)^{-1} \times GC\omega(t)+C\omega(t)\big]. \quad (29)$$

Theorem 3. For all $i,j=1,2,\cdots,p$, if there exist matrices $P>0$, K_j and scalars ε_i, $c^\star$ satisfying

$$\Gamma_{ii}<0, \quad (30)$$
$$\Gamma_{ij}+\Gamma_{ji}<0, j>i \quad (31)$$
$$\frac{\bar{\delta}_Pc^\star+\mu(\bar{\lambda}_B\bar{\lambda}_C+1)\varpi^2\mathcal{T}}{\underline{\delta}_P}<e^{-\mu\mathcal{T}}c_2 \quad (32)$$

with $\Gamma_{ij}\triangleq\begin{bmatrix}\Gamma_{ij}^{11} & *\\ \Gamma_{ij}^{21} & \Gamma_{ij}^{22}\end{bmatrix}$, $\Gamma_{ij}^{11}\triangleq PA_{ij}+A_{ij}^TP-\mu P$, $\Gamma_{ij}^{21}\triangleq[PC,-PB_i]^T$, $\Gamma_{ij}^{22}\triangleq\text{diag}\{-\mu I,-\mu I\}$, $\bar{\lambda}_C\triangleq\lambda_{max}\{C^TG^TGC\}$ the sliding mode dynamics (29) is FTB over the interval $[\mathcal{T}^\star,\mathcal{T}]$.

Proof. For the Lyapunov function $V_3(t)\triangleq x^T(t)Px(t)$, we have

$$\dot{V}_3(t)=\sum_{i=1}^p\sum_{j=1}^p\beta_i(\xi)\beta_j(\xi)\big[x^T(t)(PA_{ij}+A_{ij}^TP)x(t) - 2x^T(t)PB_i\hat{\omega}(t)+2x^T(t)PC\omega(t)\big] \quad (33)$$

with $\hat{\omega}(t)=(\sum_{i=1}^p\beta_i(\xi)GB_i)^{-1}GC\omega(t)$ satisfying

$$\hat{\omega}^T(t)\hat{\omega}(t)\le\bar{\lambda}_B\bar{\lambda}_C\omega^T(t)\omega(t). \quad (34)$$

Define

$$\mathcal{J}_2\triangleq\dot{V}_3(t)-\mu V_3(t)-\mu\omega^T(t)\omega(t)-\mu\hat{\omega}^T(t)\hat{\omega}(t)$$

and $\hat{\chi}(t)\triangleq[x^T(t),\omega^T(t),\hat{\omega}^T(t)]^T$. It yields from (33) and conditions conditions (30)-(31) that

$$\mathcal{J}_2\le\sum_{i=1}^p\sum_{j>i}\beta_i(\xi)\beta_j(\xi)\hat{\chi}^T(t)(\Gamma_{ij}+\Gamma_{ji})\hat{\chi}(t) + \sum_{i=1}^p\beta_i^2(\xi)\hat{\chi}^T(t)\Gamma_{ii}\hat{\chi}(t)<0, \quad (35)$$

which implies that

$$\dot{V}_3(t)\le\mu V_3(t)+\mu\omega^T(t)\omega(t)+\mu\hat{\omega}^T(t)\hat{\omega}(t).$$

By following the similar lines to the ones in Theorem 2 and using (34), one has

$$e^{-\mu t}V_3(t)<V_3(\mathcal{T}^\star)+\int_{\mathcal{T}^\star}^t\mu\omega^T(\tau)\omega(\tau)d\tau + \int_{\mathcal{T}^\star}^t\mu\hat{\omega}^T(\tau)\hat{\omega}(\tau)d\tau \le\bar{\delta}_Pc^\star+\mu(\bar{\lambda}_B\bar{\lambda}_C+1)\varpi^2\mathcal{T}. \quad (36)$$

On the other hand, note that

$$e^{-\mu t}V_3(t)\ge e^{-\mu\mathcal{T}}\underline{\delta}_Px^T(t)Rx(t). \quad (37)$$

By employing (32), (36) and (37), one has

$$x^T(t)Rx(t)\le\frac{\bar{\delta}_Pc^\star+\mu(\bar{\lambda}_B\bar{\lambda}_C+1)\varpi^2\mathcal{T}}{e^{-\mu\mathcal{T}}\underline{\delta}_P}\le c_2 \quad (38)$$

for $t\in[\mathcal{T}^\star,\mathcal{T}]$. Thus, the FTB of the sliding mode dynamics (29) is verified. □

3.3 Solving Algorithm

In the sequel, sufficient conditions are provided in terms of linear matrix inequalities (LMIs) to guarantee that the conditions in Theorems 2 and 3 hold simultaneously.

Theorem 4. For all $i,j=1,2,\cdots,p$, if there exist matrices $Q>0$, Y_j, W and scalars $c^\star$, $\hbar$, $\bar{\varepsilon}_i$ satisfying (17) and

$$\tilde{\Xi}_{ii}<0, \quad (39)$$
$$\tilde{\Xi}_{ij}+\tilde{\Xi}_{ji}<0, j>i \quad (40)$$
$$\begin{bmatrix}\Theta & *\\ \sqrt{c_1} & -\hbar\end{bmatrix}<0, \quad (41)$$
$$c^\star+\hbar\mu(\bar{\lambda}_B\bar{\lambda}_C+1)\varpi^2\mathcal{T}<\frac{1}{2}c_2\hbar e^{-\mu\mathcal{T}}, \quad (42)$$
$$c_1<c^\star<c_2, \quad (43)$$
$$\hbar\mathcal{R}^{-1}<Q<2\mathcal{R}^{-1} \quad (44)$$

with $\tilde{\Xi}_{ij}\triangleq\begin{bmatrix}\tilde{\Xi}_{ij}^{11} & *\\ \tilde{\Xi}_{ij}^{21} & \tilde{\Xi}_{ij}^{22}\end{bmatrix}$, $\tilde{\Xi}_{ij}^{11}\triangleq A_iQ+B_iY_j+QA_i^T+Y_j^TB_i-\mu Q$, $\tilde{\Xi}_{ij}^{21}\triangleq[C,-B_i,0,B_i]^T$, $\tilde{\Xi}_{ij}^{22}\triangleq\text{diag}\{-\mu I,-\mu I,\bar{\varepsilon}_iI-\mu I+W,-\bar{\varepsilon}_iI\}$, $\Theta\triangleq-\frac{1}{2}e^{-\mu\mathcal{T}}c^\star+\mu\varpi^2\mathcal{T}+3\mu\bar{\lambda}_B(\|GC\|^2\varpi^2\mathcal{T}+\nu^2\mathcal{T})$, the closed-loop fuzzy system is FTB over the whole time interval $[0,\mathcal{T}]$. Moreover, the control gains are obtained as $K_j=Y_jQ^{-1}$.

Proof. We need only prove that conditions (15)-(18), (30)-(32) can be derived by conditions (17) and (39)-(44).

Note that $\lambda_{min}\{R^{1/2}QR^{1/2}\}=\frac{1}{\lambda_{max}\{R^{-1/2}PR^{-1/2}\}}$. It follows from (44) that

$$\underline{\delta}_P>\frac{1}{2},\bar{\delta}_P<\frac{1}{\hbar}, \quad (45)$$

by using which conditions (18) and (32) can be obtained based on (41)-(42).

Let $Q=P^{-1}$, $\bar{\varepsilon}_i=\varepsilon_i^{-1}$ and note that $Y_j=K_jQ$. By utilizing (39)-(40), one can easily verify that conditions (15)-(16), (30)-(31) hold. □

To solve the inequalities in Theorem 4, the parameter μ need be given in prior, which should be selected within the interval $[0, \frac{1}{\mathcal{T}}\ln\frac{c_2}{c_1}]$ according to conditions (41)-(44). On the other hand, as smaller control gains usually mean fewer energy demands, it would be better if smaller control gains can be obtained by using a suitable μ. Therefore, in the sequel, a searching algorithm is further provided to acquire an ideal μ within $[0, \frac{1}{\mathcal{T}}\ln\frac{c_2}{c_1}]$ such that the norm of the control gains are minimum.

Searching Algorithm:

1) Given parameters c_1, c_2, $\mathcal{T}$, ϖ. Set $\gamma = 0$.
2) Let sp $= 0.002$, $N = \lfloor \frac{\frac{1}{\mathcal{T}}\ln\frac{c_2}{c_1}}{\text{sp}} \rfloor$.
3) **for** $\ell = 1$ **to** N
4) Let $\eta = \text{sp} * \ell$ and substitute η to LMIs (17), (39)-(44).
5) Solve LMIs (17), (39)-(44).
6) **if** LMIs (17), (39)-(44) are feasible.
7) Calculate $\sum_{i=1}^{p} \|K_i\|$ by $K_i = Y_i Q^{-1}$, where Y_i and Q are solutions of conditions (17), (39)-(44).
8) Let $K_i^{\ell} = K_i$, $\gamma = \gamma + 1$.
9) **else**
10) $\sum_{i=1}^{p}\sum_{j=1}^{p} \|K_i\| = \text{inf}$.
11) **end if**
12) **end for**
13) $K^{\ell} = \min\{\sum_{i=1}^{p}\|K_i^1\|, \sum_{i=1}^{p}\|K_i^2\|, ..., \sum_{i=1}^{p}\|K_i^N\|\}$. Denote min be the subscript of the minimum value.
14) Let $\mu = \mu_{min}$.
15) **if** $\gamma = 0$
16) Set sp $=$ sp/2. Go to step 2).
17) **end if**

4. SIMULATION

Consider a 2-rule T-S fuzzy system with the following parameters:

$$A_1 = \begin{bmatrix} -0.1 & 0.7 \\ -0.5 & 0.3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.1 \\ -1.2 \end{bmatrix},\ C = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.2 & 0.5 \\ -0.6 & 0.3 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 \\ -1 \end{bmatrix},\ w(t) = 0.1\cos(2t)$$

with $\varpi = 0.1$. The membership functions are selected as $\beta_1(x_1(t)) = \frac{b_2 - x_1^2(t)}{b_2 - b_1}$, $\beta_2(x_1(t)) = \frac{x_1^2(t) - b_1}{b_2 - b_1}$ with $b_1 = -0.5$, $b_2 = 4.5$, $x_1(t)$, $x_2(t) \in [-2, 2]$. Then, we have $\beta_1(x_1(t)), \beta_2(x_1(t)) \geq 0.1$ for $x_1(t) \in [-2, 2]$.

Let $G = [1, 0.98]$, $\mathcal{R} = \text{diag}\{0.1, 0.09\}$, $c_1 = 0.8$, $c_2 = 3$, $\mathcal{T} = 5$, $\epsilon = 0.03$, $\nu = 0.03$. By employing the searching algorithm in Section 3.3, we obtain a series of μ such that LMIs (17), (39)-(44) are feasible. Fig. 1 plots the pairs $(\mu, \|K_1\| + \|K_2\|)$ for the feasible case, and the red dot denotes the optimal value $(0.068, 1.7384)$. The optimal solutions are given as follows:

$$K_1 = [\,0.2225\ \ 0.8202\,],\ c^{\star} = 1.7067,$$
$$K_2 = [\,0.0186\ \ 0.8884\,].$$

Thus, the desirable SMC law can be computed by (8)-(9).

In the simulation, the attack function is selected as $\sigma_c(x(t)) \triangleq 0.01\cos(10t)\sqrt{x_1^2 + x_2^2 + 0.1}$ with $\zeta = 0.01$ and $\psi(x(t), t) = \sqrt{x_1^2 + x_2^2 + 0.1}$. For initial states $x(0) = [2\ -2]^T$, the simulation results are shown in Figs. 2-3. It is observed form Fig. 2 that the evolution of $x^T(t)Rx(t)$ exceeds the given threshold $c_2 = 3$ in the open-loop case, which implies that the open-loop system is not FTB. Conversely, the designed SMC law can ensure the FTB of the closed-loop systems in spite of the injection attacks. The reachability of the specified sliding surface with $\mathcal{T}^* < \mathcal{T}$ is illustrated in Fig. 3.

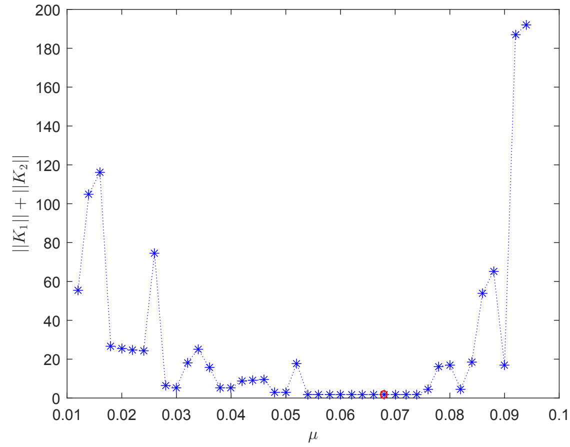

Fig. 1. The value of $\|K_1\| + \|K_2\|$

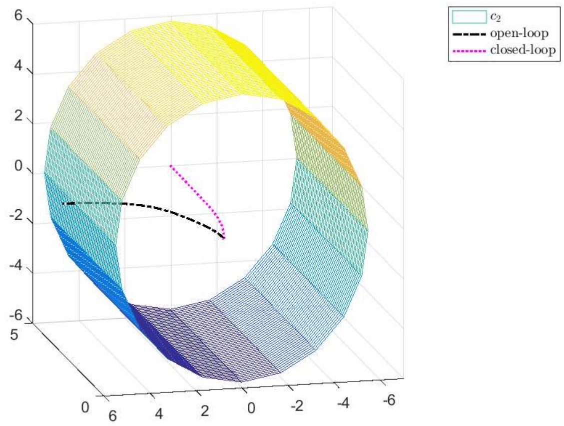

Fig. 2. The evolution of $x^T(t)\mathcal{R}x(t)$

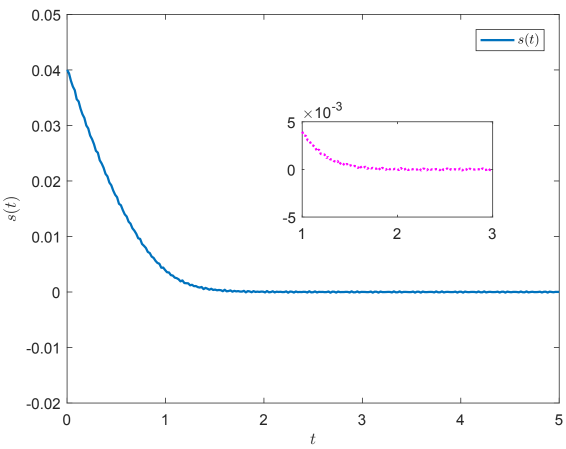

Fig. 3. The sliding variable $s(t)$

5. CONCLUSION

In this work, we have investigated the finite-time SMC of a class of T-S fuzzy systems subject to injection attacks. The reachability of the specified sliding surface can be ensured by a fuzzy sliding mode controller. The FTB of the corresponding closed-loop fuzzy systems over the whole time interval has been analyzed based on the partitioning approach. It is noted that the system membership functions in this paper are crisp functions. However, in practical applications, parameter uncertainties in membership functions may happen. Thus, future work will focus on the finite-time SMC for interval type-2 fuzzy systems.

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